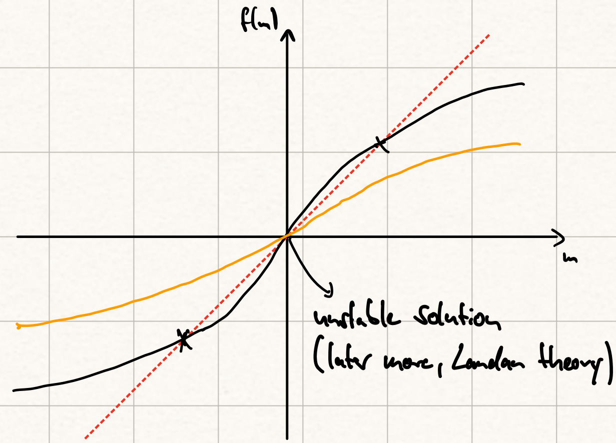
# Review of previous lecture (May 19)

equation of state and critical exponents of 1d Ising model in mean-field approximation

$$m = \tanh\left(\beta\left(B + 2Jm\right)\right)$$




expanding for small B and m:

$$m \sim |T - T_c|^{\beta} \quad \Rightarrow \quad \beta = \tfrac{1}{2}$$

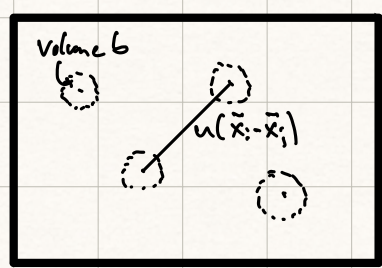
$$B \sim m^{\delta} \quad (\text{at } T = T_c) \quad \Rightarrow \quad \delta = 3$$

$$\chi_T = \frac{\partial m}{\partial B}\Big|_T \sim |T - T_c|^{-\gamma} \quad \Rightarrow \quad \gamma = 1$$

liquid-gas transition, van-der-Waals equation of state

$$p = -\frac{\partial F}{\partial V}\Big|_{T,N} = \frac{N k_B T}{V - Nb} - \frac{N^2}{V^2}\, a$$

$$a = -\frac{1}{2}\int d^3(\vec{x} - \vec{x}')\, u(\vec{x} - \vec{x}')$$




$$\left.\begin{aligned} V_g - V_e &\sim |T - T_c|^{\beta} \\ |p - p_c| &\sim |V - V_c|^{\delta} \\ \kappa_T = -\frac{1}{V}\frac{\partial V}{\partial p}\Big|_T &\sim |T - T_c|^{-\gamma} \end{aligned}\right\}$$ same critical exponents like Ising model

# Landau theory of phase transitions

there is a deeper reason why the critical exponents are identical for ferromagnetic transitions and liquid-gas transitions

Consider equations of state close to $T_c$:

**Ising**

$$\frac{B}{k_B T} = m(1-\tau) + m^3\left(\tau - \tau^2 + \frac{\tau^3}{3} + \dots\right)$$

$$= \eta t + \eta^3 + O(t\eta^3)$$

$$\left[\tau = \frac{T_c}{T},\ \eta = m,\ t = \frac{T-T_c}{T_c}\right]$$

**Van-der Waals** → exercise

$$p_R = \frac{8T_R}{3V_R - 1} - \frac{3}{V_R^2}$$

$$= \frac{8(1+t)}{3(1+\phi)-1} - \frac{3}{(1+\phi)^2}$$

$$= 1 + 4t - 6t\phi - \frac{3}{2}\phi^3 + O(t\phi^2)$$

$$\left[p_R = \frac{p}{p_c},\ V_R = \frac{V}{V_c},\ T_R = \frac{T}{T_c},\ t = \frac{T-T_c}{T_c},\ \phi = \frac{V-V_c}{V_c}\right]$$

mapping:

$$\frac{B}{k_B T} \Leftrightarrow p_R - 1 - 4t$$

$$m = \eta \Leftrightarrow -\frac{V-V_c}{V_c} = -\phi$$

⇒ identical equation of state !
(modulus prefactors)

