Minimum of η/s and the phase transition of the Linear Sigma Model in the large-N limit

Antonio Dobado, Felipe J. Llanes-Estrada and Juan M. Torres-Rincón

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Strongly Interacting Matter under Extreme Conditions,
Hirschegg 2010
Based on PRD80:114015 (2009)

η/s: What is viscosity?

For introduction, see talks by
Xu-Guang Huang, J.P. Blaizot, R. Snellings...

(...) All the sets of data obtained showed that the fluidized snow behaves as a Bingham body in the range of density (250 to 450 kg/m^3) (...) Viscosity coefficients of fluidized snow obtained ranged from 10^{-2} to 10^{-1} N s/m^2; (...).
A strong increase in the viscosity coefficient with the density was observed, which could be expressed with the exponential function.
(Nishimura, 1996)

In a gas, instead, diffusive transport

$$\eta \propto p_\perp \lambda \propto \frac{1}{\sigma}$$

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η/s: What is this ratio?

- Condition for equilibration: $\frac{\lambda}{\tau} << 1$
(mean free path/characteristic expansion time small)

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- Note η/s is a dimensionless pure number

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η/s: Why has the ratio received attention?

- Perfect fluid or superfluid: $\eta \to 0$
- KSS conjecture: η/s bounded below
- Argument based on Heisenberg's uncertainty principle $\Delta E \Delta t \geq 1$ (Danielewicz and Gyulassy 1985):

$$\frac{\varepsilon}{n}\lambda \geq 1$$

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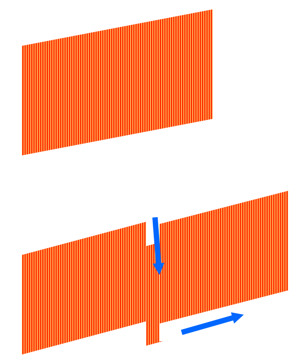

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$$\omega + i\frac{\eta k^2}{Ts} = 0$$

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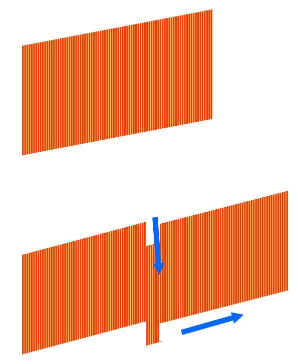

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$$\eta = \frac{\sigma}{16\pi G_N} \qquad s = \frac{\sigma}{4 G_N}$$

- Conjecture

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

- Hydrodynamics compared to RHIC (Romatschke 2008, Bozek 2009, Masui *et al.* 2009)
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- Systems with many particle species

such as $L\sigma M(N)$
Dobado & LL-E 2007,
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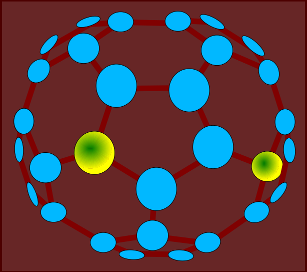

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- No system experimentally examined

(To the credit of Kovtun, Son and Starinets 2003)

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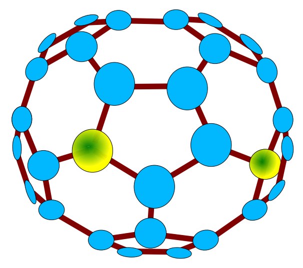

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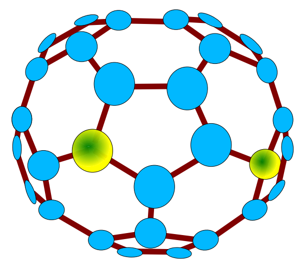

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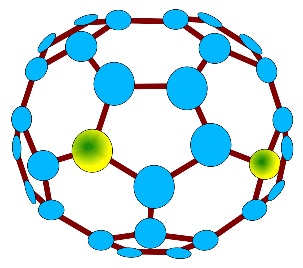

