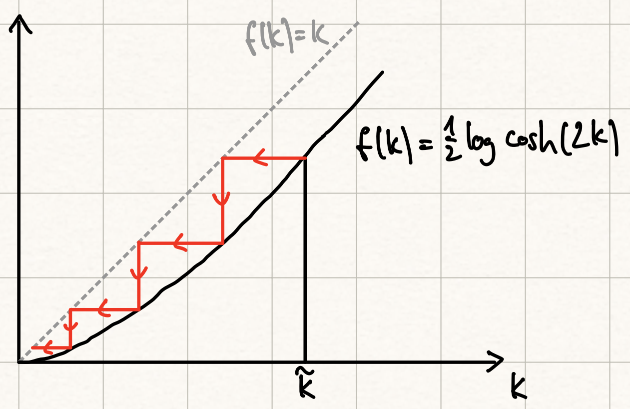
Review of previous lecture (June 27)

RG analysis of 1d Ising model

RG-transformed Hamiltonian: $\quad -\beta H' = k_0' \mathbb{1} + k_1' \sum_{\langle ij \rangle} \sigma_i \sigma_j$

only nearest-neighbor interaction!

$\Rightarrow$ RG equations exactly solvable



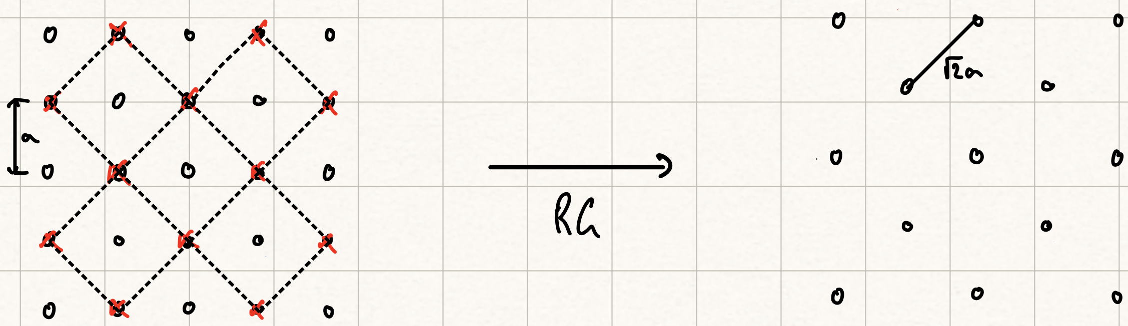
two fixed points:

$$k_1^* = \frac{1}{2} \log \cosh(2 k_1^*)$$

1.) $k_1^* = 0$ $\quad (T \to \infty)$: complete disorder

2.) $k_1^* = \infty$ $\quad (T = 0)$: perfect order

RG analysis of 2d Ising model



RG-transformed Hamiltonian:

$$-\beta H' = K_0' \mathbb{1} + \underbrace{\tfrac{1}{2} K_1' (\sigma_1\sigma_2 + \dots + \sigma_3\sigma_4)}_{\text{nearest-neighbor interaction}} + \underbrace{K_2' (\sigma_1\sigma_3 + \sigma_2\sigma_4)}_{\text{next-to-nearest neighbor int.}} + \underbrace{K_3' \sigma_1\sigma_2\sigma_3\sigma_4}_{\text{4-spin interaction}}$$

additional RG transformations will generate even more couplings!

$\Rightarrow$ RG equations NOT exactly solvable!

RG transformed correlation length:

$R_\ell[\xi] = \dfrac{\xi}{\ell}$ when block spins with spacing $\ell \cdot a$ are generated

general properties of RG transformations

consider the general form of a RG transformation

$$[k'] = R_\ell [k] \qquad \ell > 1$$

where $[k]$ and $[k']$ is the set of coupling constants before and after the RG transformation

- the RG transformation is in general a complicated non-linear transformation (see 2d Ising model)
- in general the set $[k']$ is larger than $[k]$
- RG transformations form a semi-group, i.e. products of RG transf. are also RG transformations:

$$[k'] = R_{\ell_1}[k]$$
$$[k''] = R_{\ell_2}[k'] = R_{\ell_2} \cdot R_{\ell_1}[k] = R_{\ell_1 \cdot \ell_2}[k]$$

and $R_{\ell=1}[k] = k$

however, generally the inverse R_ℓ^{-1} does not exist ("semi"-group)

