Review of previous lecture (May 30)

Basic idea of Landau theory:

Postulate a Landau free energy $L(\{k_i\})$ and an order parameter η. The ground state is given by the global minimum of L: $\left.\frac{\partial L}{\partial \eta}\right|_{\eta=\eta^*} = 0$.

constraints on L:

- symmetries of system
- for $T \to T_c$ we have $\eta \to 0$, L is analytic

$$\mathcal{L} = \frac{L}{V} = \sum_{n=0}^{\infty} a_n(\{k_i\})\,\eta^n \quad \text{(uniform system)}$$

$$L = \int d^{3}\vec{x}\, \mathcal{L}[\eta(\vec{x})] \quad \text{(non-uniform systems)}$$

- $\eta^* = 0$ for $T > T_c$, $\eta^* \neq 0$ for $T < T_c$
- coefficients a_n <u>cannot</u> be determined within Landau theory

for Ising model we obtained using only general arguments:

$$\mathcal{L} = a_2^1 t \eta^2 + a_4^0 \eta^4 - B\eta \qquad \text{with } t = \frac{T - T_c}{T}$$

$\Rightarrow$ reproduces form of Ising model EOS based on explicit calculations using the Hamiltonian

First-order phase transitions

consider now a more general form for $\mathcal{L}$:

$$\mathcal{L} = a_2^1 t \eta^2 + a_4^0 \eta^4 + a_3^0 \eta^3 \cancel{- B\eta}$$

new

consider $B=0$ in the following

note: linear term in η associated with parameter a_1 not allowed since $\eta = 0$ for $T > T_c$

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0 \Rightarrow \eta = 0 \quad \text{or} \quad \eta = -c \pm \sqrt{c^2 - \frac{a_2^1 t}{2a_4^0}}$$

real solution for

$$c^2 > \frac{a_2^1 t}{2a_4^0} \Leftrightarrow t < \frac{2a_4^0 c^2}{a_2^1} \equiv t^*$$

$$\left[c = \frac{3a_3^0}{8a_4^0} \right]$$

- lowering t below t^* leads to 2nd minimum in $\mathcal{L}$
- lowering t below t_1 leads to a new <u>global</u> minimum (see figures)

$\Rightarrow$ value for η jumps <u>discontinuously</u> from $\eta = 0$ to $\eta(t_1)$

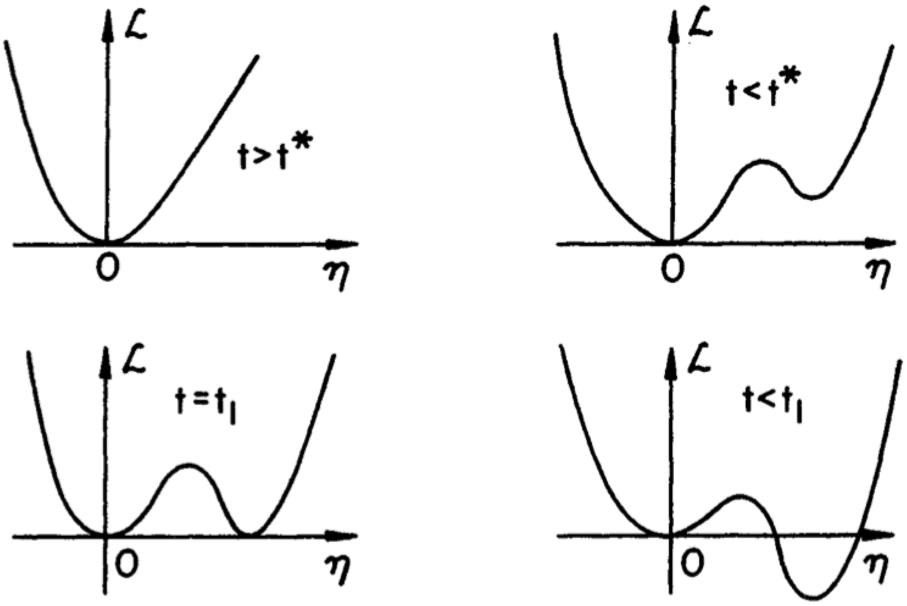


Figure 5.2 $\mathcal{L}$ as a function of η for various temperatures, showing the Landau theory description of a first order transition.

Goldenfeld, p. 146

$\Rightarrow$ in general a cubic term in $\mathcal{L}$ leads to a first order phase transition, absence of cubic term guarantees a continuous phase transition

<u>However:</u> note that Landau theory is in general not valid for first-order phase transitions!

why?

$\eta \not\to 0$ for $T \to T_c$, i.e. η not neccessarily small

Landau theory and coarse graining

- for the physical interpretation of $\mathcal{L}$ it is instructive to allow for spatially varying order parameters, i.e. for Ising model $m(\vec{x})$
- allows to connect microscopic theories with classical effective theories, illustrates role of degrees of freedom at different length scales and their systematic treatment