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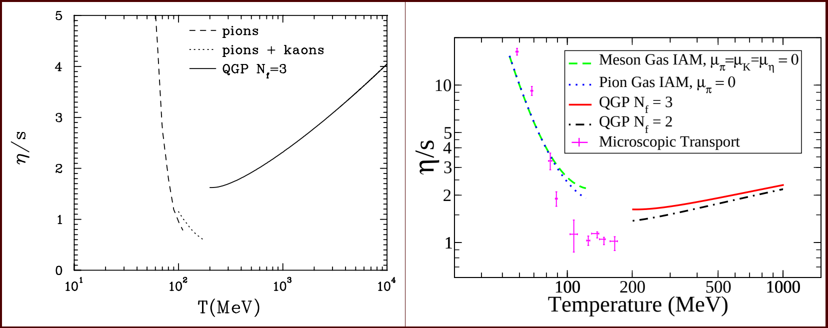
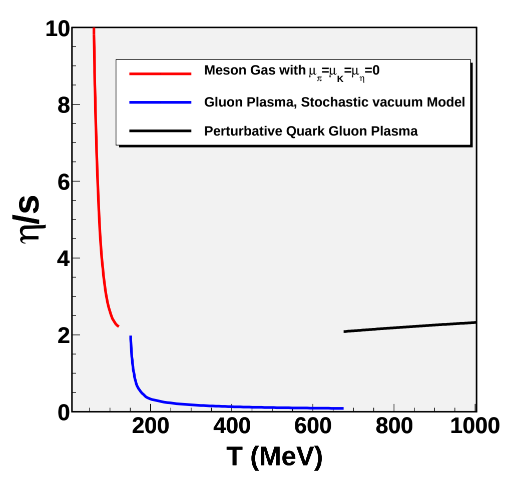
η/s and phase transitions



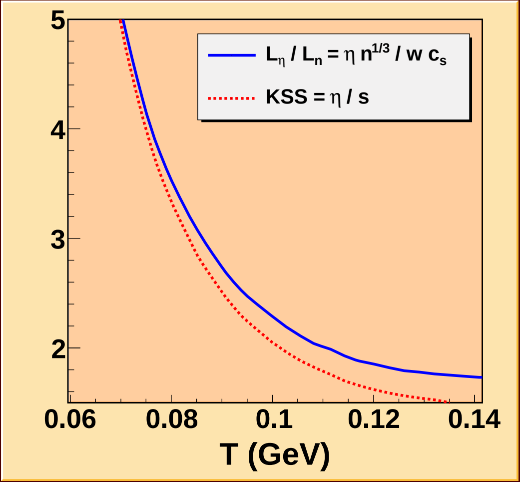
(Csernai, McLerran, Kapusta 2006; Dobado, LL-E, Torres-Rincon, 2008, data also from Bass and Demir 2008)

η/s and phase transitions

A different result by Antonov 2009



Fluidity parameter



(Proposed by Volker Koch, 2009

Btw, Reynolds number $\rho V L/\eta$)

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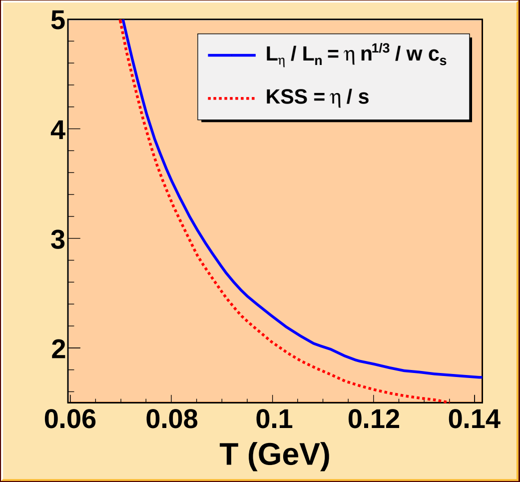

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The question

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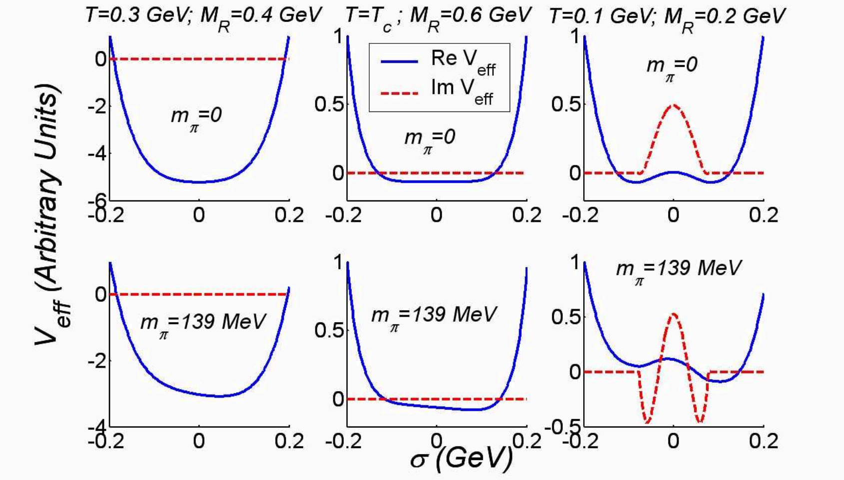
Outline