- in general any RG transformation can be defined via a projection operator in the partition function:

$$e^{-\beta H([k'], \sigma_I')} = \mathrm{Tr}'_{\{\sigma_i\}} \, e^{-\beta H([k], \sigma_i)}$$

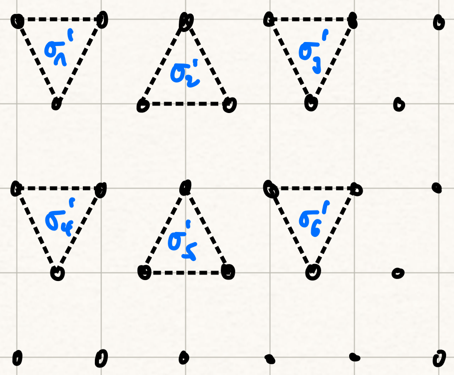
block spins

restricted trace

$$\equiv \mathrm{Tr}_{\{\sigma_i\}} \, P(\sigma_i, \sigma_I') \, e^{-\beta H([k], \sigma_i)}$$

↳ projection operator, constructed such that block spins σ_I' have same range of values like σ_i

example: 2d Ising model on a triangular lattice



block spins

$$\sigma_I' = \operatorname{sign}\Big(\sum_{i \in I} \sigma_i\Big) = \pm 1$$

$$\Rightarrow P(\sigma_i, \sigma_I) = \prod_I \delta\Big(\sigma_I' - \operatorname{sign}\Big(\sum_{i\in I} \sigma_i\Big)\Big)$$

in general the projection operator must satisfy the following conditions:

a) $P(\sigma_i, \sigma_I') \geq 0 \quad \Rightarrow$ guarantees $e^{-\beta H([k'], \sigma_I')} \geq 0$

b) $P(\sigma_i, \sigma_I')$ reflects symmetries of system

$\Rightarrow$ guarantees that $H([k'])$ and $H([k])$ have the same form, only couplings are changing

c) $\sum_{\{\sigma_I'\}} P(\sigma_i, \sigma_I') = 1$

$\Rightarrow$ guarantees invariance of partition functions

$$Z_{N'}[k'] \equiv \operatorname{Tr}_{\{\sigma_I'\}} e^{-\beta H([k'], \sigma_I')}$$

$$= \operatorname{Tr}_{\{\sigma_I'\}} \operatorname{Tr}_{\{\sigma_i\}} P(\sigma_i, \sigma_I')\, e^{-\beta H([k], \sigma_i)}$$

$$= \operatorname{Tr}_{\{\sigma_i\}} e^{-\beta H([k], \sigma_i)} = Z_N[k]$$

in practice condition c) is not fulfilled as the set of couplings $[k']$ needs to be truncated the power of the RG is based on the fact that it is much easier and systematic to approximate $[K']$ than the partition function Z

fixed points

at each RG step only a finite number of degrees of freedom are being coarse grained

$\Rightarrow$ RG transformations have to be analytic

for calculation of partition function in the thermodynamic limit $N \to \infty$ an infinite number of RG steps is required

$\Rightarrow$ free energy can become non-analytic $\to$ phase transitions

consider a series of n RG transformations:

- coarse graining length is ℓ^n and system is described by a set of couplings $K_0^{(n)}, K_1^{(n)}, \dots$
- at each stage (for each n) the system can be characterized by a point in the space labeled by the axes $K_0, K_1, K_2, \dots$
- RG transformations trace out a trajectory in coupling constant space
- set of all trajectories, generated by different initial sets of coupling constants define the RG flow

- for $n \to \infty$ it is found that trajectories become attracted to fixed points with $[k^*] = R_\ell [k^*]$
- set of all initial conditions $[k_0]$ whose trajectories flow to a given fixed point is called the basin of attraction of that fixed point
- in general we know that $\xi[k'] = \frac{\xi[k]}{\ell}$

at a fixed point we have $\xi[k^*] = \frac{\xi[k^*]}{\ell}$

$$\Rightarrow \underbrace{\xi[k^*] = 0}_{\text{trivial fixed point}} \quad \text{or} \quad \underbrace{\xi[k^*] = \infty}_{\text{critical fixed point}}$$

