# Review of previous lecture (June 23)

RG transformations $R_\ell$:

$$R_\ell\left[H(\{\sigma_i\},[K_i])\right] = H'(\{\tilde{\sigma}_i\},[K_i'])$$

$\{\sigma_i\}$: N spins

$\{\tilde{\sigma}_i\}$: $\frac{N}{\ell^D}$ block spins

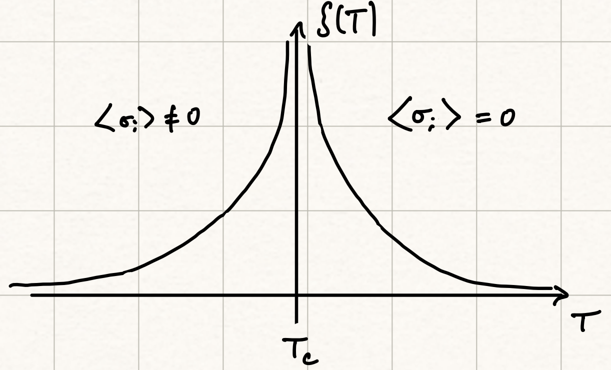
$[K_i']$: modified couplings

when RG transformations can be evaluated exactly:

$$R_\ell[Z_c] = Z_c$$

correlation length $\xi$:

$$G(i,j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$
$$\equiv \exp\left(-\frac{|i-j|}{\xi}\right)$$



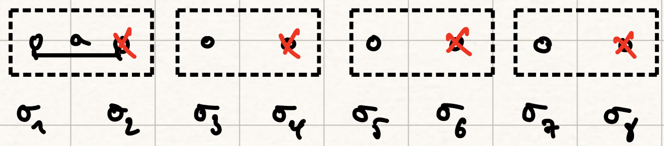

fixed points:

$$R_\ell\left[H(\{\sigma_i\},[K_i^*])\right] = H(\{\tilde{\sigma}_i\},[K_i^*])$$

trivial fixed point: $\xi[K_i^*] = 0$

critical fixed point: $\xi[K_i^*] = \infty$

# application of the RG to the 1d Ising model ($B=0$)

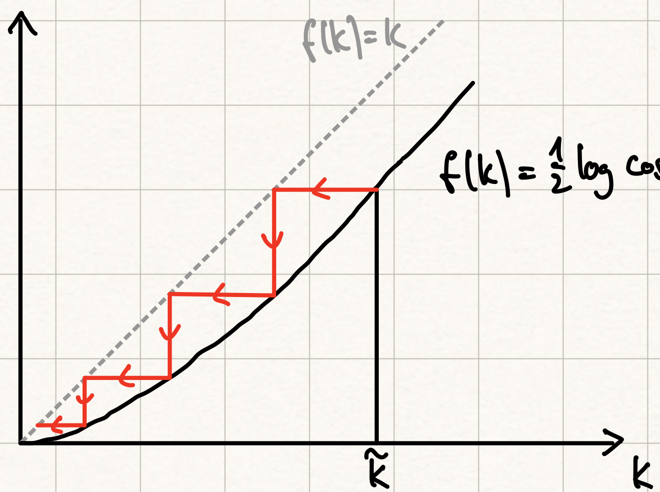



RG-transformed Hamiltonian:
$$-\beta H' = K_0' \mathbb{1} + K_1' \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

renormalized couplings:
$$K_1^{(n)} = \frac{1}{2} \log \cosh\left(2 K_1^{(n-1)}\right)$$
$$K_0^{(n)} = \log 2 + K_1^{(n)}$$

what are the fixed points of this RG transformation?