Effective potential for $\langle\sigma\rangle$



Phase transition in the model

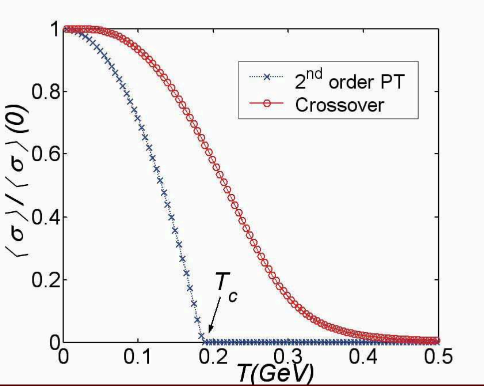
$m_\pi = 0 \rightarrow$ 2nd order Phase transition

$$\langle\sigma(T)\rangle = \begin{cases} \langle\sigma(T)\rangle \propto \sqrt{1-\frac{T^2}{T_c^2}} & T < T_c \\ 0 & T \geq T_c \end{cases}$$

$$m_\pi^2(T) = \begin{cases} 0 & T < T_c \\ \frac{N}{3}\lambda_R(T^2 - T_c^2) & T \geq T_c \end{cases}$$

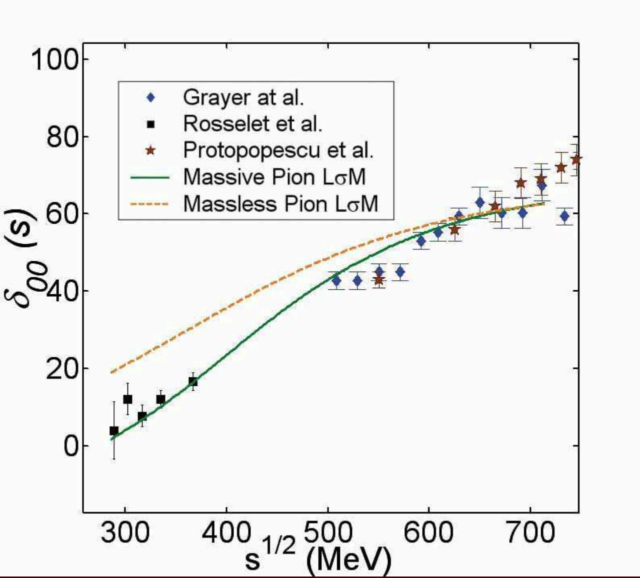
$m_\pi \neq 0 \rightarrow$ Crossover

Phase transition in the model



$$T_c = 2\sqrt{3}F \qquad (f_\pi = F\sqrt{N})$$

Phase shifts



What we sacrifice

- Bc of mean field treatment: Critical behavior near end-point, see Nakano's talk. (But not worrying since the η exponent small; revisit for ζ).
- Bc of large-N, non $I = 0$ channels -but we are making no claim to reality.