↳ basic idea of RG

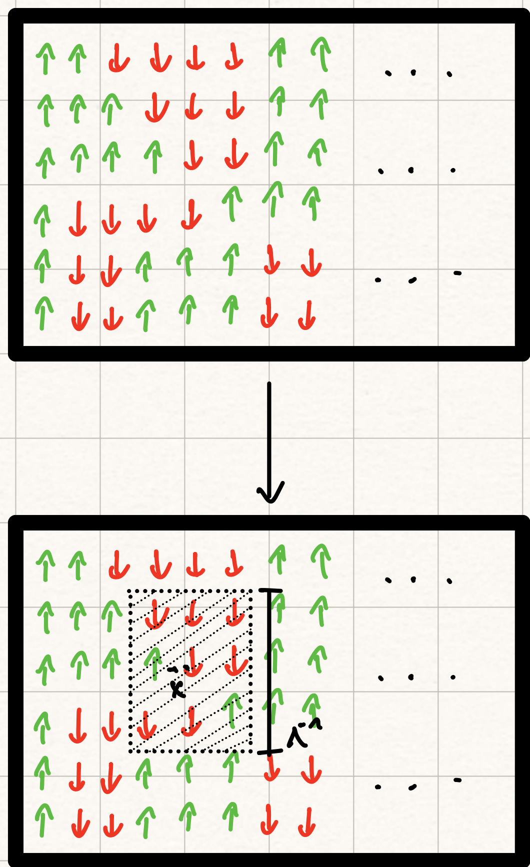
consider spin system with a correlation length ξ

assign a local magnetization $m(\vec{x})$ to each point in space via <u>coarse graining</u>:

$$m_\Lambda(\vec{x}) = \frac{1}{N_\Lambda(\vec{x})} \sum_{i \in \vec{x}_\Lambda} \langle \sigma_i \rangle$$

$N_\Lambda(\vec{x}) = \dfrac{\Lambda^{-d}}{a^d}$: number of spins in block of size Λ^{-1}

a: lattice spacing



$a \ll \Lambda^{-1} \leq \xi(T)$

(<u>not</u> the case in the illustration)

$\Rightarrow$ $m_\Lambda(\vec{x})$ is a smooth and slowly varying function, does contain Fourier components of wave numbers $k \leq \Lambda$

What is the general form of $\mathcal{L}$ for a system given by $m_\Lambda(\vec{x})$?

- a function of the form $L = \sum_{\vec{x}_i} \mathcal{L}(m_\Lambda(\vec{x}_i))$ <u>cannot</u> be complete since minimization with respect to $m_\Lambda(\vec{x})$ would just result in a independent minimization at each point $\vec{x}_i$
- need to take into account that domain walls, i.e. differences in $m_\Lambda(\vec{x})$ in adjacent blocks cost energy
- Simplest term that takes this into account:

$$\sum_{\vec{x}_i} \sum_{\vec{\delta}} \left(\frac{m_\Lambda(\vec{x}) - m_\Lambda(\vec{x}-\vec{\delta})}{\Lambda^{-1}} \right)^2 \rightarrow \int d^d\vec{x} \, (\nabla m_\Lambda(\vec{x}))^2$$

$$\Rightarrow \quad L = \int d^d\vec{x} \left[\mathcal{L}(m_\Lambda(\vec{x})) + \frac{\gamma}{2} (\nabla m_\Lambda(\vec{x}))^2 \right]$$

L is a functional of $m_\Lambda(\vec{x})$, i.e. it depends on entire function at all $\vec{x}$, also called <u>effective Hamiltonian</u>, short-distance physics integrated out, effective degree of freedom $m_\Lambda(\vec{x})$ low-energy/long-distance degrees of freedom

How is $\mathcal{L}$ related to the Hamiltonian and the free energy $\mathcal{F}$?

$$e^{-\beta \mathcal{F}} = \mathrm{Tr}\, e^{-\beta H}$$

$$\Rightarrow \quad e^{-\beta \mathcal{L}(m_\Lambda(\vec{x}))} = \mathrm{Tr}\, e^{-\beta H(\{\sigma_i\})}\, \delta\Big[\Big(\sum_{i \in \vec{x}_\Lambda} \sigma_i\Big) - m_\Lambda(\vec{x})\, N_\Lambda(\vec{x})\Big]$$

$$= \mathrm{Tr}'\, e^{-\beta H(\{\sigma_i\})}$$

↳ partial trace, only configurations with local magnetization $m_\Lambda(\vec{x})$ included

Hence, by introducing $\mathcal{L}[m_\Lambda(\vec{x})]$ we have effectively divided the sum over all states ("Tr") into two steps:

$$\begin{aligned} Z_c &= e^{-\beta \mathcal{F}} \\ &= \sum_{\{\sigma_i\}} e^{-\beta H} \\ &= \sum_{\{\sigma_i^*\}} \underbrace{\sum_{\{\bar{\sigma}_i\}}} e^{-\beta H} = \sum_{\{\sigma_i^*\}} e^{-\beta \mathcal{L}[m_\Lambda(\vec{x})]} \end{aligned}$$

all microscopic configurations consistent with magnetization $m_\Lambda(\vec{x})$

remaining configurations not consistent with $m_\Lambda(\vec{x})$

in the continuum limit the sum amounts to the sum/integral over all coarse-grained functions $m_\Lambda(\vec{x})$ and is written as

$$Z_c = \int \mathcal{D}m_\Lambda(\vec{x})\, e^{-\beta L[m_\Lambda(\vec{x})]}$$

functional integral

$\mathcal{D}m_\Lambda(\vec{x})$ involves an infinite number of integrals in the continuum limit, a discretized version takes the form

$$\int \mathcal{D}m_\Lambda(\vec{x}) \to \prod_{i=1}^{N} \int dm(\vec{x}_i)$$

in practice the integration over coarse-grained degrees of freedom is often done in momentum space:

$$m_\Lambda(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3\vec{k}\, m_\Lambda(k)\, e^{i\vec{k}\cdot\vec{x}}$$

$$m_\Lambda(\vec{k}) = \int d^3\vec{x}\, m_\Lambda(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}$$

$$\int \mathcal{D}m_\Lambda(\vec{k}) \to \prod_{|k_i|<\Lambda} \int dm_\Lambda(\vec{k}_i)$$

the functional $L[m_\Lambda(\vec{x})]$ can in principle be computed starting from the Hamiltonian of a system

problem set 3: for Ising model