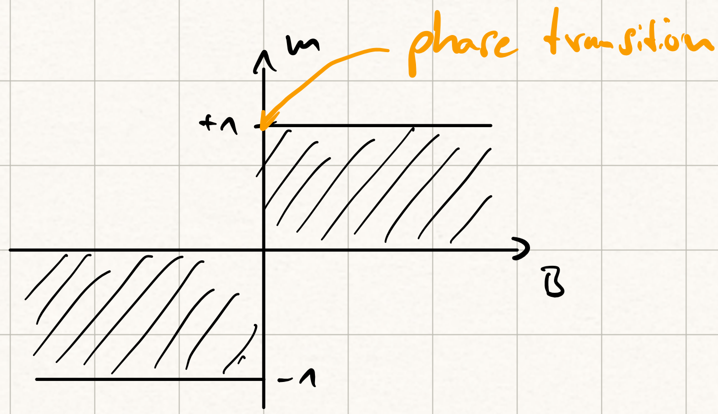
Review of previous lecture (May 05)

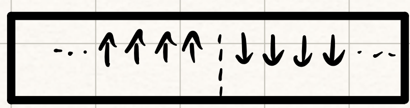
Phase transitions at $T=0$ and the Ising model:

$$m = \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle = -\frac{1}{N}\frac{\partial F}{\partial B} = -\frac{\partial f}{\partial B}$$

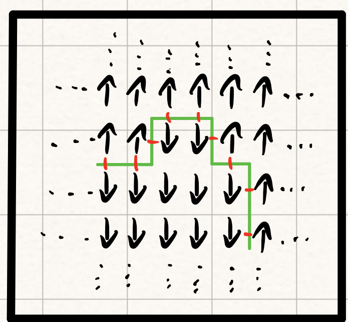
ground state:



at $T>0$ long-range order is unstable against thermal fluctuations in 1d (for short-range interactions and discrete symmetry of H): $m(B=0) = 0$



in 2d long-range order is stable for $T<T_c$: $m(B=0, T<T_c) \neq 0$



continuous symmetries

consider a spin system with a continuous symmetry, for example the Heisenberg model for ferromagnets:

$$H(\{\sigma_i\}) = -\sum_{i=1}^{N} \vec{B}_i \cdot \vec{\sigma}_i - \sum_{\langle i,j\rangle} J_{ij}\, \vec{\sigma}_i \cdot \vec{\sigma}_j$$

with $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$.

This model has the continuous rotational $O(3)$ symmetry:

$$H(\{R(\Omega)\vec{\sigma}_i\}) = H(\{\vec{\sigma}_i\})$$

How do the arguments regarding the presence of long-range order change at $T>0$ compared to the discrete symmetry

$$H(\{\sigma_i\}) = H(\{-\sigma_i\}) \; ?$$

- it is easier for thermal fluctuations to destroy long-range order in case of a continuous symmetry since spins can rotate by an arbitrary small angle, i.e. the symmetry group has a larger parameter space

- it is possible to follow the same arguments like for discrete symmetry, again we need $\Delta F = F^{(2)} - F^{(1)} \overset{!}{>} 0$, for $O(3)$ symmetry the entropy increase is larger

- one can show that long-range order is only possible in $d>2$ dimensions (more dimensions imply more interacting neighbor spins and hence a larger energy penalty when rotating individual spins)

Spontaneous symmetry breaking and ergodicity breaking

Consider the Hamiltonian of the Ising model for $B=0$ and nearest neighbor interaction:

$$H = J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

the Hamiltonian has the symmetry $H(\{\sigma_i\}) = H(\{-\sigma_i\})$.

despite this symmetry, the statistical expectation values are <u>not</u> invariant under this symmetry:

$$\langle \sigma_i \rangle \neq 0 \quad \text{for } T < T_c \quad \text{(see previous section)}$$

$$m = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle \neq 0$$

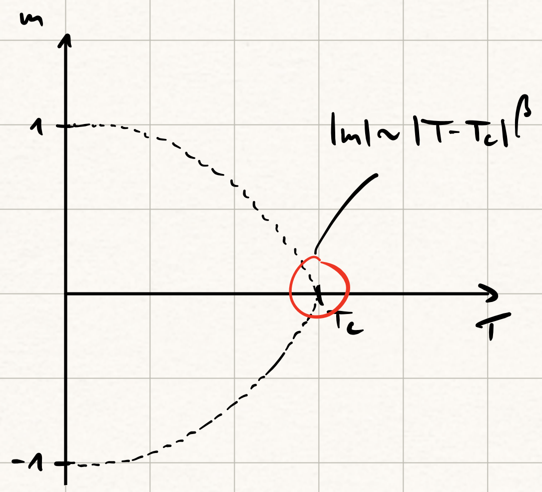
a non-vanishing expectation value implies that the majority of spins is pointing up ($m>0$) or down ($m<0$).

This phenomenon is called <u>Spontaneous symmetry breaking</u>.

- the sign of m is determined by the <u>initial conditions</u> of the system before undergoing the phase transition below T_c.