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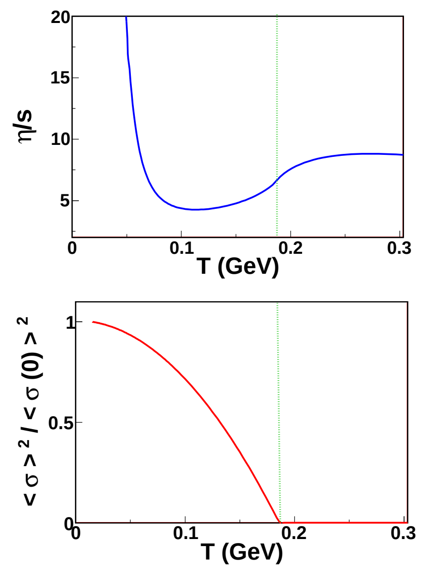
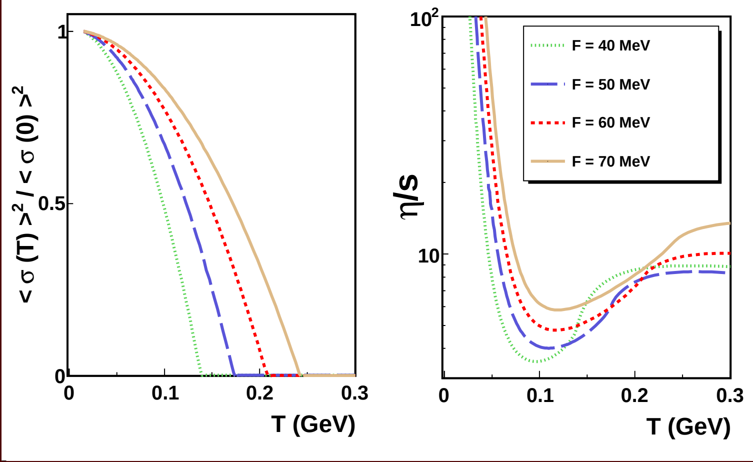
Minimum just before the phase transition

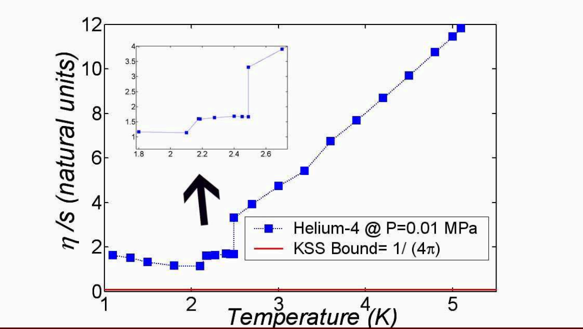


Movement of minimum with f_π



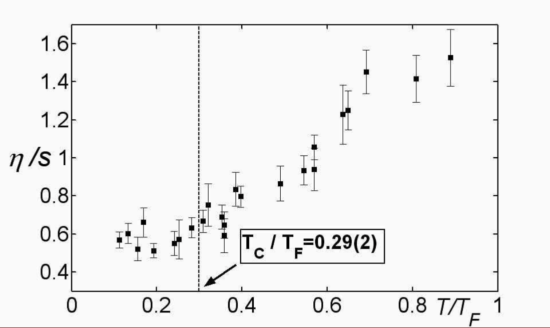
The minimum does follow the displacement of T_c

Helium gas



"The minimum is found at a temperature below T_c (liquid side) but is accompanied by a discontinuous change across the phase transition" (Lacey *et al.* 2007).

Fermions near the unitarity limit



(T. Schaefer 2007)

Outline

Linear sigma model Lagrangian

$$\Phi = (\sigma, \pi^1 \dots \pi^{N-1})$$

$$\mathcal{L}[\Phi, \partial_\mu \Phi] = \frac{1}{2}\,\partial_\mu \Phi^T \partial^\mu \Phi + \overline{\mu}^2(\Phi^T \Phi) - \lambda(\Phi^T \Phi)^2$$

$$\mathcal{L}_{\rm SB} \equiv m_\pi^2 f_\pi \sigma = m_\pi^2 f_\pi^2 - \frac{1}{2} m_\pi^2 \pi^a \pi^a + \cdots$$

Auxiliary variable to make pion integration Gaussian

$$\chi \equiv \Phi^2 \sqrt{\frac{2\lambda}{N}}$$

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Computation of the effective potential

The new variable appears only in the combination

$$G^{-1}[q,\chi] \equiv q^2 - 2\overline{\mu}^2 + \chi\sqrt{8\lambda N}$$

Integrating over π, effective potential for σ

$$V_{\rm eff} = \frac{1}{2}\left(\sigma^2 - NF^2\right) G^{-1}[0,\chi]$$
$$-\frac{(G^{-1}[0,\chi])^2}{16}\left(\frac{1}{\lambda_R} + \frac{N}{4\pi^2}\log\frac{\mu^2}{G^{-1}[0,\chi]}\right)$$
$$+\frac{N(G^{-1}[0,\chi])^2}{8(4\pi)^2} - m_\pi^2 f_\pi \sigma - \frac{N}{2} g_0(T, G^{-1}[0,\chi])$$