# Landau theory of phase transitions

the equations of state can be derived from an energy function

$$L(p, t, \phi) \quad \text{resp.} \quad L(B, t, m)$$

via a minimization with respect to $\phi$ resp. $m$ (order parameter)

for example, mean-field solutions for Ising model:

$$L(B,t,m) = L_0(B,t) + c\left(-\frac{\eta B}{k_B T} + \frac{t\eta^2}{2} + \frac{\eta^4}{4}\right)$$

$$\frac{\partial L}{\partial \eta} = 0 \quad \Rightarrow \quad \frac{B}{k_B T} = \eta t + \eta^3 + O(t\eta^3)$$

Note: the energy function $L$ contains all leading terms in the order parameter $\eta \to 0$ consistent with the symmetries of the system (details below).

Basic idea of Landau theory:

Postulate a free energy ("Landau free energy") $L$ that depends on a set of coupling constants $\{k_i\}$ (e.g. $B, T$, J for the Ising model) and an order parameter $\eta$ ($\eta = m$ for the Ising model). The ground state of state is given by the global minimum of $L$. Assume that any continuous phase transition can be described in this way.

constraints on $L$:

a) $L$ has to be consistent with symmetries of system

b) for $T \to T_c$ we have $\eta \to 0$, assume that $L$ is analytic function of $\eta$ and $\{k_i\}$, i.e. for a uniform system:

$$\mathcal{L} = \frac{L}{V} = \sum_{n=0}^{\infty} a_n(\{k_i\})\, \eta^n$$

c) for inhomogeneous system the order parameter becomes a function of position, $\eta(\vec{x})$. $\mathcal{L}(\vec{x})$ with $L = \int d^d\vec{x}\, \mathcal{L}(\vec{x})$ is a local function, i.e. it is a function of $\eta(\vec{x})$ plus gradients of $\eta(\vec{x})$.

d) in the disordered phase we have $\eta = 0$ and $\eta \neq 0$ in the ordered phase for $T < T_c$.

construction of $\mathcal{L}$: Ising model

* for $\underline{B=0}$ the system has the symmetry $H(\{\sigma_i\}) = H(\{-\sigma_i\})$, hence $\mathcal{L}(\eta) = \mathcal{L}(-\eta)$

$$\Rightarrow \mathcal{L} = a_0 + \cancel{a_1 \eta} + a_2 \eta^2 + \cancel{a_3 \eta^3} + a_4 \eta^4 + \dots$$

$a_1 = 0$ since $\eta = 0$ for $T > T_c$ $\left(\frac{\partial \mathcal{L}}{\partial \eta} = 0\right)$
i.e. even without $\eta \leftrightarrow -\eta$ symmetry this term has to vanish

* if we truncate at $a_4$ we need $a_4 > 0$, otherwise we can minimize $\mathcal{L}$ for $\eta \to \pm\infty$

* each parameter $a_i$ generally depends on $T$ : $a_i(T)$

* at large $T$ we have $\eta = 0$ and hence

$$\mathcal{L}(T \gg T_c) = a_0(T)$$

smooth background term, does not involve order parameter

$\Rightarrow$ can set $a_0(T) = 0$ without loss of generality

* expand $a_2(T)$ and $a_4(T)$ in temperature around $T_c$:

$$a_2(T) = a_2^0 + \frac{T-T_c}{T_c} a_2^1 + O((T-T_c)^2)$$

$$a_4(T) = a_4^0 + \frac{T-T_c}{T_c} a_4^1 + O((T-T_c)^2)$$

$$\Rightarrow \mathcal{L} = \left(a_2^0 + \frac{T-T_c}{T_c} a_2^1\right)\eta^2 + \left(a_4^0 + \frac{T-T_c}{T_c} a_4^1\right)\eta^4$$

