III Mean field theory

only very few interacting systems in statistical physics can be solved exactly

even just extending the very simple exact solution for the nearest-neighbour Ising model in 1d to 2d is a highly nontrivial task

↳ Lars Onsager, Phys. Rev. 65, 117
(one of the most famous paper in theoretical physics)

⇒ need methods that allow to solve general interacting systems in a approximate way.

two main options:

1.) <u>mean-field theory</u>: replace interaction with other particles by an averaged interaction ("mean field")

2.) <u>Renormalization Group</u>: more systematic framework that allows to treat interaction effects beyond the mean-field level (more later)

mean field theory for the nearest-neighbor Ising model (1d)

$$H = -B\sum_i \sigma_i - J\sum_{\langle i,j\rangle} \sigma_i\sigma_j$$

consider $J=0$:

here we have $H = -B\sum_i \sigma_i$ $\Rightarrow$ an effective one-body problem

$\Rightarrow$ partition function factorizes into product of single-particle partition functions

$$Z_c(J=0,B,T) = \prod_{i=1}^{N} \left(e^{\beta B} + e^{-\beta B}\right)$$
$$= \left[2\cosh(\beta B)\right]^N$$

$$\Rightarrow \quad m = -\frac{1}{N}\frac{\partial F}{\partial B} = \tanh(\beta B)$$

basic idea of mean-field theory:

transform H for $J \neq 0$ to a form $H = -\sum_i B^i_{eff}\,\sigma_i$

site-dependent effective field due to magnetic moment of all other spins

self consistent problem:

- B^i_{eff} generated by magnetic moment of all other spins: $B^i_{eff} \sim m$
- however m is unknown a priori
- all spins feel external field plus mean field

$$B_{\text{eff}}^i = B_i + {\sum_j}' J_{ij} \, \sigma_j$$

↳ couples only nearest neighbors, restricts sum to 2 terms

$$= \underbrace{B_i}_{\text{external field}} + \underbrace{{\sum_j}' J_{ij} \langle \sigma_j \rangle}_{\text{mean field contribution}} + \underbrace{\cancel{\sum_j \left(\sigma_j - \langle \sigma_j \rangle \right)}}_{\text{fluctuations (neglected in mean field theory)}}$$

$$\overset{B_i = B,\; J_{ij} = J}{=} B + 2Jm$$

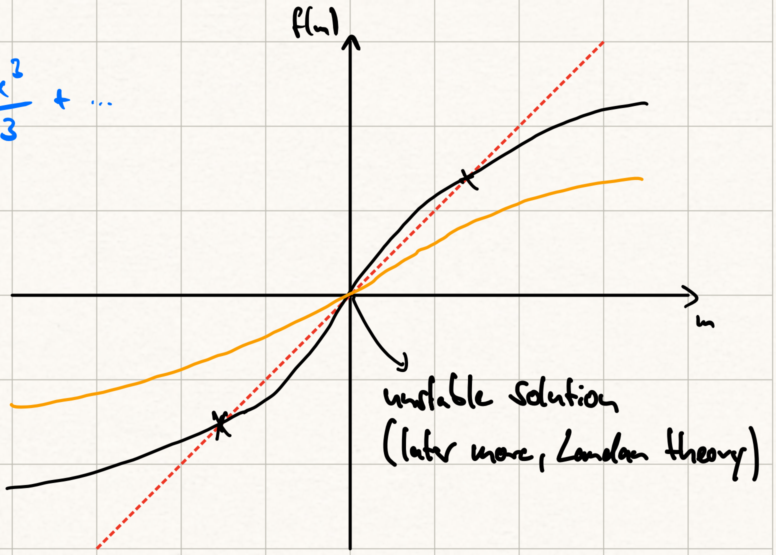
$\Rightarrow$ from free Ising model: $m = \tanh\left(\beta(B + 2Jm)\right)$

↓ implicit solution for m (self consistency) study spontaneous magnetization $(B=0)$: $m = \tanh(2\beta J m)$

for solutions $m \neq 0$ we need

$$\tanh(2\beta J m) > m$$

for small m: $\tanh x \sim x - \frac{x^3}{3} + \dots$



$$\Rightarrow \quad m = \frac{2Jm}{k_B \cdot T_c}$$

$$T_c = \frac{2J}{k_B}$$

phase transition
(in contrast to exact solution!)

determine critical exponents: $m \sim |T-T_c|^\beta$

$B \sim m^\delta \quad (\text{at } T=T_c)$

$\chi_T = \frac{\partial m}{\partial B}\Big|_T \sim |T-T_c|^\gamma$

expand equation of state around $T=T_c$:

$$m = \tanh\left(\beta\left(B + 2Jm\right)\right)$$

$$= \tanh\left(\frac{B}{k_B T} + m \cdot \tau\right) \quad \text{with} \quad \tau = \frac{T_c}{T} = \frac{2J}{k_B T}$$

$$\sim \left(\frac{B}{k_B T} + m\tau\right) - \frac{1}{3}\left(\frac{B}{k_B \cdot T} + m\tau\right)^3 + \dots$$

small B and m

for $B=0$ we obtain: $m(1-\tau) = -\frac{1}{3}m^3\tau^3$

$$\Rightarrow m^2 = -\frac{3(1-\tau)}{\tau^3} \sim (T-T_c) \quad \Rightarrow \underline{\beta = \tfrac{1}{2}}$$

<u>for δ:</u> $m(\underset{\substack{\downarrow\\0}}{1-\tau}) = \frac{B}{k_B \cdot T} - \frac{1}{3}\left(\frac{B}{k_B \cdot T} + m\tau\right)^3$

$$= \frac{B}{k_B \cdot T} - \frac{1}{3}m^3\tau^3 \underbrace{\cancel{- \frac{1}{3}\frac{B^3}{(k_B \cdot T)^3} - \frac{B}{k_B \cdot T}m^2\tau^2}}_{\text{higher order terms in powers of } B}$$

$$\Rightarrow \frac{B}{k_B T} \sim m^3 \quad \Rightarrow \underline{\delta = 3}$$

justify a posteriori

for γ: $\left.\frac{\partial m}{\partial B}\right|_T = \chi_T = \frac{1}{k_B \cdot T} + \chi_T \cdot \tau + m^2$

$$\Rightarrow \chi_T(1-\tau) = \frac{1}{k_B T} \quad \Rightarrow \chi_T \sim \frac{1}{T-T_c} \quad , \quad \underline{\gamma = 1}$$

liquid-gas transition, van-der-Waals equation of state

no phase transition for a free non-interacting gas

<u>basic idea:</u> modify ideal gas equation of state,

$$p \cdot V = N \cdot k_B \cdot T$$

so that a <u>finite volume</u> and an <u>averaged interaction</u> is taken into account