Hadron decays with lattice QCD

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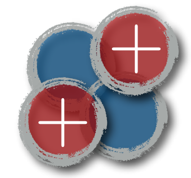
Recipe for strong force predictions

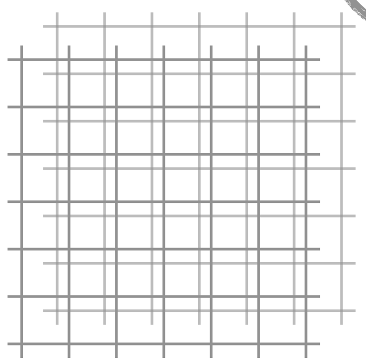
1. Lagrangian defining QCD
2. Formal / numerical machinery (lattice field theory)
3. A few experimental inputs (e.g. M_π, M_K, M_Ω)

$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$



Wide range of precision pre-/post-dictions

$m_q^{\overline{\mathrm{MS}}}(\mu)$ $\quad m_p$ $\quad \alpha_{\overline{\mathrm{MS}}}(\mu)$ $\quad m_n$ $\quad f_\pi$ $\quad g_A$ $\quad f_K$ $\quad (g-2)_\mu^{\mathrm{HVP}}$ $\quad f_D$ $\quad f_B$



FLAG 2019 Flavour Lattice Averaging Group

$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1182(8)$ lattice average

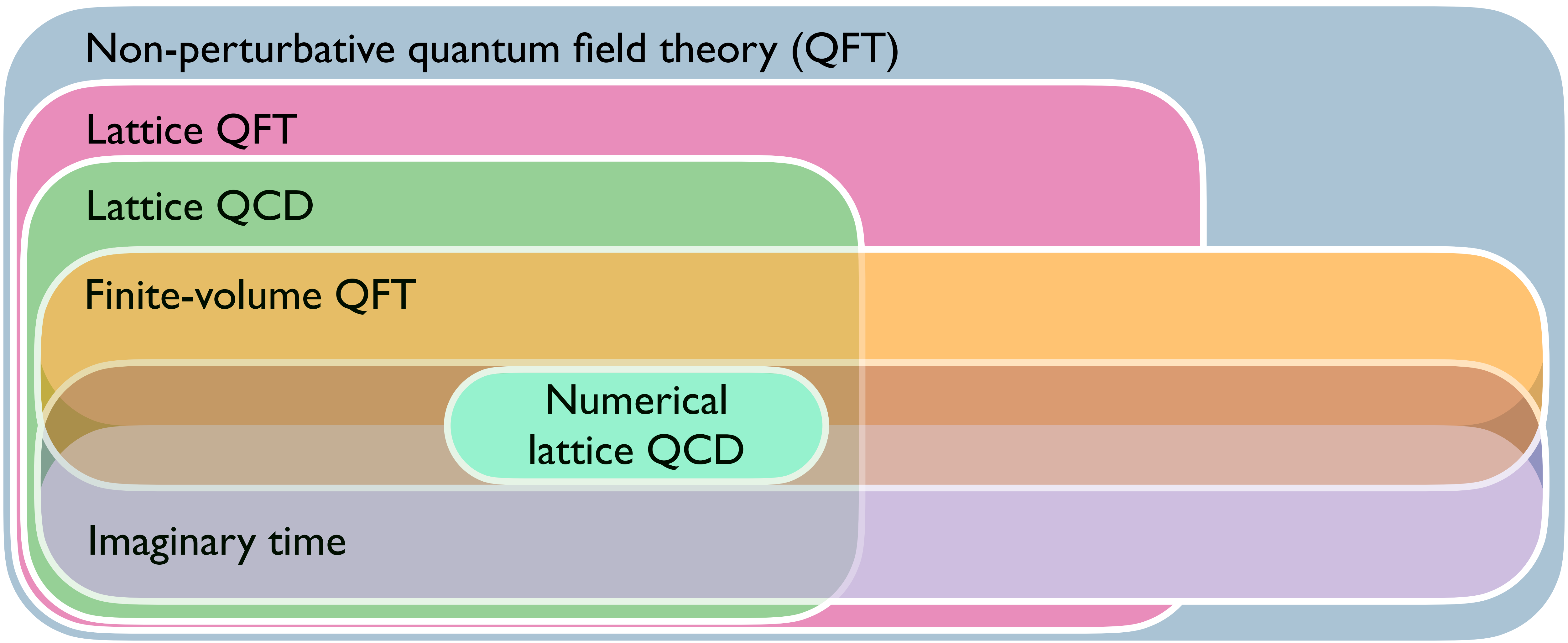
$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1174(16)$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ Tool for new-physics searches ✓

Lattice QCD

- a non-perturbative regularization of QCD
- a definition that is well-suited to numerical evaluation

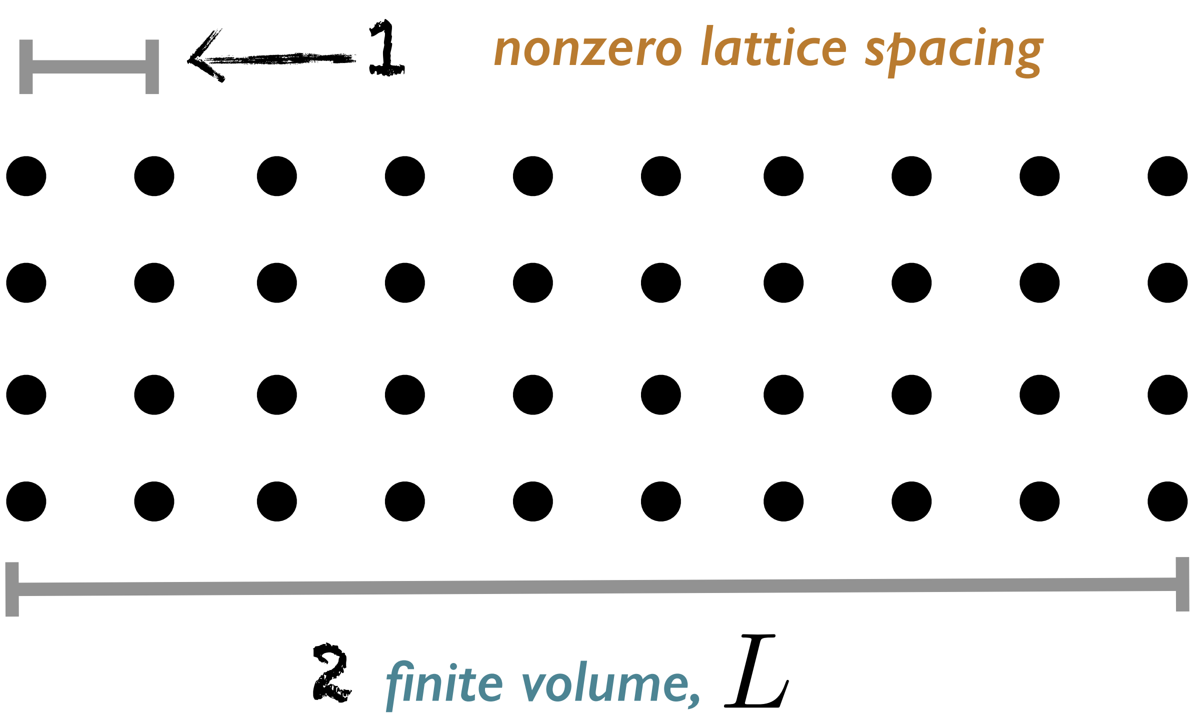
render the quantum path-integral finite-dimensional → *evaluate using Monte Carlo importance sampling*

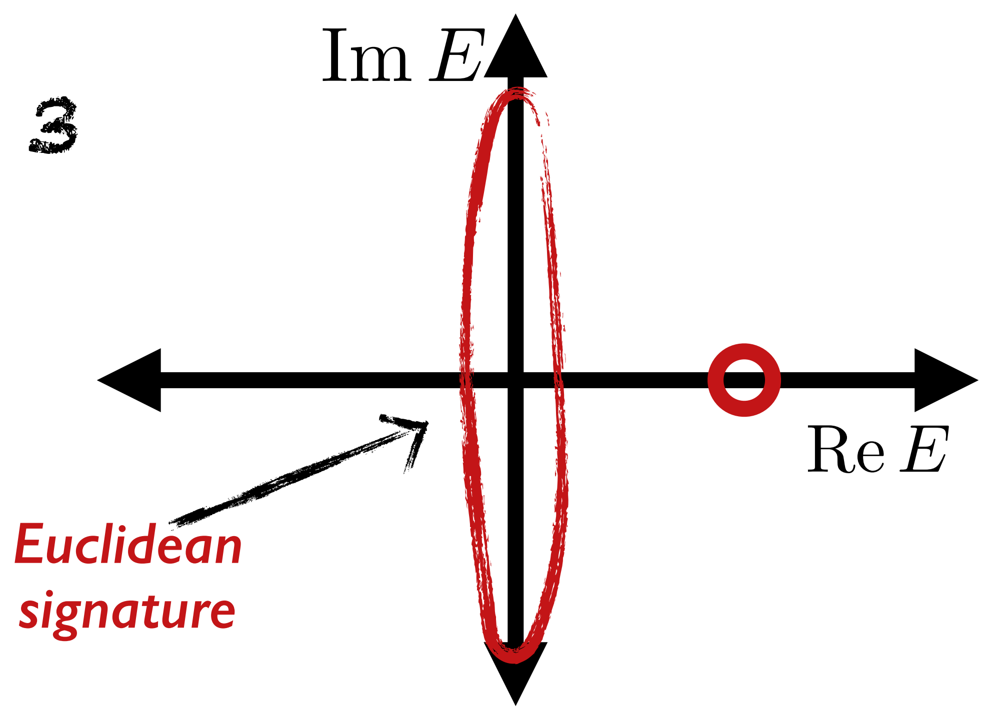


Limitations of lattice QCD

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*





Also… $M_{\pi,\mathrm{lattice}} > M_{\pi,\mathrm{our\ universe}}$

(but physical masses → increasingly common)

