Review of previous lecture (May 12)

One dimensional Ising model : exact solution

$$f = \lim_{N\to\infty} \frac{F(h,k,T)}{N}$$

$$= -k_B \cdot T \log e^{k} \left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4k}}\right)$$

$h = \beta \cdot B \,,\; k = \beta \cdot J$

for $T \to 0$ $(k \to \infty)$: $m_{T=0}(h) = \frac{\sinh(h)}{|\sinh(h)|} = \begin{cases} 1 & \text{for } h > 0 \\ -1 & \text{for } h < 0 \end{cases}$

$B = 0$: $f(h=0,k) = \begin{cases} -J & \text{for } T = 0 \; (k \to \infty) \\ -k_B \cdot T \log 2 & \text{for } T \to \infty \; (k = 0) \end{cases}$

cf. $F = E - T \cdot S$

$S = k_B \cdot \log 2^N$

$\downarrow$ total disorder

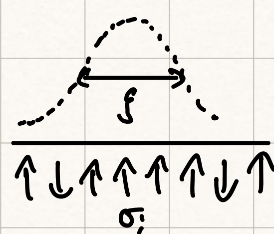
$\cdots \uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow \cdots$

two-body correlation function:

$$G(i,j) = \langle (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle) \rangle = [\tanh(k)]^{i-j}$$

$$= \exp\left(-\frac{i-j}{\xi}\right)$$

$\hookrightarrow$ correlation length



analytic properties of the free energy in the Ising model

Z_c for the Ising model consists of 2^N positive terms for any finite T:

$$F = -k_B T \log Z_c = -k_B T \log \mathrm{Tr}\, e^{-\beta H} \quad , \quad f \equiv \frac{F}{N}$$

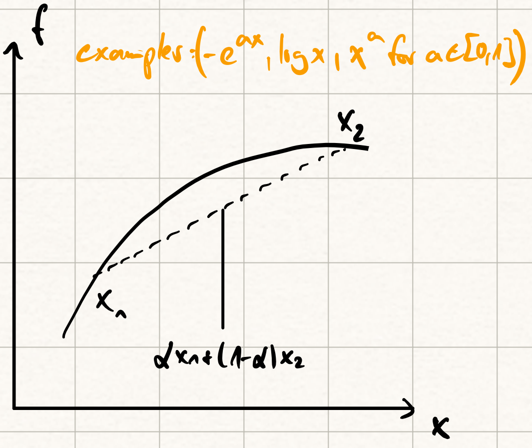
$F<0$: **($f<0$)**
- need to show that $Z_c > 1$
- this is true if there is one spin configuration $\{\sigma_i^*\}$ for which $\langle H \rangle_* < 0$ since then $\langle e^{-\beta H} \rangle_* > 1$ for $\beta > 0$
- for $B>0$ and $J>0$ the configuration $\sigma_i^* = +1$ is such a configuration (for $B<0$, $\sigma_i^* = -1$)
- all other contributions to $Z_c > 0$ $\Rightarrow$ q.e.d.

F is analytic: logarithm only develops singularities when argument vanishes $\Rightarrow$ q.e.d. (see above).

F and f convex up in B:

for this we use the Hölder inequality:

$$\sum_i a_i^\alpha b_i^{(1-\alpha)} \leq \left(\sum_i a_i\right)^\alpha \left(\sum_i b_i\right)^{1-\alpha}$$



$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2)$$
$$\alpha \in [0,1]$$

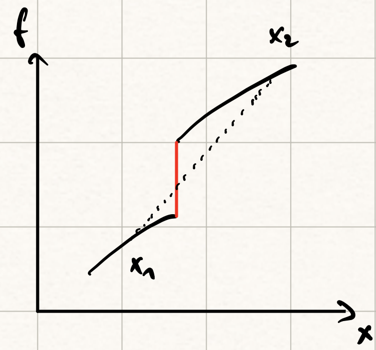
$$Z_c(B) = \mathrm{Tr}\left[e^{\beta B \sum_i \sigma_i} \underbrace{e^{\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j}}_{G[\sigma]} \right]$$