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0 \Rightarrow 2\left(a_2^0 + \frac{T-T_c}{T_c} a_2^1\right)\eta + 4\left(a_4^0 + \frac{T-T_c}{T_c} a_4^1\right)\eta^3 = 0$$

for $T < T_c$ we have $\eta \neq 0$ $\Rightarrow$ $\eta = \sqrt{\frac{-\boxed{\left(a_2^0 + \frac{T-T_c}{T_c} a_2^1\right)}}{2\left(a_4^0 + \cancel{\frac{T-T_c}{T_c} a_4^1}\right)}}$

for real solution we need $a_2^0 = 0$ and $a_2^1 > 0$

subleading can set $a_4^1 = 0$

$$\Rightarrow \eta = \sqrt{\frac{-\frac{T-T_c}{T_c} a_2^1}{2a_4^0}}, \quad \mathcal{L} = a_2^1 \frac{T-T_c}{T_c}\eta^2 + a_4^0 \eta^4$$

* for $B \neq 0$ we obtain an additional term in $\mathcal{L}$:

$$\mathcal{L} = a_2^1 t \eta^2 + a_4^0 \eta^4 - B\eta$$

$\hookrightarrow$ since $H \sim -B \sum_i \sigma_i$

microscopic parameters $a_i$ cannot determined within Landau theory, but need to be determined from the underlying Hamiltonian or experimentally

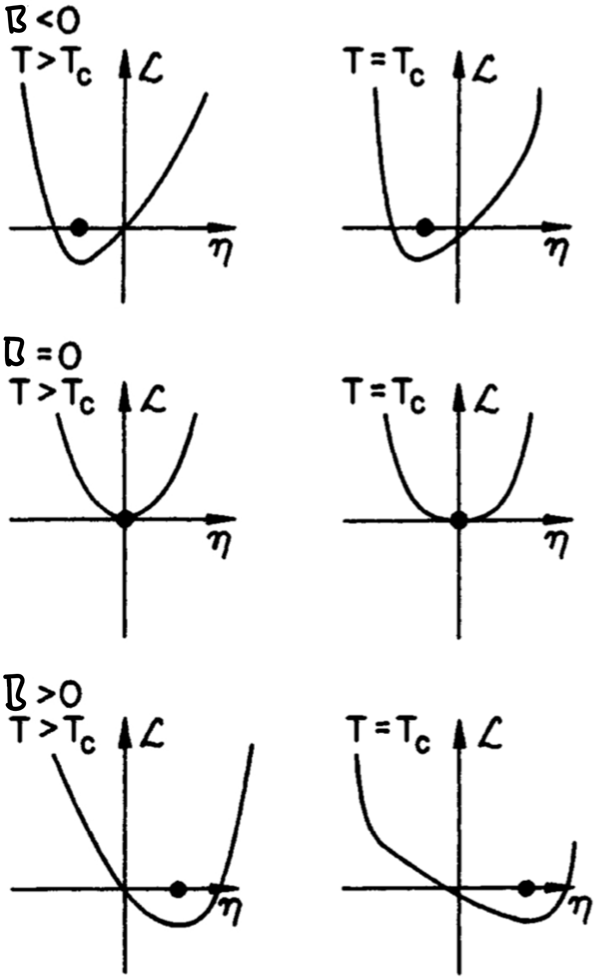



**Figure 5.1** The Landau free energy density for various values of $T$ and $H$. The • indicates the value of $\eta$ at which $\mathcal{L}$ achieves its global minimum. The right-most column of graphs depicts the first order transition, which occurs for $T < T_c$ as $H$ is varied from a negative to a positive value. The central row depicts the continuous transition, which occurs for $H = 0$ as $T$ is varied from above $T_c$ to below $T_c$.

Goldenfeld, p. 143

critical exponents:

- $\eta \sim |T-T_c|^\beta$, we found $\eta(T) = \sqrt{\frac{-a_2 t}{2a_4}} \Rightarrow \beta = \frac{1}{2}$
- for $B \neq 0$ we found $2a_2^1 t\eta + 4a_4^0 \eta^3 = B$

  at $T = T_c$ we have $t = 0 \Rightarrow B \sim \eta^3 \Rightarrow \delta = 3$

$$-\chi_T(B) = \frac{\partial \eta}{\partial B}\Big|_T \Rightarrow 2a_2^1 t \frac{\partial \eta}{\partial B} + 12 a_4^0 \eta^2 \frac{\partial \eta}{\partial B} = 1$$

$$\Rightarrow \chi_T = \frac{1}{2}\Big(a_2^1 t + 6 a_4^0 \underbrace{\eta^2}_{\text{scales} \sim t \text{ for } t>0 \text{ (see above)}}\Big)^{-1} \sim t^{-1} \Rightarrow \gamma = 1$$