Processes with QCD-stable hadrons

- Three categories:
  - Decay constants
  - Form factors
  - Mixing parameters

$$\langle 0|\mathcal{J}|\mathbf{1}\rangle \qquad \langle \mathbf{1}|\mathcal{J}|\mathbf{1}'\rangle \qquad \langle \overline{\mathbf{1}}|\mathcal{H}^{\Delta F=2}|\mathbf{1}\rangle$$

$$f_\pi, f_K, f_B \qquad f_+^{K^0\pi^-}(q^2),\ f_{B\to\pi}(q^2) \qquad B_{B_d}^{(n)},\ B_{B_s}^{(n)}$$

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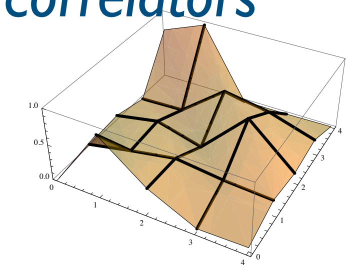
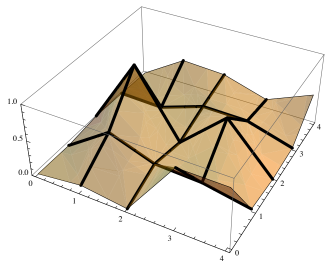
- Summary of the approach…
  - Importance sampling QCD gauge fields → *correlators*

$$\langle A_\mu^{\text{bare}}(0)\ \pi_{\boldsymbol{p}}(-\tau)\rangle_{T,L,m_q,a} =$$  $+$  $+\cdots$

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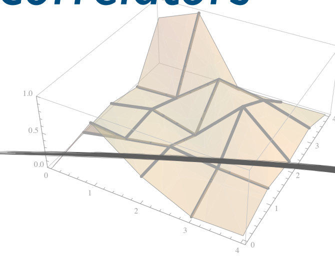

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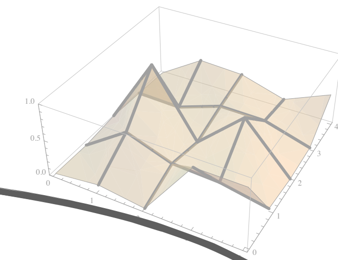

$+\cdots$

temporal length — volume — quark masses — lattice spacing

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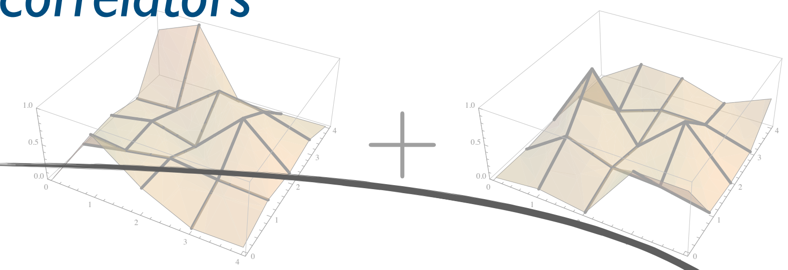

$+ \cdots$

temporal length — volume — quark masses — lattice spacing

  - *Renormalization* of currents required (typically non-perturbative)

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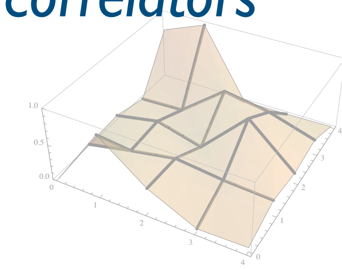

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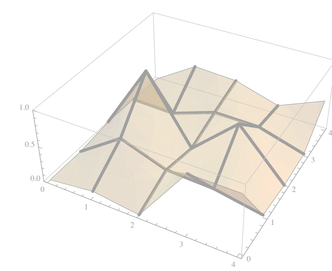

+ ⋯

temporal length   volume   quark masses   lattice spacing

  - *Renormalization* of currents required (typically non-perturbative)
  - *Large time separation* filters excited states

$$Z^{\mathrm{renorm}}\,\langle A_\mu^{\mathrm{bare}}(0)\;\pi_{\boldsymbol{p}}(-\tau)\rangle_{T,L,m_q,a} = \langle A_\mu^{\mathrm{renorm}}(0)\;e^{-\hat{H}\tau}\;\pi_{\boldsymbol{p}}(0)\rangle$$

$$\underset{\tau\gg\delta E_\pi}{\longrightarrow}\; Z_\pi e^{-E_\pi\tau}\;ip_\mu f_\pi(T,L,m_q,a)$$

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temporal length　　volume　　quark masses　　lattice spacing

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$$Z^{\rm renorm}\,\langle A_\mu^{\rm bare}(0)\ \pi_{\boldsymbol{p}}(-\tau)\rangle_{T,L,m_q,a} \xrightarrow[\tau\gg\delta E_\pi]{} Z_\pi e^{-E_\pi\tau}\ ip_\mu f_\pi(T,L,m_q,a)$$

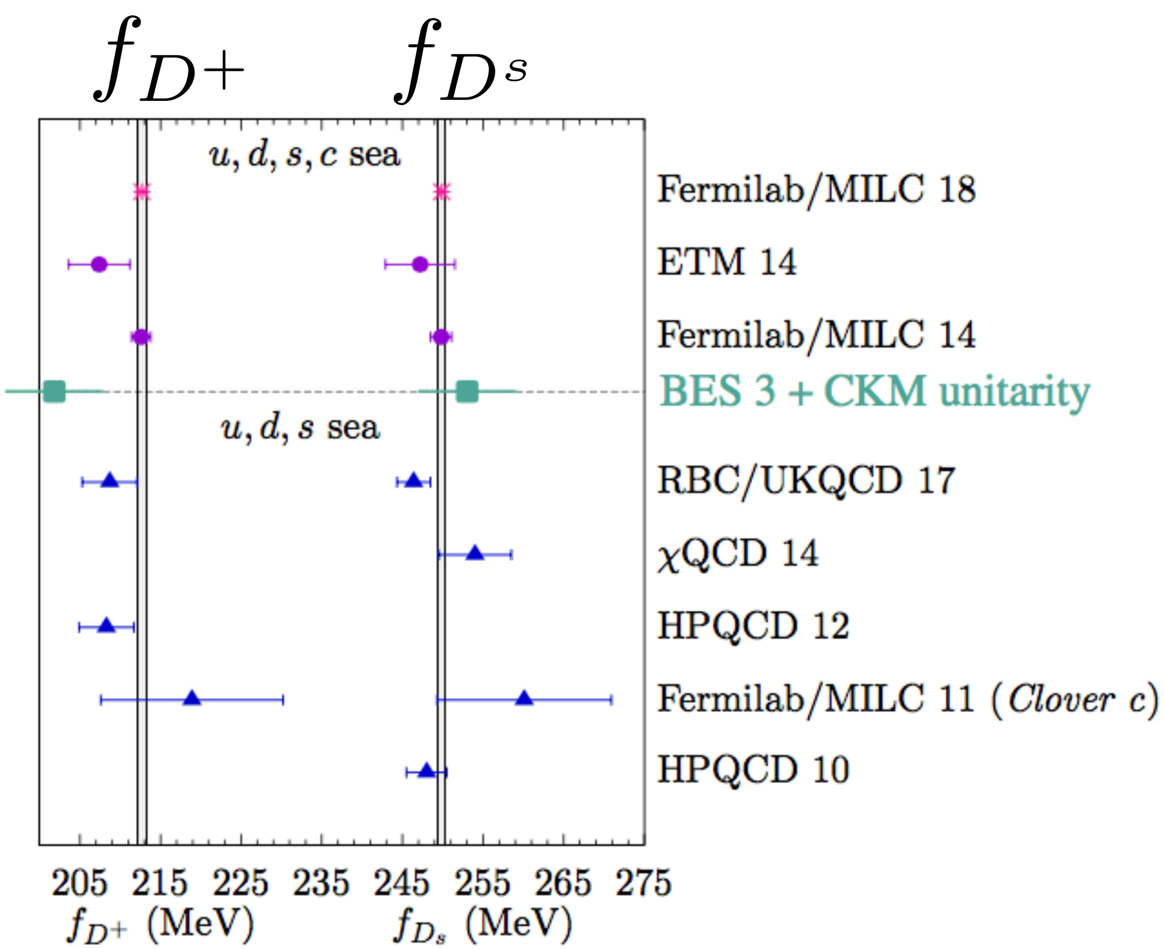
temporal length — volume — quark masses — lattice spacing