Equations of motion

Saddle point for a constant condensate $\langle\sigma\rangle$

$$\frac{dV_{\rm eff}}{d\sigma} = 0; \quad \frac{dV_{\rm eff}}{d(G^{-1}[0,\chi])} = 0$$

$$\begin{cases} 0 = \sigma \; G^{-1}[0,\chi] - m_\pi^2 f_\pi , \\ 0 = \frac{1}{2}\left(\sigma^2 - NF^2\right) - \frac{G^{-1}[0,\chi]}{8}\left(\frac{1}{\lambda_R} - \frac{N}{4\pi^2}\log\frac{e \;\, G^{-1}[0,\chi]}{\mu^2}\right) \\ \qquad +\frac{N}{2} g_1(T, G^{-1}[0,\chi]). \end{cases}$$

Entropy density

$$s(T) = -\frac{\partial V_{\rm eff}}{\partial T} = \frac{N}{2}\frac{\partial g_0(T, G^{-1}[0,\chi])}{\partial T}$$

Boltzmann equation

$$f = f_0 \left[1 + \frac{g(p)}{T} \Delta_{ij} \tilde{V}^{ij} \right]$$

$\Delta_{ij} \equiv p_i p_j - \frac{1}{3}\delta_{ij} p^2,$
$\tilde{V}_{ij} = \frac{1}{2}(\partial_i V_j + \partial_j V_i) - \frac{1}{3}\partial_k V^k \delta_{ij}$

$$\eta = -\frac{1}{10T} \int \frac{d\mathbf{p}}{E(p)} f_0 p_i p_j \Delta^{ij} g(p)$$

Green-Kubo formula

$$\eta = \frac{1}{20} \lim_{\omega \to 0^+} \lim_{|\mathbf{p}| \to 0^+} \frac{1}{\omega} \int d^4\mathbf{x}\ e^{i\mathbf{p}\cdot\mathbf{x}}\ \langle \left[\hat{\pi}^{ij}(x), \hat{\pi}_{ij}(0) \right] \rangle$$

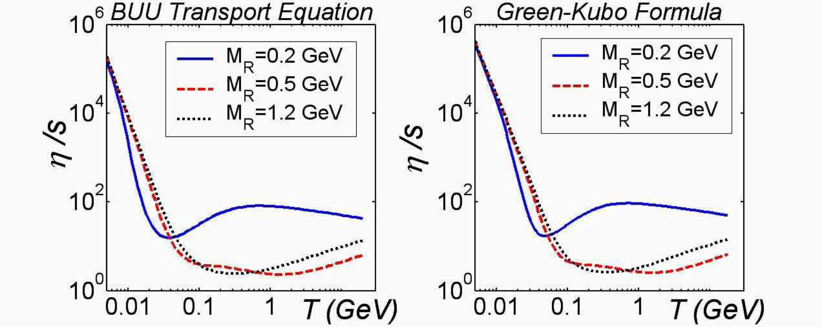
$$\hat{\pi}^{ij} = \hat{T}^{ij} - g^{ij} \hat{T}^k_k / 3$$

$$\eta^{(0)} = \frac{1}{10\pi^2 T} \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|^6}{E_p^2\ \Gamma(p)} n_B(E_p)[1 + n_B(E_p)]$$

$\Gamma(p) = -\frac{\mathrm{Im}\ \Pi_R(E_p, \mathbf{p})}{2E_p}$

(Fernández-Fraile and Nicola 2009 for ChPT).