## First-order phase transitions

consider now a more general form for $\mathcal{L}$:

$$\mathcal{L} = a_2^1 t \eta^2 + a_4^0 \eta^4 + \underbrace{a_3^0 \eta^3}_{\text{new}} - \underbrace{\cancel{B\eta}}_{\text{consider } B=0 \text{ in the following}}$$

note: linear term in $\eta$ associated with parameter $a_1$ not allowed since $\eta = 0$ for $T > T_c$

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0 \Rightarrow \eta = 0 \quad \text{or} \quad \eta = -c \pm \sqrt{c^2 - \frac{a_2^1 t}{2a_4^0}}$$

real solution for

$$c^2 > \frac{a_2^1 t}{2a_4^0} \Leftrightarrow t < \frac{2a_4^0 c^2}{a_2^1} \equiv t^*$$

$$\left[c = \frac{3a_3^0}{8a_4^0}\right]$$

- lowering $t$ below $t^*$ leads to 2nd minimum in $\mathcal{L}$
- lowering $t$ below $t_1$ leads to a new <u>global</u> minimum (see figures)

$\Rightarrow$ value for $\eta$ jumps <u>discontinuously</u> from $\eta = 0$ to $\eta(t_1)$

$\downarrow$

first order phase transition

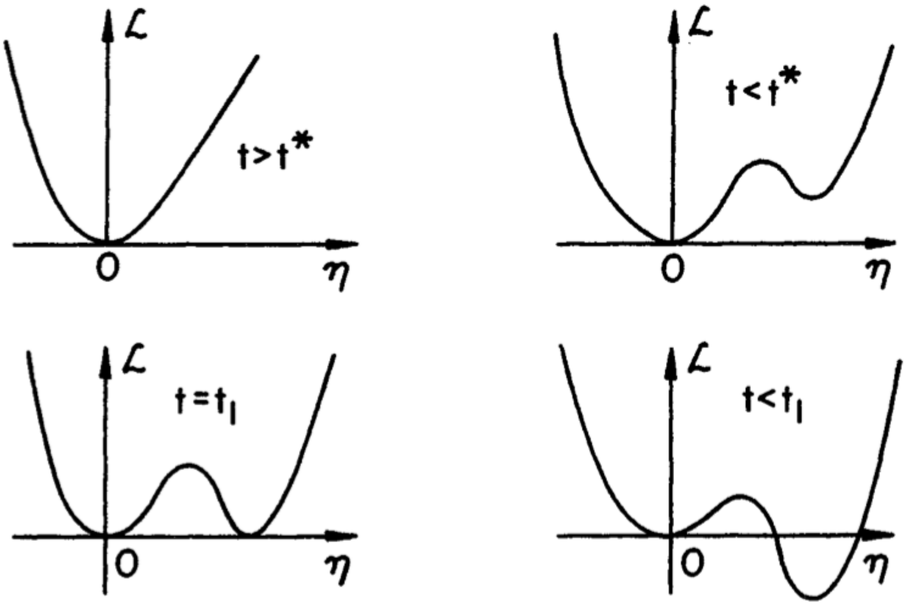



**Figure 5.2** $\mathcal{L}$ as a function of $\eta$ for various temperatures, showing the Landau theory description of a first order transition.

Goldenfeld, p. 146

$\Rightarrow$ in general a cubic term in $\mathcal{L}$ leads to a first order phase transition, absence of cubic term guarantees a continuous phase transition

<u>However:</u> note that Landau theory is in general not valid for first-order phase transitions!
why?
$\eta \not\to 0$ for $T \to T_c$, i.e. $\eta$ not neccessarily small