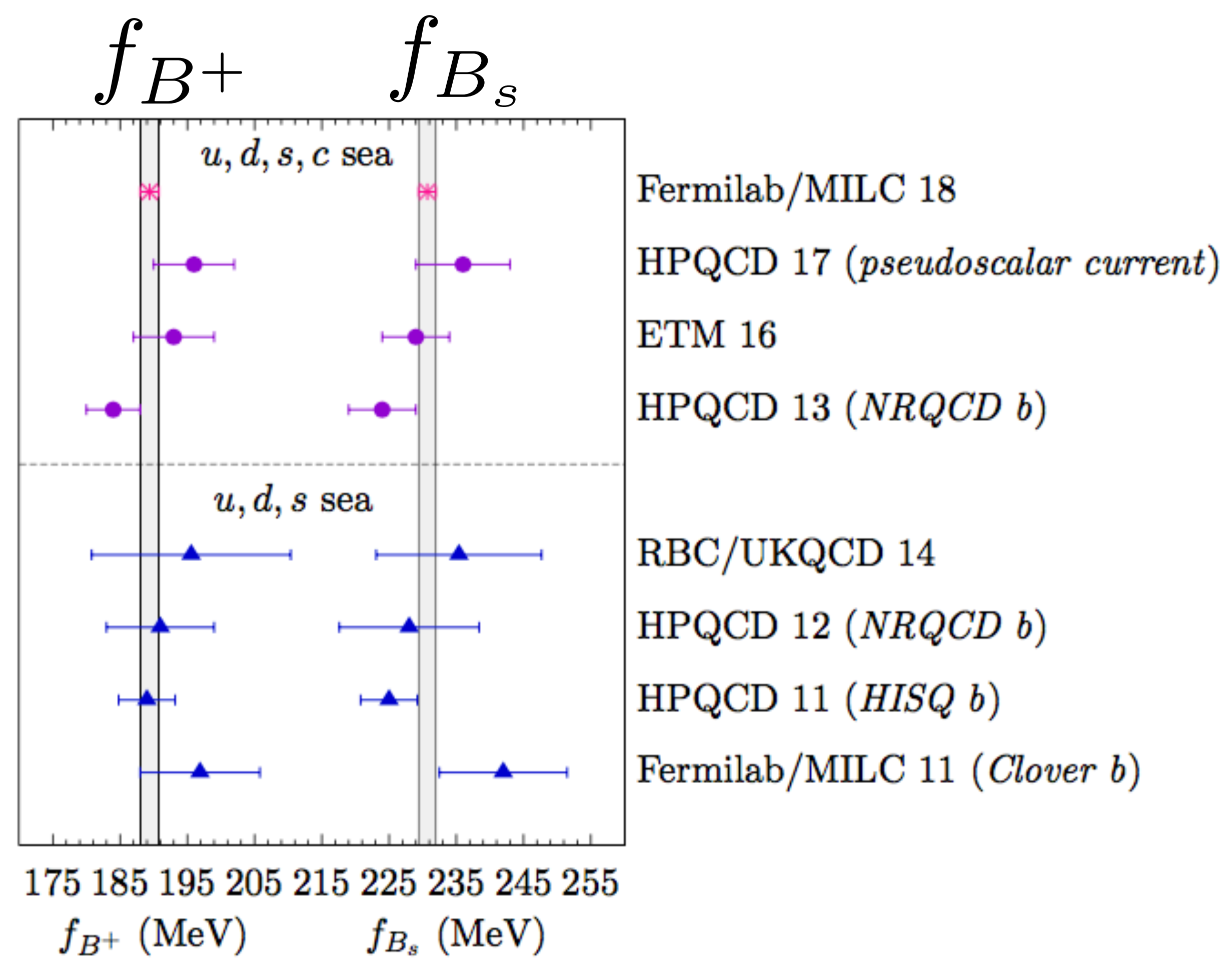
  - *Renormalization* of currents required (typically non-perturbative)
  - *Large time separation* filters excited states
  - *Extrapolation/interpolation* to physical point

$$\lim_{T,L\to\infty}\lim_{a\to 0} f_\pi(T,L,m_q^{\rm phys},a) = f_\pi^{\rm phys}$$

Decay constants $\langle 0|\mathcal{J}|\mathbf{1}\rangle$

- Summary (from Bazavov et. al. [Fermilab/MILC] 2018)

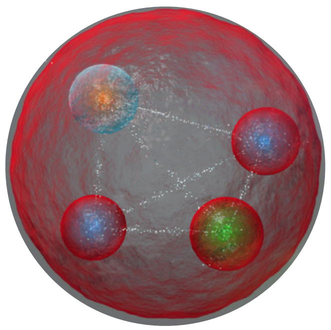




Overwhelming evidence for QCD ✓ Tool for new-physics searches ✓

Multi-hadron observables

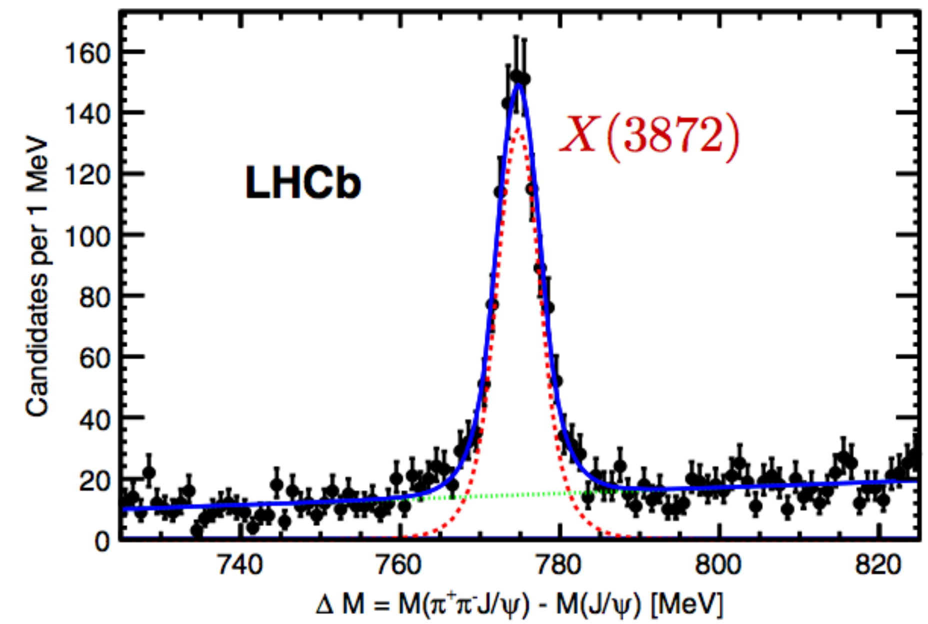
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$?

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}
• Soni (2017) •

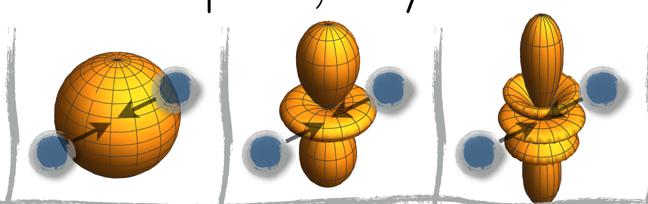
Resonant B decays $B \to K^* \ell\ell \to K\pi\,\ell\ell$

$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \notin$ **QCD Fock space**

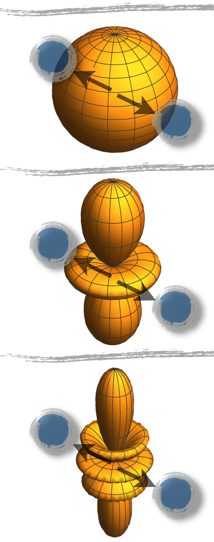
QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$|\pi\pi, \mathrm{in}\rangle$



$S(s) \equiv \langle \pi\pi, \mathrm{out}|$



$e^{2i\delta_0(s)}$	0	0
0	$e^{2i\delta_1(s)}$	0
0	0	$e^{2i\delta_2(s)}$

depends on $s = E^2_{\mathrm{cm}}$ and angular variables

diagonal in angular momentum

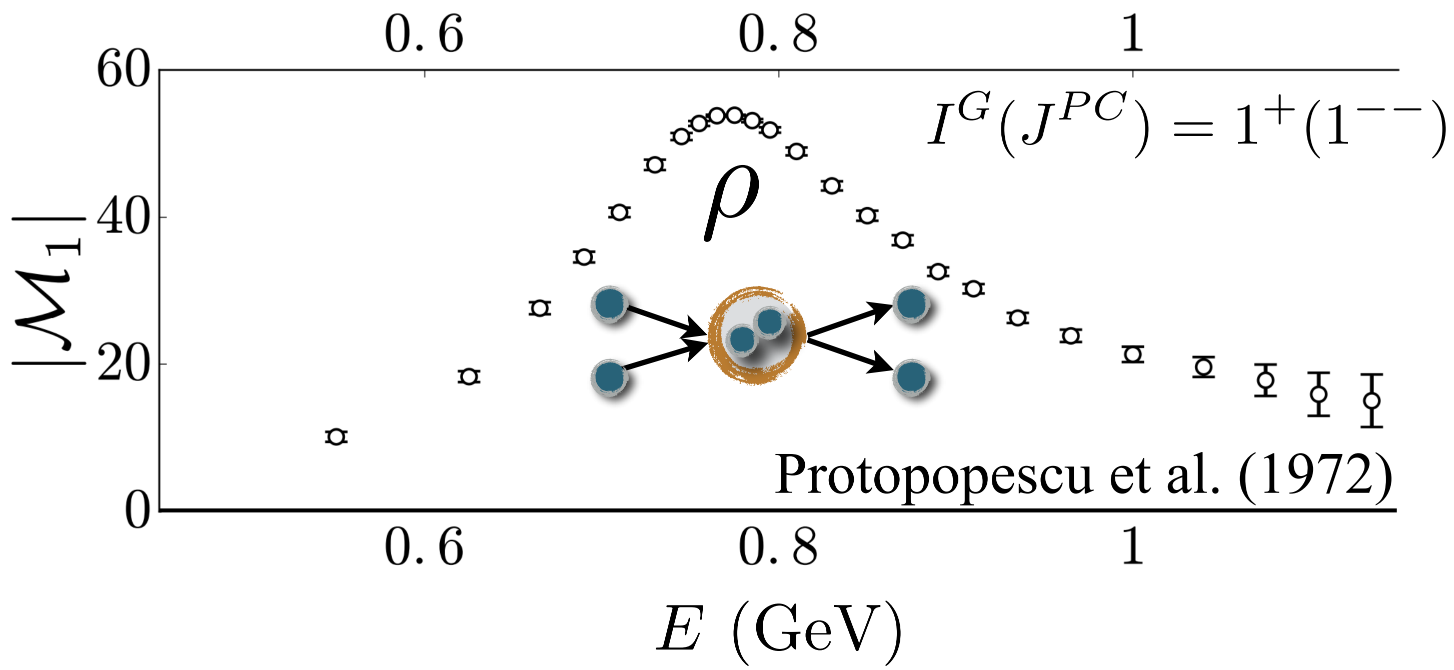
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

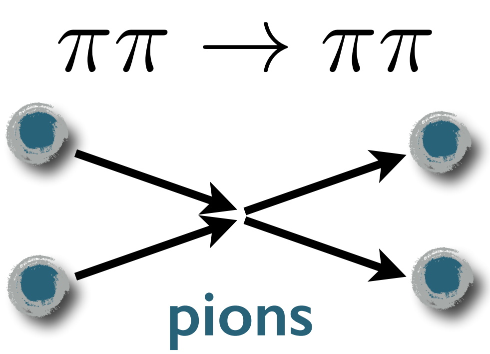
- ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$

QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





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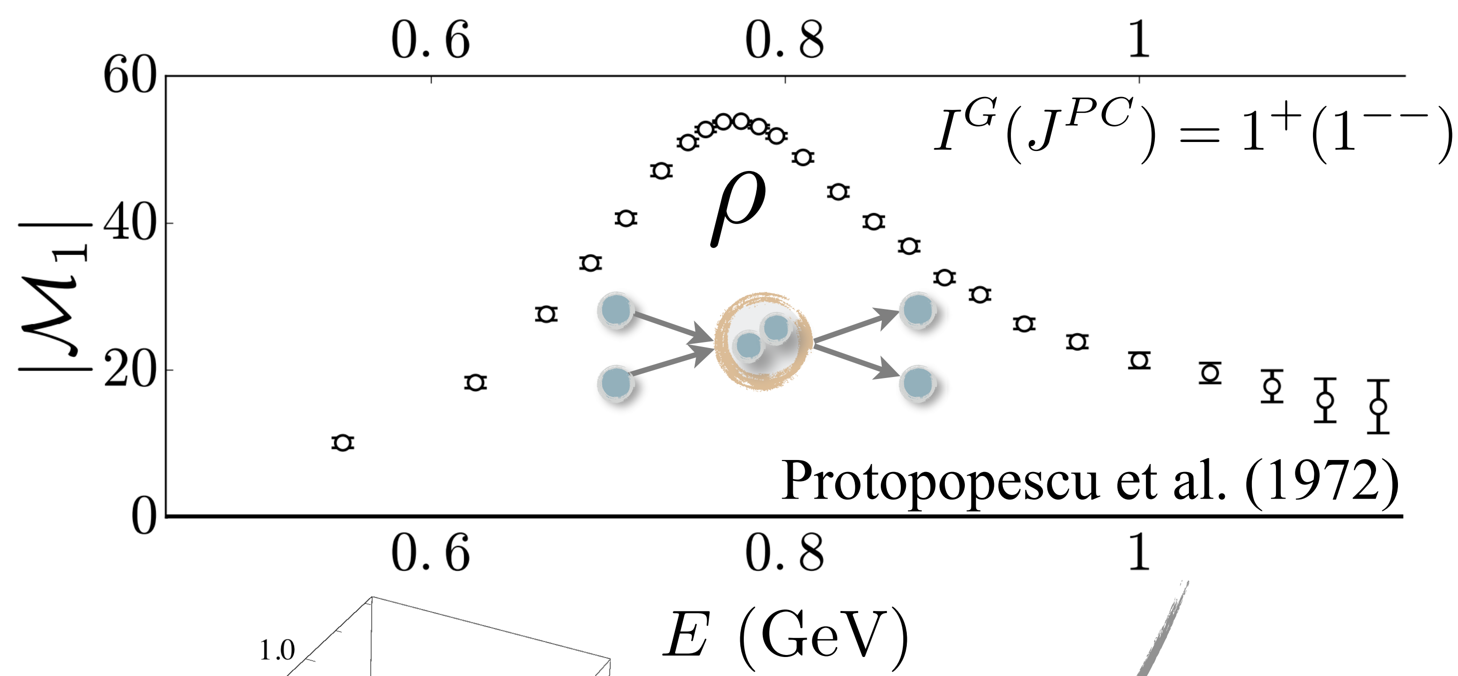