$$K_1^* = \frac{1}{2} \log \cosh(2 K_1^*)$$




for any finite initial value $\tilde{k}$ $k^{(n)}$ decreasing with every RG step

two fixed points: 1.) $K_1^* = 0$ ($T \to \infty$): complete disorder

2.) $K_1^* = \infty$ ($T = 0$): perfect order partition function:

$$Z_N(K_1) = e^{\frac{N}{2}K_0'} \; Z_{\frac{N}{2}}(K_1')$$

$$= e^{\frac{N}{2}K_0' + \frac{N}{4}K_0''} \; Z_{\frac{N}{4}}(K_1'')$$

$$= \exp\left[\frac{N}{2}\sum_{i=1}^{n}\frac{K_0^{(i)}}{2^{i-1}} + \log Z_{N/2^n}(K_1^{(n)})\right]$$

here we can use the fact that $K_1^{(n)}$ is rapidly decreasing with $n$ and we can hence set $K_1^{(n)} \sim 0$ for some large enough $n$ and we obtain for the

free energy density:

$$f^{(n)} = -k_B T \left( \frac{1}{2}\sum_{i=1}^{n}\frac{K_0^{(i)}}{2^{i-1}} + \frac{1}{2^n}\log 2 \right)$$

↳ free energy density of the non-interacting Ising model

$\Rightarrow$ rapidly approaches the exact solution

$$f_{exact} = -k_B T \log(2\cosh K_1)$$

↳ Mathematica plots

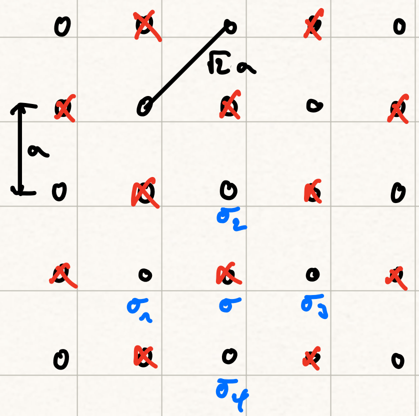
note that the form of the Hamiltonian does NOT change while applying the RG transformations (apart from the fact that the initial Hamiltonian does not contain a $K_0$ coupling ($K_0^{(0)} = 0$)

$\Rightarrow$ RG (nearest-neighbor model) = nearest-neighbor model

$\Rightarrow$ RG transformation can be evaluated exactly these properties are a particular feature of the 1d Ising model and are not true in general!

## RG analysis of the 2d Ising model

apply same strategy like in 1d:




perform partial spin sum for all crossed lattice sites

lattice spacing after RG transformation : $a \to \sqrt{2}\,a$

consider a typical term in the partition function

$$\sum_{\sigma=\pm 1} e^{K_1(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\cdot\sigma} = 2\cosh\left[K_1(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\right]$$

$$= e^{\log\left[2\cosh\left(K_1(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\right)\right]}$$

question: what is the form of the RG-transformed Hamiltonian?

$$e^{\log\left[2\cosh\left(K_1(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\right)\right]}$$

$$= e^{K_0' + \underbrace{\frac{1}{2}K_1'(\sigma_1\sigma_2+\dots+\sigma_3\sigma_4)}_{\text{nearest-neighbor interaction}} + \underbrace{K_2'(\sigma_1\sigma_3+\sigma_2\sigma_4)}_{\text{next-to-nearest neighbor int.}} + \underbrace{K_3'\sigma_1\sigma_2\sigma_3\sigma_4}_{\text{4-spin interaction}}}$$

with:

$$K_0' = \log 2 + \frac{1}{8}\left(\log\cosh 4K_1 + 4\log\cosh 2K_1\right)$$

$$K_1' = \frac{1}{4}\log\cosh 4K_1$$

$$K_2' = \frac{K_1'}{2}$$

$$K_3' = \frac{1}{8}\left(\log\cosh 4K_1 - 4\log\cosh 2K_1\right)$$

↳ exercise

for the 2d nearest-neighbor Ising model the first RG transformation induces interaction terms that were <u>NOT</u> present in the original Hamiltonian!

additional RG transformations will induce even more interaction terms

⇒ RG analysis cannot be performed exactly anymore (in contrast to 1d case).
for practical calculations we need to choose a set of interactions that are treated in the RG flow, accuracy of results will depend on this choice

in the following we restrict ourselves to a small number of terms, here analytical solutions are still possible, sufficient to illustrate general features of RG flow consider set of couplings $\{K_1, K_2\}$, i.e. taking a next-to-nearest neighbor interaction into account, the relations for small $K_1$ take the form:

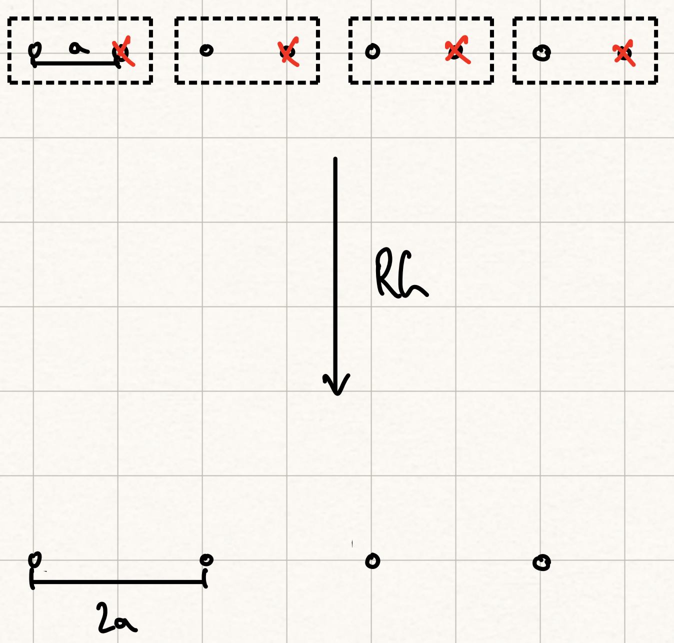
$$K_1' = 2K_1^2 + K_2$$
$$K_2' = K_1^2$$

investigation of RG flow and the significance of the different fixed points

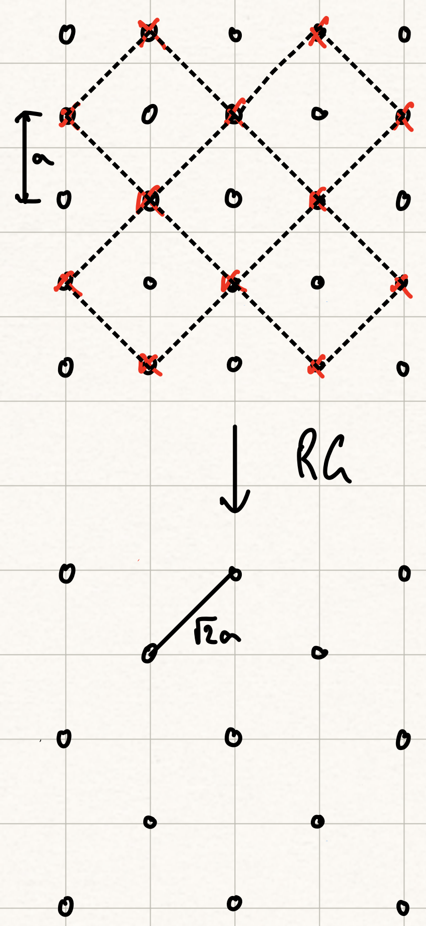
exercise

So far we have not performed any rescaling yet

1d




2d



for computation of the partition function no rescaling necessary as length scales do not enter here explicitly

what about the free energy density $f$ ?

each RG transformation changes the fundamental length scale of the system by a factor 2 (for 1d) and by a factor $\sqrt{2}$ (for 2d)

when computing $\xi$ this has to be taken into account: !

- physical value of $\xi$ remains invariant under RG transf.
- however, $\xi$ measured in units of lattice spacing changes

$$G(i,j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = e^{-|i-j|/\xi}$$

defined in lattice units!

$$\Rightarrow \quad R_\ell(\xi) = \frac{\xi}{\ell}$$

when block spins with spacing $\ell \cdot a$ are generated

with each step the system moves away from criticality for finite $\xi$!