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Phase transitions and the Renormalization Group

Summer term 2022

Problem set 4

Discussion of problems: Monday, July 4 (or Thursday, July 7)

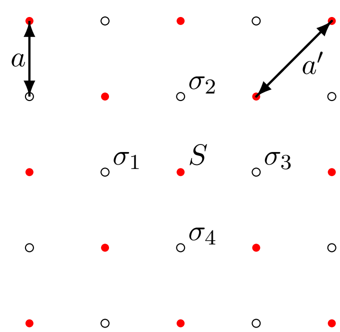
June 27, 2022

Problem 7: Renormalization Group analysis of the 2 dimensional Ising model

In this problem we investigate the nearest-neighbor Ising model in two dimensions on a rectangular lattice with spacing a via the Renormalization Group (RG) using successive *decimation* of degrees of freedom. The Hamiltonian of the system is given by

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j. \tag{1}$$

Here J denotes the coupling strength of a spin to its 4 nearest neighbors.



1. We define a Renormalization Group transformation as the spin sum over all red marked lattice sites (see Figure). What is the lattice spacing a' of the coarse grained system? Consider a typical term of the spin sum in the partition function and show that after one RG step this term takes the following form ($K_1 = \beta J$):

$$\sum_{S=\pm 1} e^{K_1 S(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} = 2\cosh K_1(\sigma_1+\sigma_2+\sigma_3+\sigma_4)$$
$$= e^{K_0' + \frac{1}{2}K_1'(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)+K_2'(\sigma_1\sigma_3+\sigma_2\sigma_4)+K_3'\sigma_1\sigma_2\sigma_3\sigma_4}, \tag{2}$$

with

$$K_0'(K_1) = \log 2 + \frac{1}{8}\left(\log\cosh 4K_1 + 4\log\cosh 2K_1\right)$$
$$K_1'(K_1) = 2K_2'(K_1) = \frac{1}{4}\log\cosh 4K_1$$
$$K_3'(K_1) = \frac{1}{8}\left(\log\cosh 4K_1 - 4\log\cosh 2K_1\right).$$

and the RG-transformed Hamiltonian consequently takes the following form:

$$\beta H' = -K_0' - K_1'\sum_{\langle i,j\rangle}\sigma_i\sigma_j - K_2'\sum_{\langle\langle i,j\rangle\rangle}\sigma_i\sigma_j - K_3'\sum_{\text{4 spin}}\sigma_i\sigma_j\sigma_k\sigma_l. \tag{3}$$

Why did we include a factor 1/2 in front of K_1' in Eq.(2)? Discuss the physical meaning of the different coupling constants and the implications for the following RG steps. Do you expect that the RG transformations can be computed exactly like for the one dimensional Ising model? What can you say about the numerical size of these couplings? For the analysis you can use the value at the critical point $K_1 = K_1^c = J/(k_B T_c) = 0.44006$ obtained from the exact solution by Onsager.

HINT: Derive relations for the coupling constants by fixing the spin orientations of σ_i in Eq.(2) and solve the resulting system.

2. Now consider a Hamiltonian including nearest neighbor and next-to-nearest neighbor interactions:

$$\beta H = -K_1\sum_{\langle i,j\rangle}\sigma_i\sigma_j - K_2\sum_{\langle\langle i,j\rangle\rangle}\sigma_i\sigma_j. \tag{4}$$

Here the couplings K_1 and K_2 are defined in complete analogy to the interactions defined by the couplings K_1' and K_2' respectively in Eq.(2). Show that the couplings K_1' and K_2' after one RG transformation are given by the following relations up to quadratic order in K_1:

$$K_1'(K_1) = 2K_1^2 + K_2, \quad K_2' = K_1^2. \tag{5}$$

Study the RG flow in the parameter space (K_1, K_2) using the relations (5). For this choose different starting values (K_1, K_2) and visualize the flow graphically. Determine all fixed points (K_1^*, K_2^*) of the RG flow and interpret their physical meaning. Discuss why the fixed point at finite K_1^* and K_2^* corresponds to the critical point. Study in particular the flow around this fixed point.

3. Study the RG flow around the critical fixed point analytically by linearizing the RG transformation: $K_1 = K_1^* + \delta K_1, K_2 = K_2^* + \delta K_2$. Show that the RG transformation takes the following form:

$$\delta K_1' = \frac{4}{3}\delta K_1 + \delta K_2, \quad \delta K_2' = \frac{2}{3}\delta K_1. \tag{6}$$

Compute the eigenvalues $\lambda_{1/2}$ and eigenvectors $\mathbf{e}_{1/2}$ of the RG transformation and consider a pair of coupling constants (K_1^0, K_2^0) by expanding the deviation from the non-trivial fixed point in terms of the eigenvectors:

$$\begin{pmatrix} K_1^0 \\ K_2^0 \end{pmatrix} = \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} + u_1\mathbf{e}_1 + u_2\mathbf{e}_2. \tag{7}$$

Address the following questions:

- analyze the flow of this set of coupling constants after n RG transformations

- determine the critical trajectory, i.e. the set of all couplings that flow into the critical point and identify the relevant and irrelevant fields
- argue that all points on the critical trajectory correspond to the critical point, i.e. $\xi = \infty$ for all points on the critical manifold
- estimate the value of the critical coupling K_1^c of the nearest neighbor Ising model by computing the intersection point of the linearized critical RG trajectory with the $K_2 = 0$ axis, compare your results with the exact value $K_1^c = 0.44006$ and the mean field result $K_1^c = 0.25$
- determine the critical exponent ν (reminder: $\xi \sim |T - T_c|^{-\nu}$).
- discuss the RG flow qualitatively for the case of a nonvanishing external magnetic field
- how could our calculations be improved?