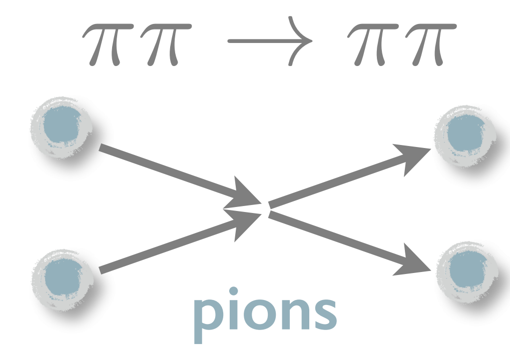

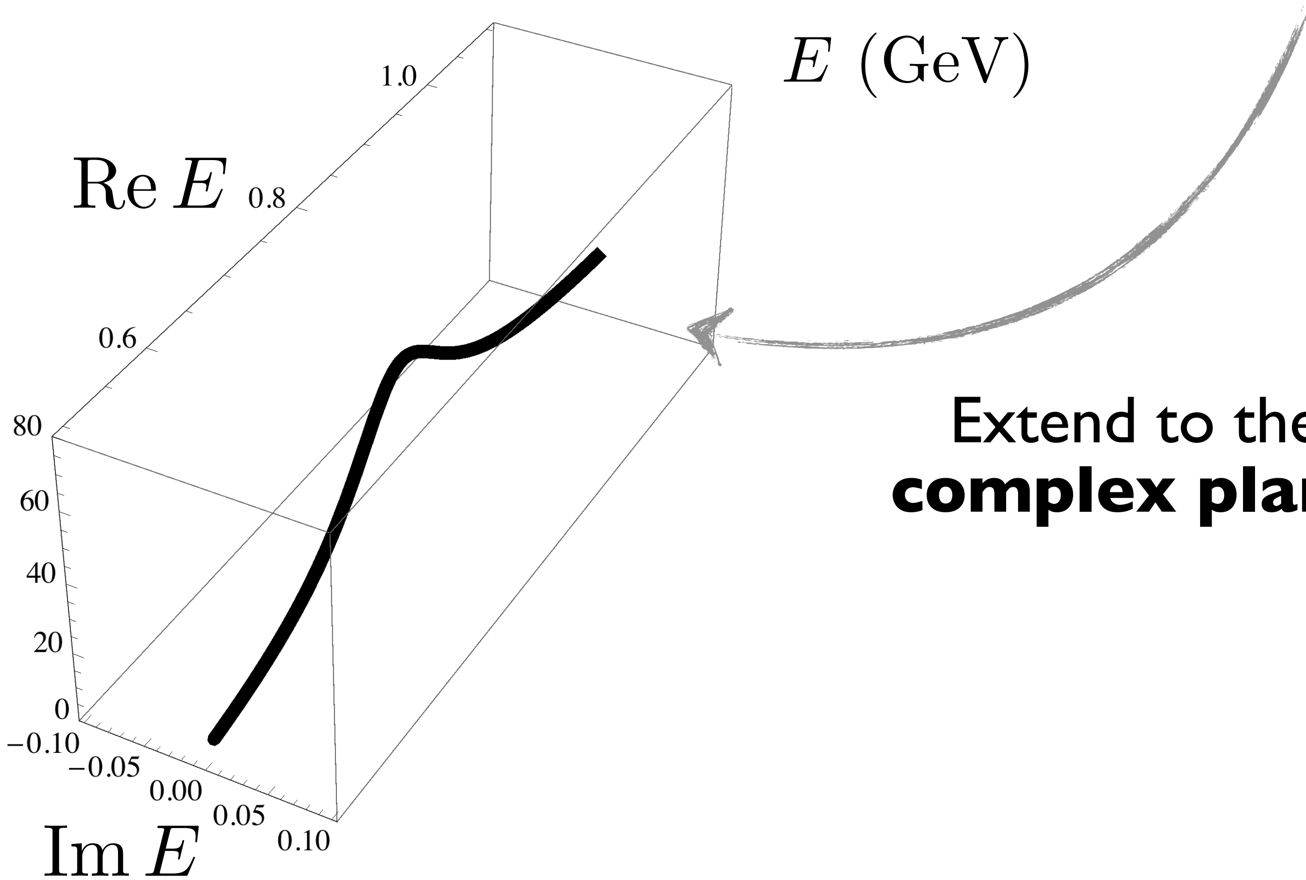

Extend to the **complex plane**

QCD resonances

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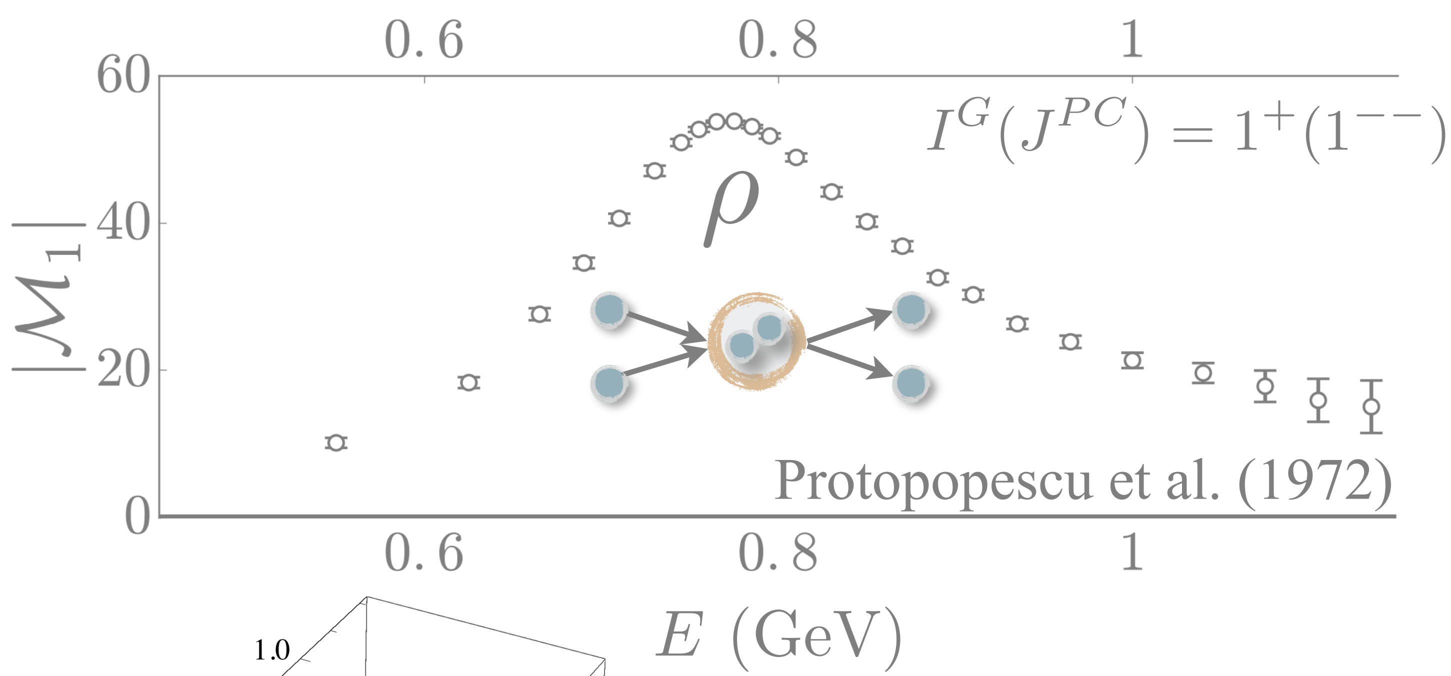
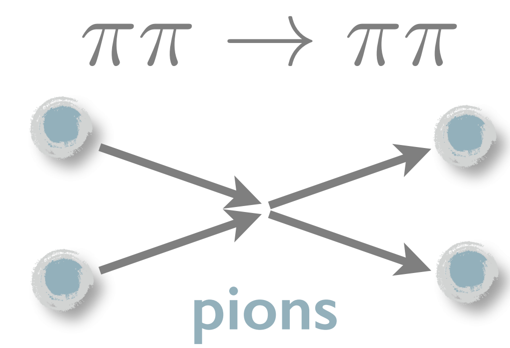
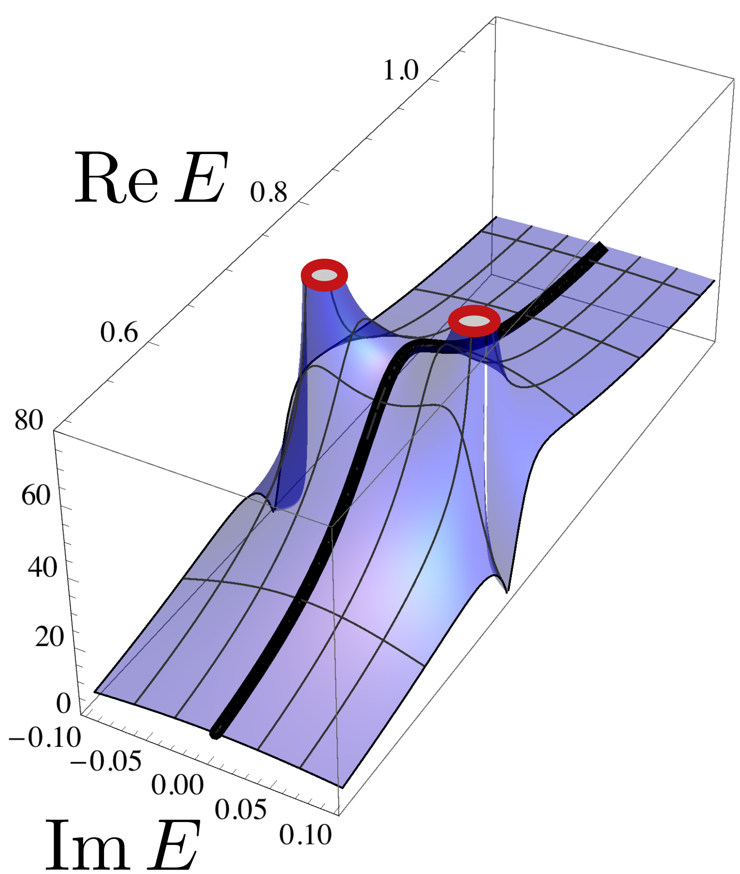
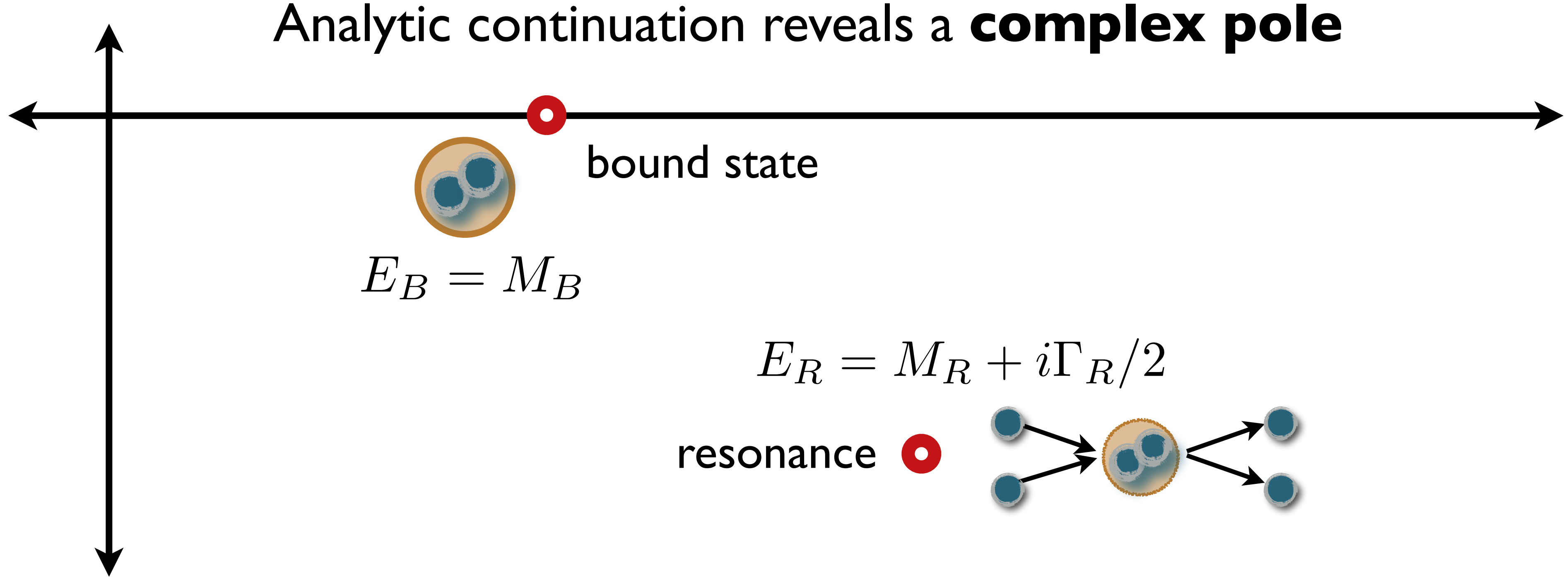




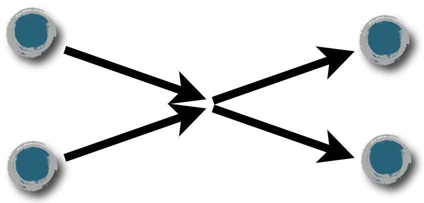


Analytic continuation reveals a **complex pole**



Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself



For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \operatorname{Im}\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

- Unique solution is…

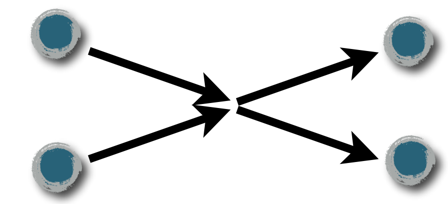
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

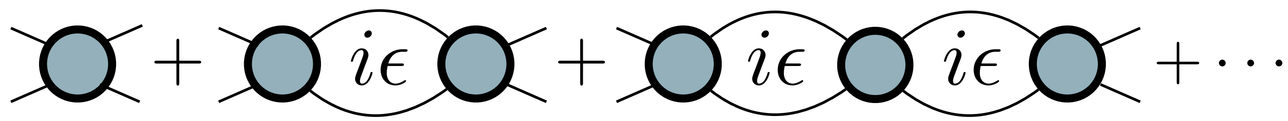
K matrix (short distance)

phase-space cut (long distance)

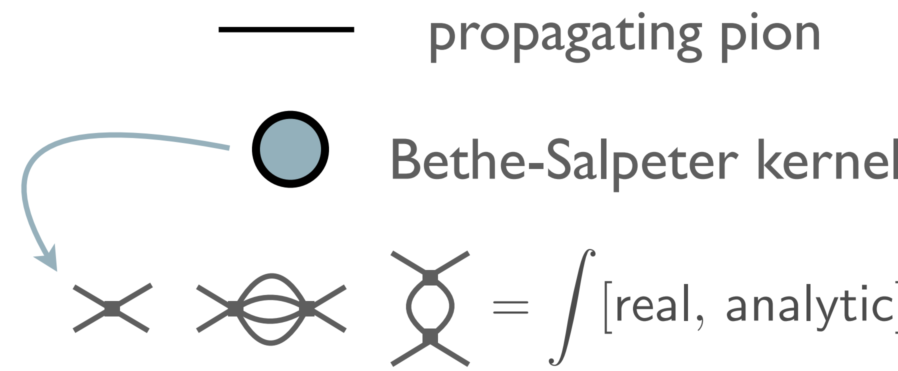
Key message: *The scattering amplitude has a square-root branch cut*

Analyticity (diagrammatic)