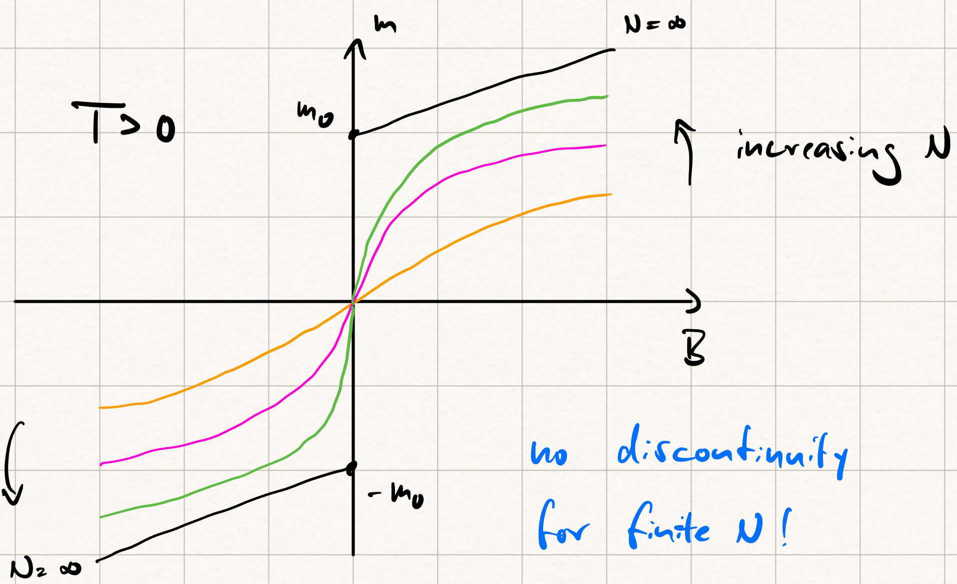
- the value of m depends on T:
  * at $T=0$, $m = \pm 1$ (all spins aligned)
  * at $T>0$ thermal fluctuations reduce m until it vanishes at $T = T_c$

- m is called an <u>order parameter</u> (vanishes above the phase transition and is vanishing below T_c)



investigate the role of limits at $T>0$ in more detail:

$\boxed{\lim_{N\to\infty}}$ $\boxed{\lim_{B\to 0}}$



no discontinuity for finite N!

- limits $N\to\infty$ and $B\to 0$ do <u>NOT</u> commute:

$$\lim_{B\to 0^+}\lim_{N\to\infty}\left(-\frac{\partial f}{\partial B}\right) = m_0$$

$$\lim_{B\to 0^-}\lim_{N\to\infty}\left(-\frac{\partial f}{\partial B}\right) = -m_0$$

$$\lim_{N\to\infty}\lim_{B\to 0}\left(-\frac{\partial f}{\partial B}\right) = 0$$

consider probability of finding the system in a state $\{\sigma_i\}$:

$$p(\{\sigma_i\}) = Z_c^{-1} e^{-\beta H(\{\sigma_i\})}$$

the fact that $H(\{\sigma_i\}) = H(\{-\sigma_i\})$ would naively imply:

$$\langle \sigma_i \rangle = \mathrm{Tr}\, \underbrace{p(\{\sigma_i\})}_{\text{even}} \underbrace{\sigma_i}_{\text{odd}} = 0$$

(where is the loophole in this argument?)

in the thermodynamic limit we have to be more careful:

- consider Ising model for $B = 0$
- denote the two system configurations that are related by a flip of all spins by 1 and 2 (majority of spins up/down)
- both states have the same energy and are equally likely, the magnetizations are $m_1 = -m_2$
- now apply external field $B > 0$, due to the term $-B \sum_i \sigma_i$ the probabilities for both states read

$$\frac{p_1}{p_2} = \frac{e^{-\beta(-B \cdot N \cdot m)}}{e^{-\beta(+B \cdot N \cdot m)}} = e^{2\beta B \cdot Nm}, \text{ i.e. } \frac{p_2}{p_1} \to 0 \text{ for } N \to \infty$$

$\Rightarrow$ system is in configuration 1 with magnetization $m_1 = m$ even for arbitrarily small $B = 0^+$

- limits $B \to 0^{+/-}$ together with limit $N \to \infty$ provides macroscopic weighting of state with magnetization $\pm m$ over the spin-flipped state

- the use of $B \to 0^{+/-}$ plus $N \to \infty$ is equivalent to using $B = 0$ and using a restricted ensemble in which microstates with magnetization $\mp m$ are not included connection to ergodicity:

- for a finite Ising system with N spins the phase space with magnetization $\pm m$ are sampled equally for $0 < T < T_c$
- a system first in a state with magnetization $+m$ might form clusters of down spins after some (possibly long) time
- the life time of a state with a given magnetization scales like $\tau \sim \exp(N)$
  - * energy of system is unchanged when flipping <u>all</u> spins at once
  - * however, only flipping a subset of all spins will <u>increase</u> energy
  - * hence, in the thermodynamic limit the system is effectively trapped in a subpart of the phase space

    ↳ ergodicity breaking (time average ≠ ensemble average)

  - * subparts of phase space are related by symmetry that is spontaneously broken