$$Z_c(\alpha B_1 + (1-\alpha)B_2) = \mathrm{Tr}\, \exp\left(\beta \alpha B_1 \sum_i \sigma_i + \beta(1-\alpha) B_2 \sum_i \sigma_i\right) G[\sigma]$$
$$= \mathrm{Tr}\left(\exp\left(\beta B_1 \sum_i \sigma_i\right) \cdot G[\sigma]\right)^\alpha \left(\exp\left(\beta B_2 \sum_i \sigma_i\right) G[\sigma]\right)^{1-\alpha}$$

$$\le \mathrm{Tr}\left(\exp\left(\beta B_1 \sum_i \sigma_i\right) \cdot G[\sigma]\right)^{\alpha} \; \mathrm{Tr}\left(\exp\left(\beta B_2 \sum_i \sigma_i\right) G[\sigma]\right)^{1-\alpha}$$

$$= Z_c(B_1)^{\alpha} \; Z_c(B_2)^{1-\alpha}$$

$$\Rightarrow F(\alpha B_1 + (1-\alpha) B_2) \ge \alpha F(B_1) + (1-\alpha) F(B_2)$$ q.e.d.

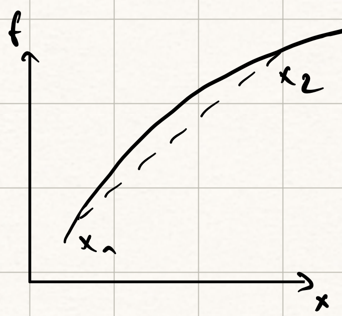


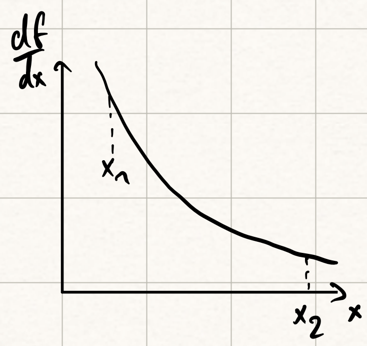
F and f are continuous (implied by convexity)

any discontinuity violates convexity:

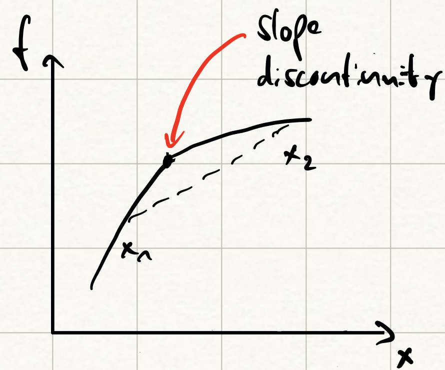
$\frac{\partial F}{\partial x}, \frac{\partial f}{\partial x}$ monotonically non-increasing

direct consequence of convexity:





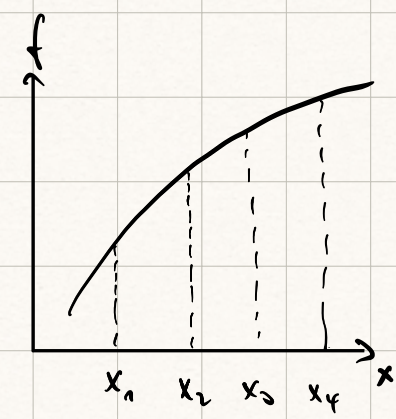
f is differentiable almost everywhere



convexity implies:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \ge \frac{f(x_4) - f(x_3)}{x_4 - x_3}$$

i.e.: $f'_-(x) \ge f'_+(x) \ge f'_-(y) \ge f'_+(y)$, for $x < y$



for every point x where $f'_-(x) > f'_+(x)$

pick rational number q_i with $f'_-(x) < q_i < f'_+(x)$

all q_i have to be distinct $\Rightarrow$ countable non-analytic points