- using $\xi[k] = \ell \xi[k'] = \ell^2 \xi[k''] = \ell^n \xi[k^{(n)}]$

$\Rightarrow$ all points in the basin of attraction of a critical fixed point have infinite scattering length!
(also called critical manifold)

study RG flow near a fixed point in more detail:

$$K_n = K_n^* + \delta K_n$$

δK_n ↳ small

$$\Rightarrow K_n' = R_\ell[K_n] = K_n^* + \delta K_n'$$

$$= K_n^* + \sum_m \underbrace{\left.\frac{\partial K_n'}{\partial K_m}\right|_{K_m = K_m^*}}_{\equiv M_{nm}^\ell} \cdot \delta K_m + O(\delta K^2)$$

$\equiv M^\ell_{nm}$: linearized RG transformation

for the following we assume M^ℓ_{nm} to be real, symmetric and diagonalizable

consider eigenvectors and eigenvalues of M^ℓ_{nm}:

$$\sum_m M^\ell_{nm} e_m^{(i)} = \lambda_\ell^{(i)} e_n^{(i)}$$

since $R_{\ell_1} \cdot R_{\ell_2} = R_{\ell_1 \cdot \ell_2}$: $M^{\ell_1} \cdot M^{\ell_2} = M^{\ell_1 \cdot \ell_2}$

$$\Rightarrow \lambda_{\ell_1}^{(i)} \cdot \lambda_{\ell_2}^{(i)} = \lambda_{\ell_1 \cdot \ell_2}^{(i)}$$

$$\Rightarrow \lambda_\ell^{(i)} = \ell^{y_i}$$ (y_i unknown, but independent of ℓ)

expand $\delta \vec{K}$ in terms of the eigenvectors $\vec{e}^{(i)}$:

$$\delta K_m = \sum_i a^{(i)} e_m^{(i)} \Rightarrow a^{(i)} = \sum_m e_m^{(i)} \delta K_m$$

$a^{(i)}$ ↓ scaling variables

for the sake of simplicity we will assume for the following discussion that the eigenvectors $\vec{e}^{(i)}$ are orthonormal, that is generally not true (see 2d Ising model), but does not change the main arguments

$$\Rightarrow \delta k_n' = \sum_m M_{nm}^l \delta k_m$$

$$= \sum_m M_{nm}^l \sum_i a^{(i)} e_m^{(i)}$$

$$= \sum_i a^{(i)} \lambda_l^{(i)} e_n^{(i)} = \sum_i \underbrace{a'^{(i)}}_{\substack{\text{projection of } \delta \vec{k}' \\ \text{on eigenvectors } \vec{e}^{(i)}}} e_n^{(i)}$$

depending on $\lambda_l^{(i)}$, some components of δk grow under M^l while others shrink:

a) $|\lambda_l^{(i)}| > 1$, i.e. $y_i > 0$: $a^{(i)}$ grows during RG flow

$\rightarrow$ relevant eigenvectors / eigenvalues / directions

b) $|\lambda_l^{(i)}| < 1$, i.e $y_i < 0$: $a^{(i)}$ shrinks during RG flow

$\rightarrow$ irrelevant eigenvectors

c) $|\lambda_l^{(i)}| = 1$, i.e. $y_i = 0$: $a^{(i)}$ invariant

$\rightarrow$ marginal eigenvectors

$\Rightarrow$ for $\vec{k}$ near $\vec{k}^*$ (<u>not</u> on the critical manifold), the RG flows <u>away</u> from $\vec{k}^*$ are associated with <u>relevant</u> eigenvectors, <u>irrelevant</u> eigenvectors correspond to directions of flow <u>into</u> the fixed point

$\Rightarrow$ eigenvectors corresponding to irrelevant eigenvalues span critical manifold

global and local properties of the RG flow

global behaviour of RG flow determines phase diagram:

a) start at a given point $\vec{K}^{(0)} = (K_0^{(0)}, K_1^{(0)}, K_2^{(0)}, \dots)$

b) iterate RG transformations: $\vec{K}^{(0)} \to \vec{K}^{(1)} \to \vec{K}^{(2)} \to \dots \to \vec{K}^{(n)}$

c) identify fix points to which the system flows,
state of system described by this fixed point corresponds to the phase at the original point $\vec{K}_0$ (note that partition function is preserved along RG trajectory if RG transformations are performed exactly)

classification of fixed points:

a) sinks: all trajectories flow into fix point (no relevant directions), sinks correspond to bulk phases

example: Ising model in 2d

sink at $B = \pm\infty$, $T = 0$: at finite B there is a finite magnetization for all T

b) discontinuity / continuity fixed points (1 relevant direction)