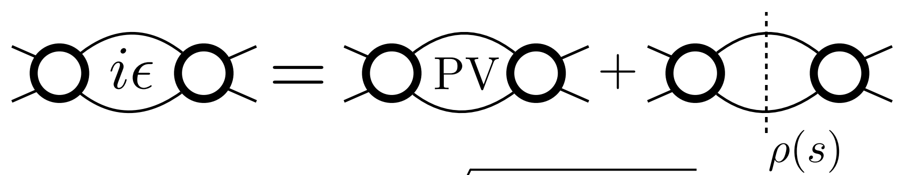
$\mathcal{M}(s) \equiv$ 

on-shell particles = singularities:
non-analytic for $(2m)^2 < s < (4m)^2$



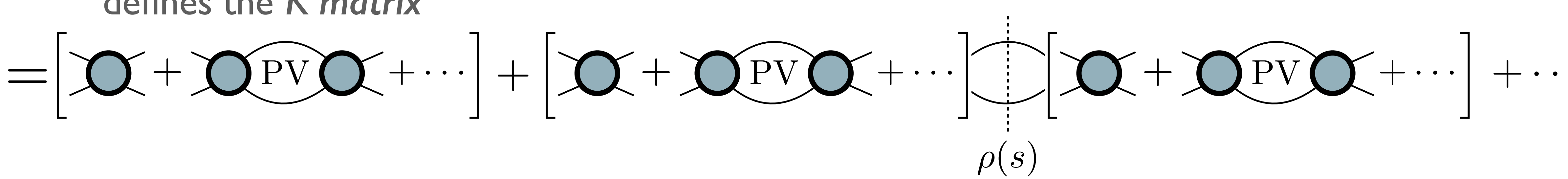
for $(2m)^2 < s < (4m)^2$

cutting rule



$\rho(s) \propto i\sqrt{s-(2m)^2}$

defines the *K* ***matrix***



$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

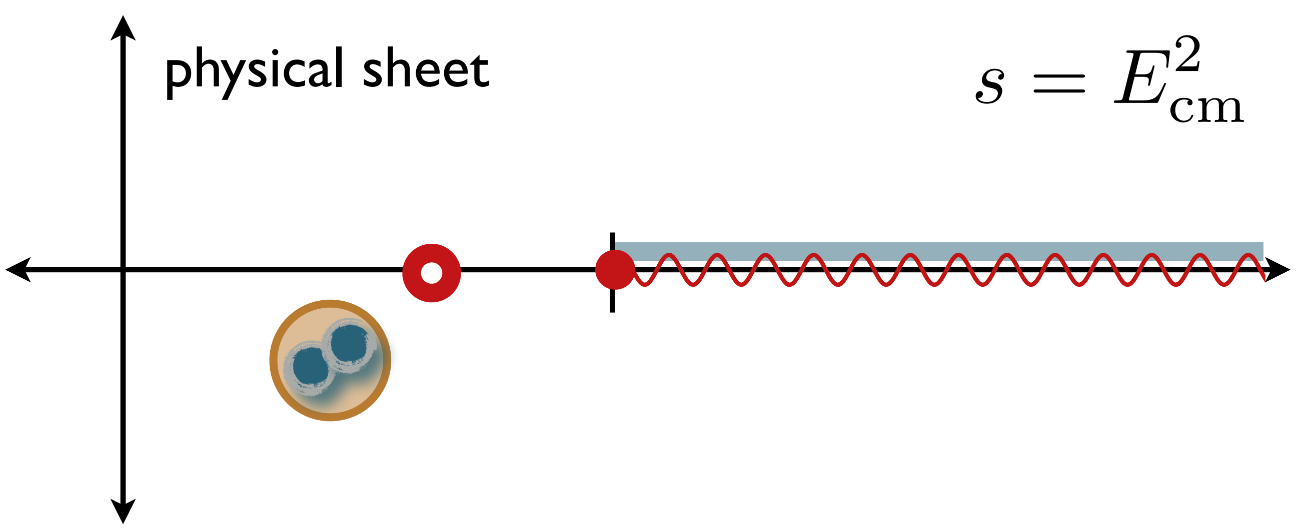
K matrix (short distance)

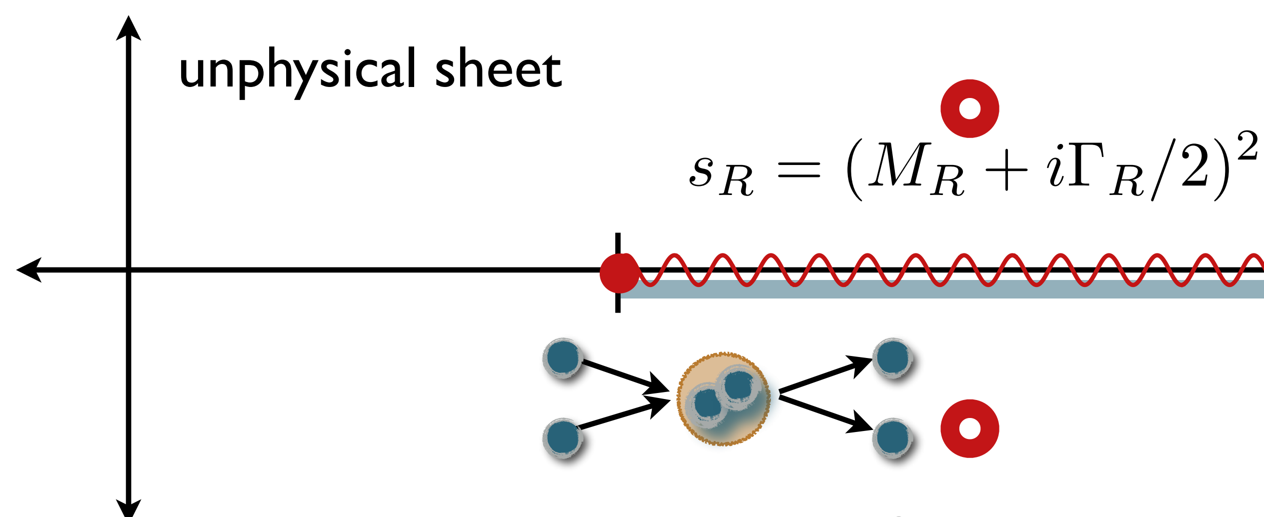
phase-space cut (long distance)

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - \rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto i\sqrt{s-(2m)^2}$$

- Each channel generates a *square-root cut* → doubles the number of sheets





resonance poles only on *2nd* sheet

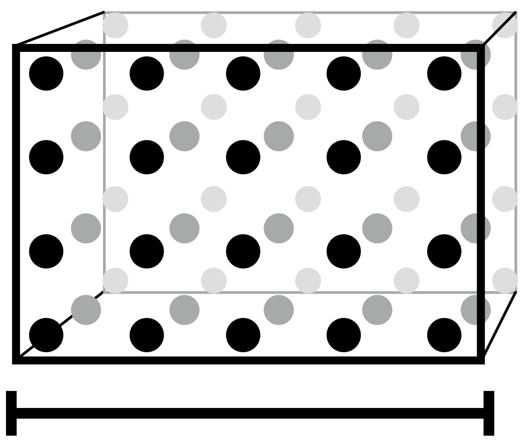
- Important lessons:

  Details of analyticity = important for quantitative understanding

  Possible to separate…
  (i) long-distance kinematic singularities
  (ii) short-distance/microscopic physics (depending on interaction details)

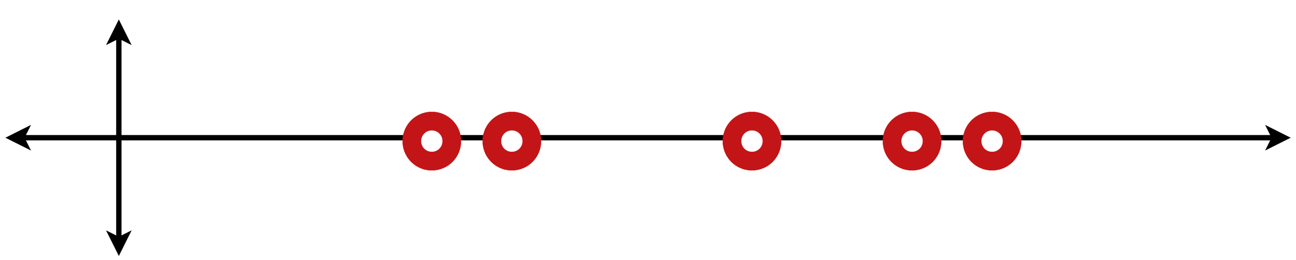
Difficulties for multi-hadron observables

- The ***Euclidean signature / imaginary time…***
  - ***Obscures*** real time evolution (that defines scattering)
  - ***Prevents*** normal LSZ (want $p_4^2 = -(p^2 + m^2)$, but we have only $p_4^2 > 0$)

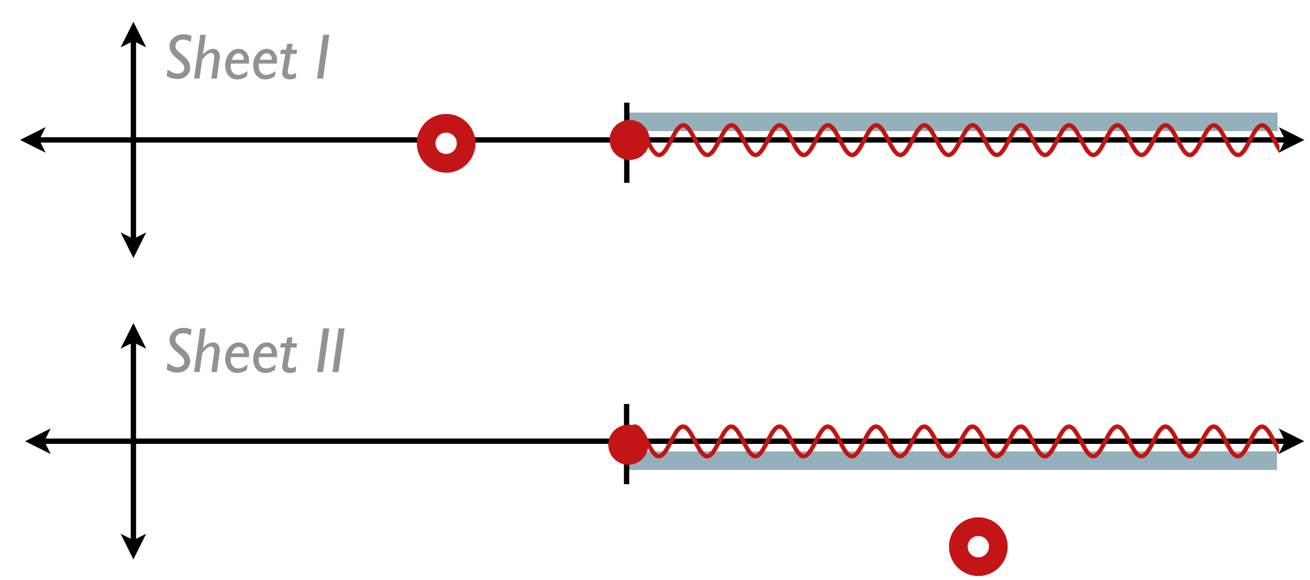


- The ***finite volume…***
  - ***Discretizes*** the spectrum
  - ***Eliminates*** the branch cuts and extra sheets
  - ***Hides*** the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