Boltzmann equation vs. Green-Kubo formula

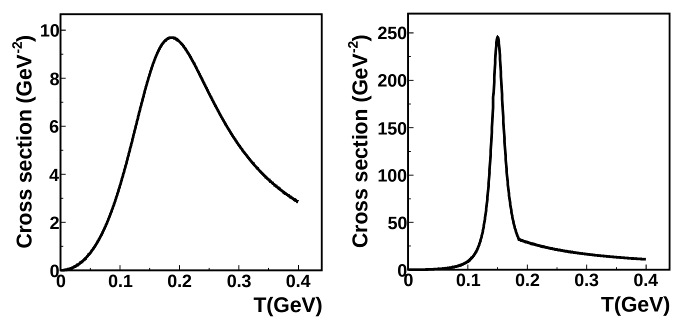


Both formulations give compatible results

Outline

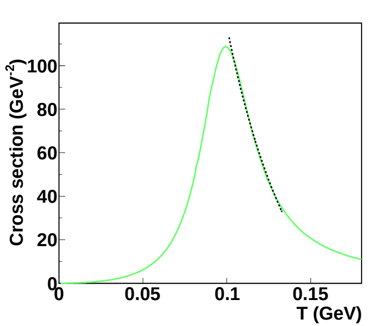
Cross section for $\pi\pi$ scattering on σ background

Consider some average collision energy $E \simeq 10T$



Without and with the T-dependence of $\langle\sigma(T)\rangle$

Cross section peaks before phase transition



Power law on the trailing edge allows a scaling analysis

$$\sigma = A^{2+k}\frac{1}{T^k}; \qquad \eta \sim \frac{T}{\sigma}; \qquad s \sim T^3$$

Hence, the KSS number must grow before the phase transition $\eta/s \sim T^{0.53}$

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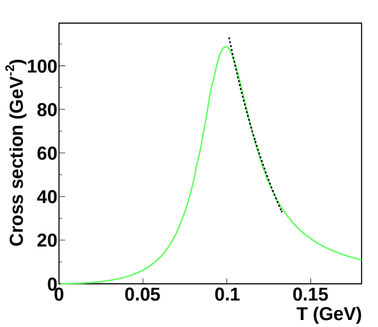

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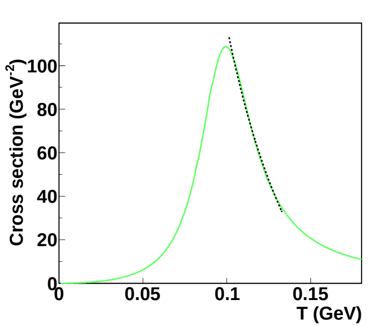

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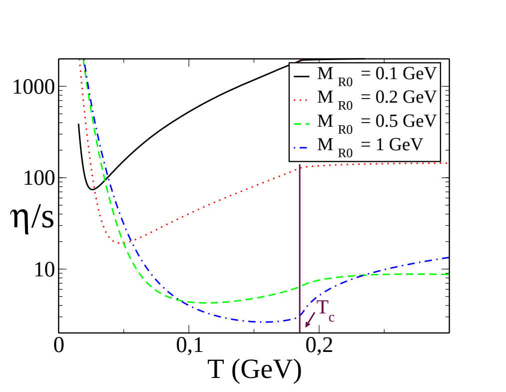
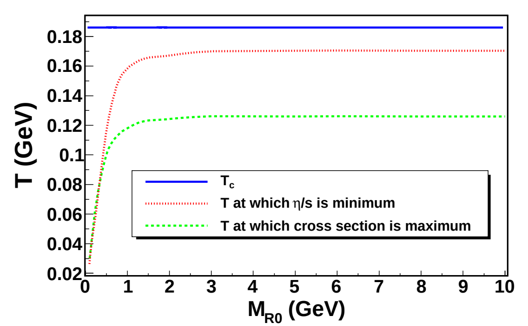
Movement of minimum with M_R



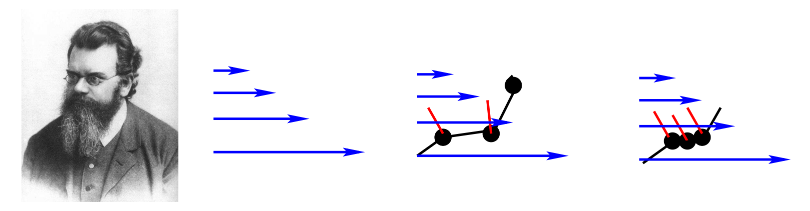
Movement of minimum with M_{R0}



Saturation of pole position:

for large M_{R_0} there is no M_{R_0}-dependence

Validity of the Boltzmann approximation



- Boltzmann's "Molecular chaos hypothesis":
  Successive collisions uncorrelated
  Keep $f(p)$, neglect $f_n(p_1 \dots p_n)$
- $$\lambda = \frac{1}{n(T)\bar{\sigma}} >> a_0 = \sqrt{\bar{\sigma}}$$
  $$n(T)\bar{\sigma}^{3/2} << 1 \ .$$