$\downarrow$

phase boundary	phase of system
	no transition in vicinity
for Ising model:	
all points on line $B=0$ for $T<T_c$	all points on line $B=0$, $T>T_c$
$\to$ flow to $B=0$, $T=0$	$\to$ flow to $B=0$, $T=\infty$
first order phase transition when crossing $B=0$ from $B>0$ or $B<0$	

both fixed points unstable with respect to $B=0 \to B=0^{\pm}$
(relevant direction)

$\downarrow$
flows to sinks (a)

c) critical points, multiphase coexistence (2 relevant directions)
2 couplings must be tuned to place system at the critical point ($B=0$, $T=T_c$)

$\downarrow$
flow into critical fixed point

Table 9.1 CLASSIFICATION OF FIXED POINTS

Codimension	Value of ξ	Type of Fixed Point	Physical Domain
0	0	Sink	Bulk phase
1	0	Discontinuity FP	Plane of coexistence
1	0	Continuity FP	Bulk phase
2	0	Triple point	Triple Point
2	∞	Critical FP	Critical manifold
Greater than 2	∞	Multicritical point	Multicritical point
Greater than 2	0	Multiple coexistence FP	Multiple coexistence

Goldenfeld p. 247

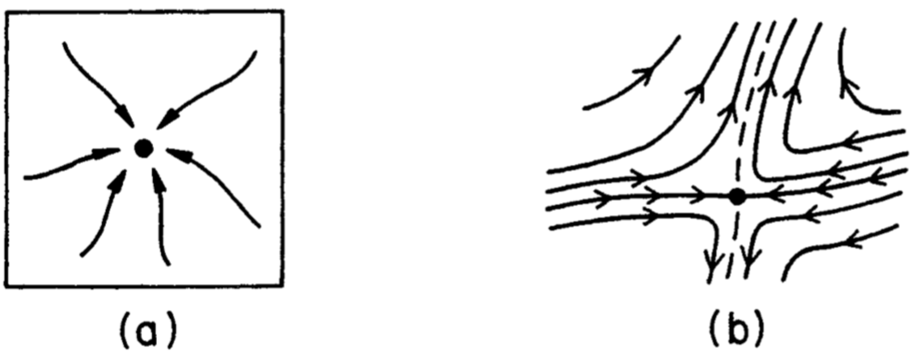


Figure 9.2 Renormalisation group flows near a critical fixed point: (a) View of flows on the critical manifold. (b) View of flows off the critical manifold.

Goldenfeld p. 248

$$\mathcal{H} = K_1 \sum_{<ij>} S_i S_j + K_2 \sum_{ij=n.n.n.} S_i S_j \quad K_1 = J_1/k_B T \quad K_2 = J_2/k_B T$$

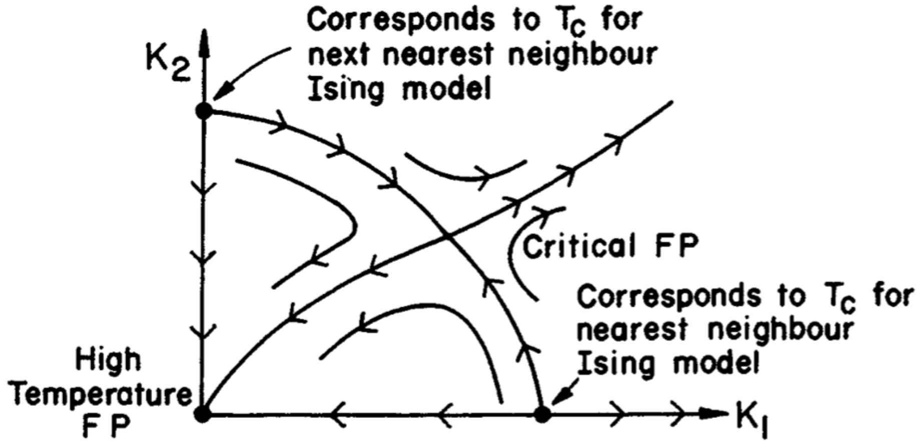


Figure 9.3 Flow diagram for an Ising model with nearest and next nearest neighbour interactions.

Goldenfeld p. 249

local properties of RG flow around critical fixed point determines critical behavior

- trajectories on the critical manifold remain on the manifold and flow to critical fixed point
- trajectories that start slightly off the critical manifold initially flow towards critical fixed point, but are ultimatelly repelled due to relevant couplings
- the same relevant eigenvalues (see below) drive all slightly off-oriented systems away from critical manifold (for a given universality class), independent of original values of coupling constants

↳ universality