Two strategies…

- Finite-volume as a tool
  - LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
  - Applicable only in limited energy range for two- and three-hadron states

- Spectral function method
  - Formally applies for any number of particles / any energy range
  - An answer to the question… “Can’t you just analytically continue?”
  - Still important challenges and limitations to consider

Two strategies…

- Finite-volume as a tool
  - LQCD → Energies and matrix elements

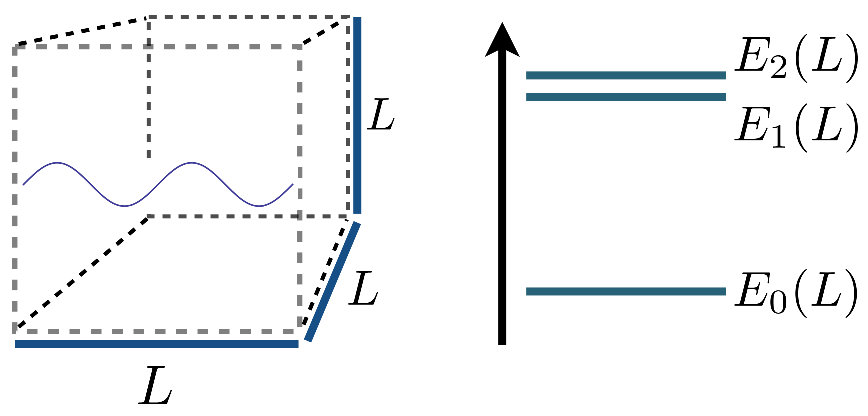
$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
  - Applicable only in limited energy range for two- and three-hadron states

- Spectral function method
  - Formally applies for any number of particles / any energy range
  - An answer to the question… "Can't you just analytically continue?"
  - Still important challenges and limitations to consider

The finite-volume as a tool

- Finite-volume set-up

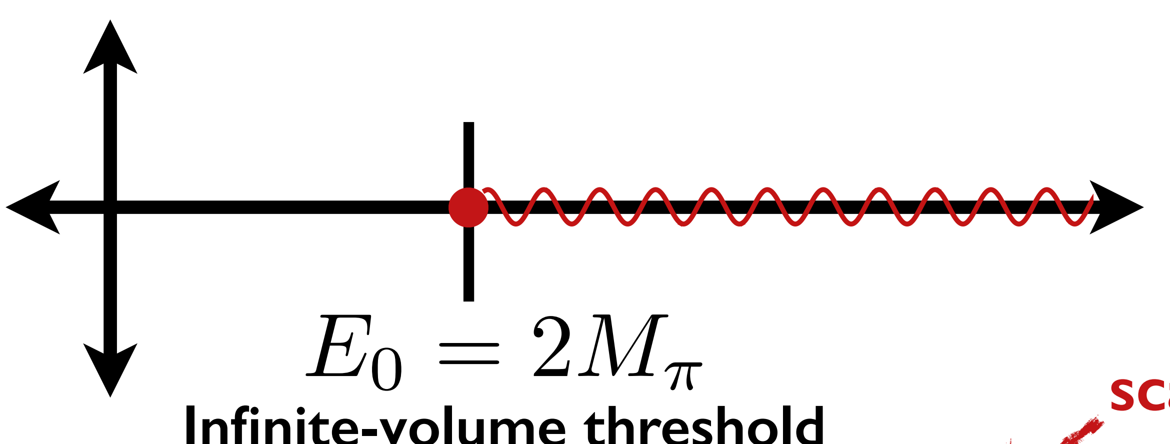


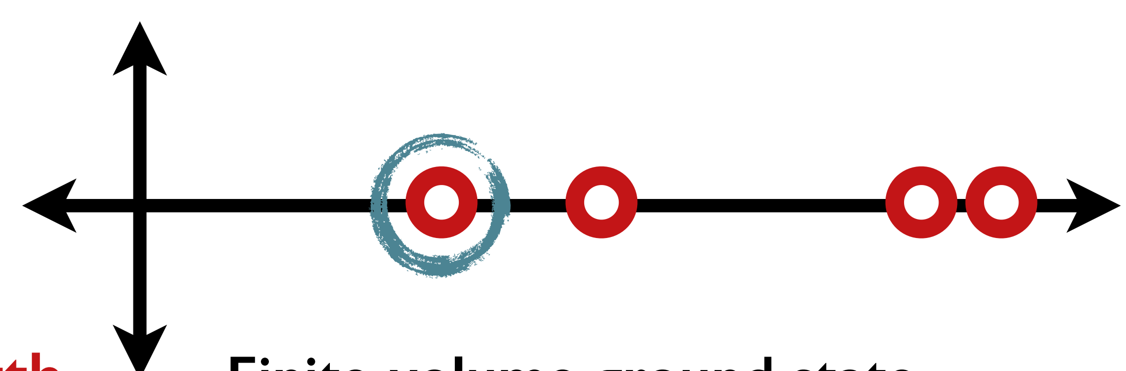
- **cubic**, spatial volume (extent *L*)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$
- *T* and lattice also negligible

- Scattering leaves an ***imprint*** on finite-volume quantities





scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

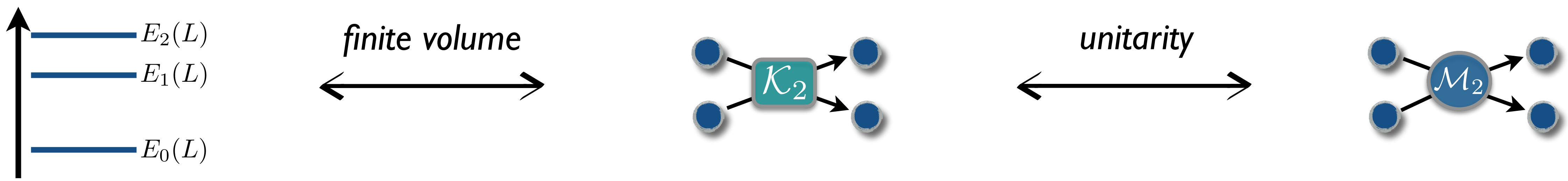
$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

General relation

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to ***non-degenerate masses, multiple channels, spinning particles***

Encodes angular momentum mixing

• Lüscher (1989) • *many others* •

Scattering processes and resonances from lattice QCD

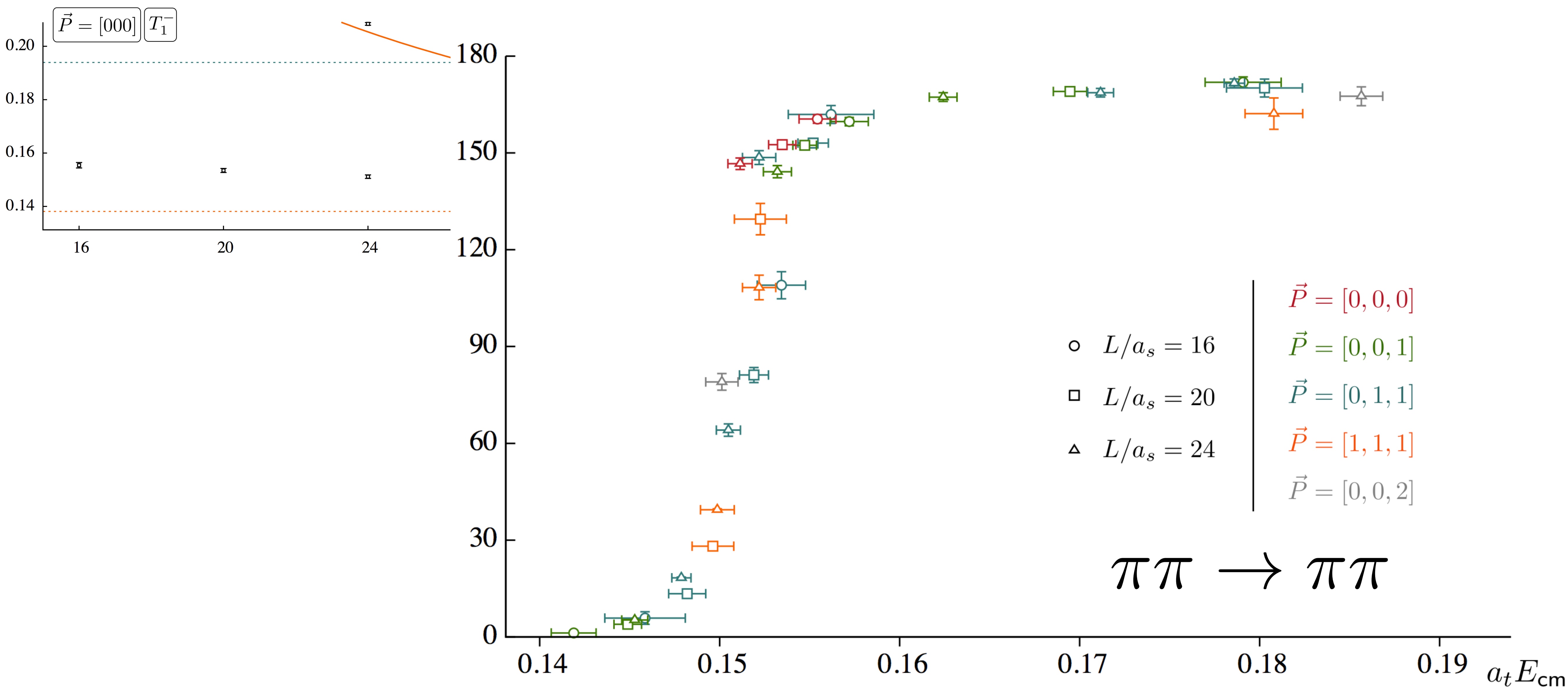
Raúl A. Briceño,[1,*] Jozef J. Dudek,[1,2,†] and Ross D. Young[3,‡]