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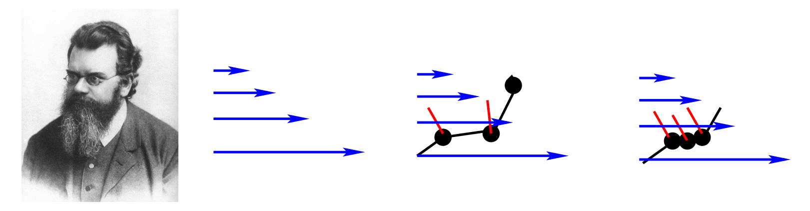

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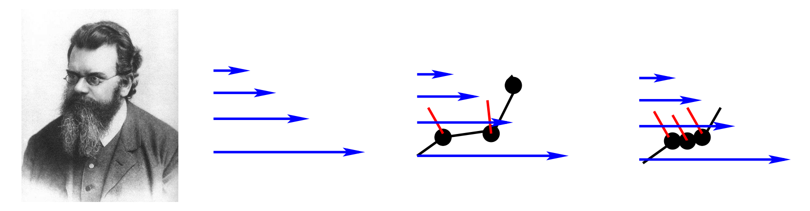

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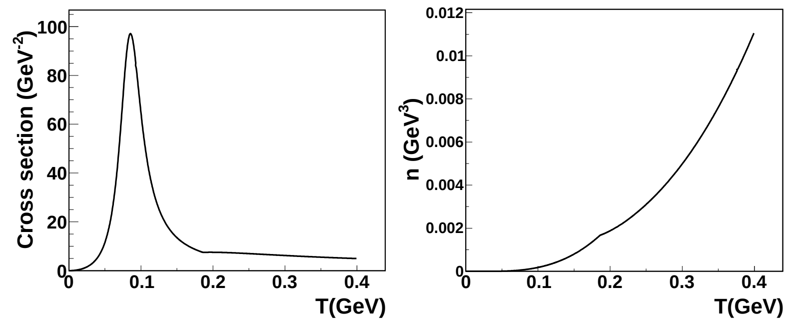
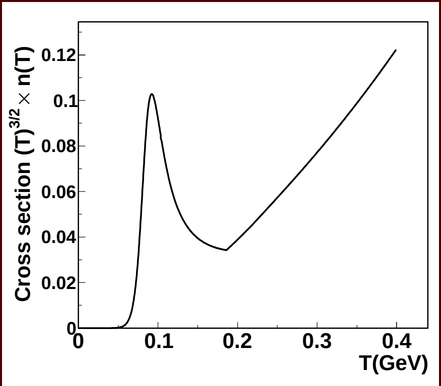
Is the gas dilute enough?

Plot $n(T)\bar{\sigma}^{3/2}$



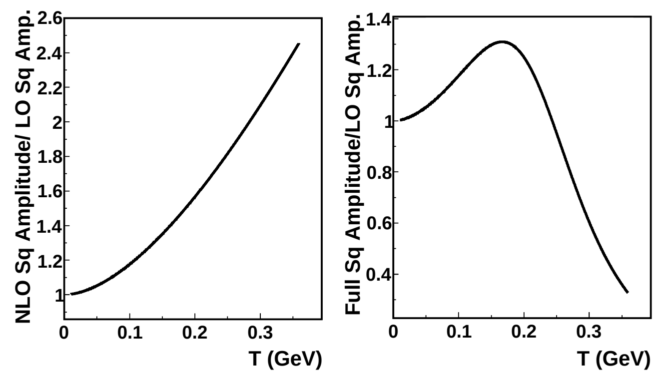
Is the gas dilute enough?

YES!

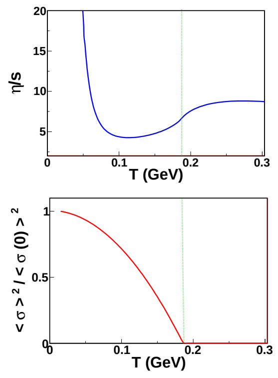


Strength of interaction

This in spite of the theory being strongly coupled



Conclusion



A theoretically controlled example where the minimum of η/s occurs around but not exactly at a phase transition,

the large-N Linear σ Model

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