REVIEWS OF MODERN PHYSICS

Using the result

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot\delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

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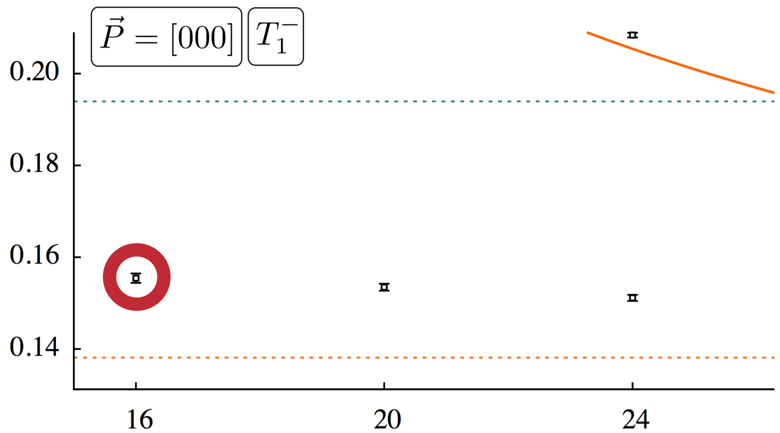

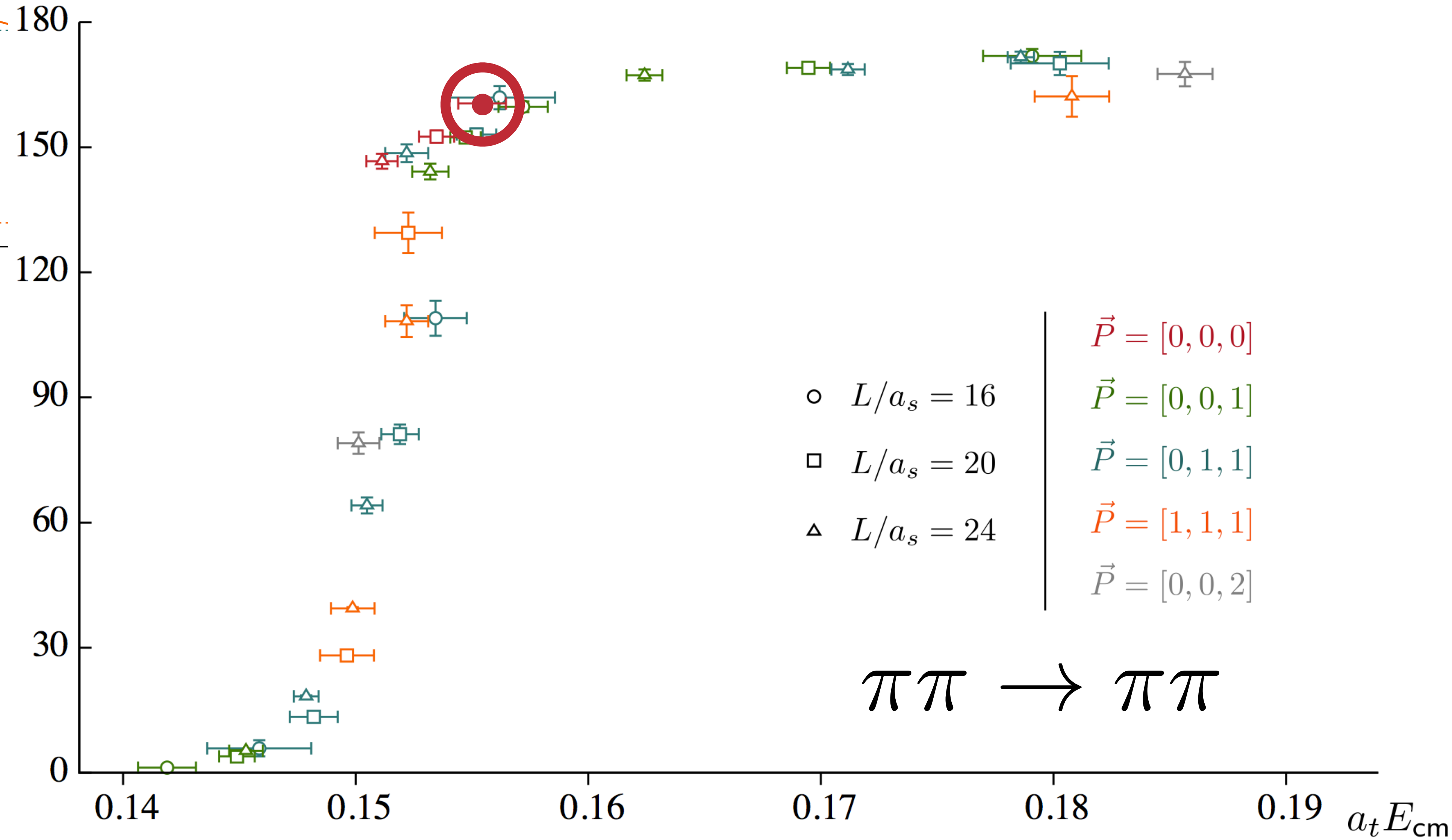

• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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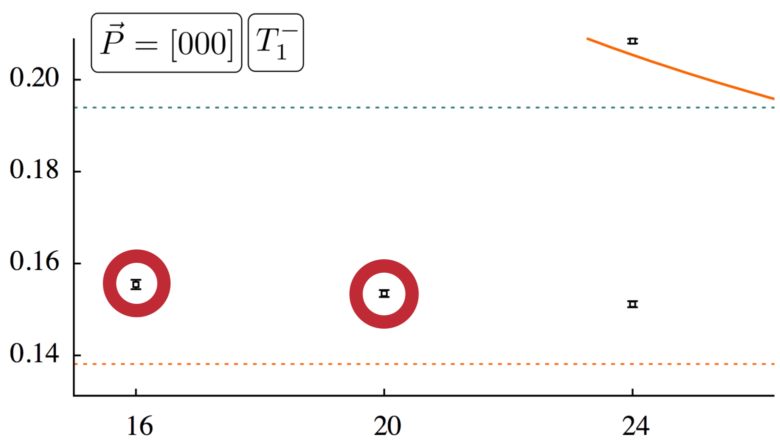

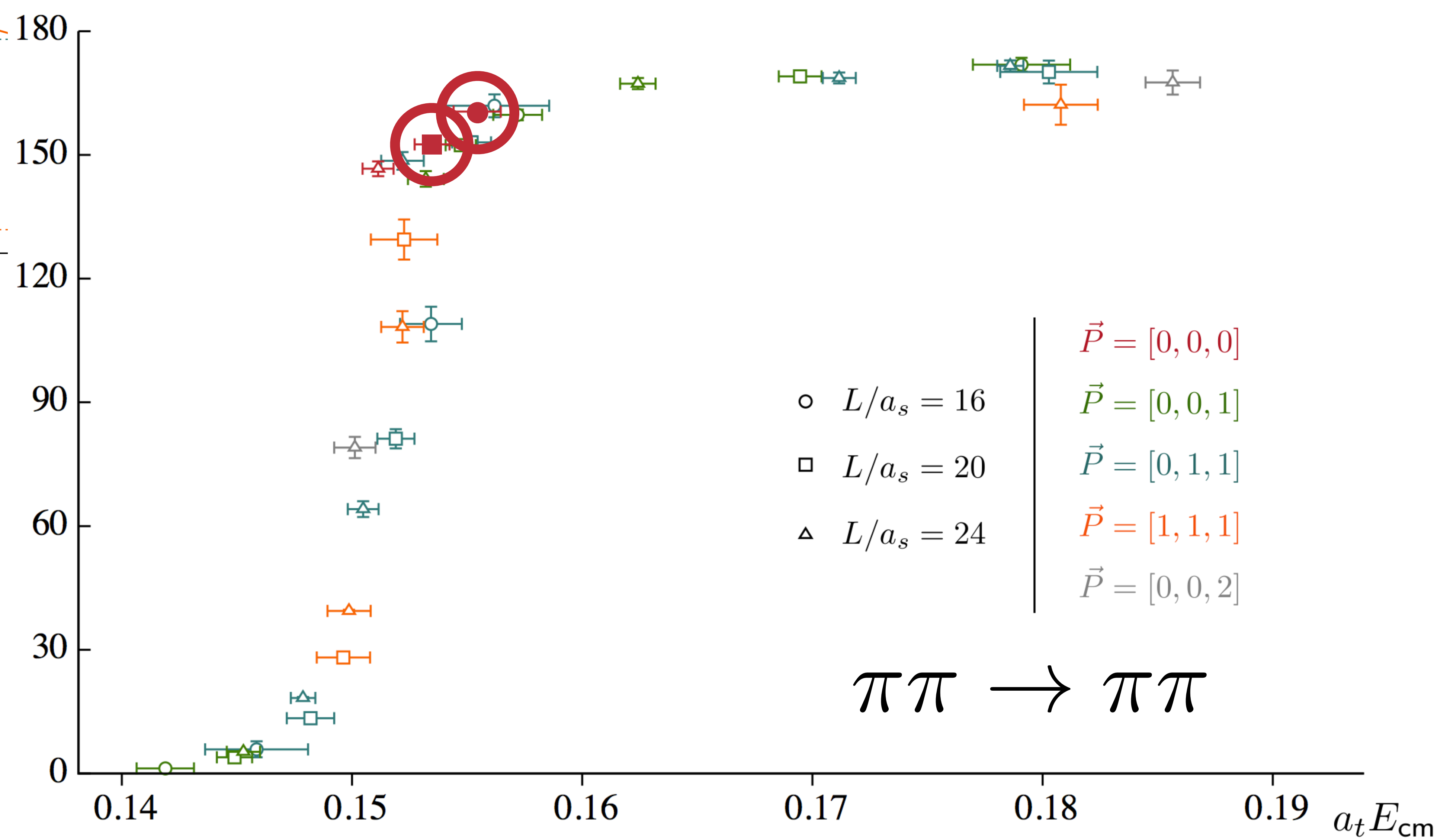

• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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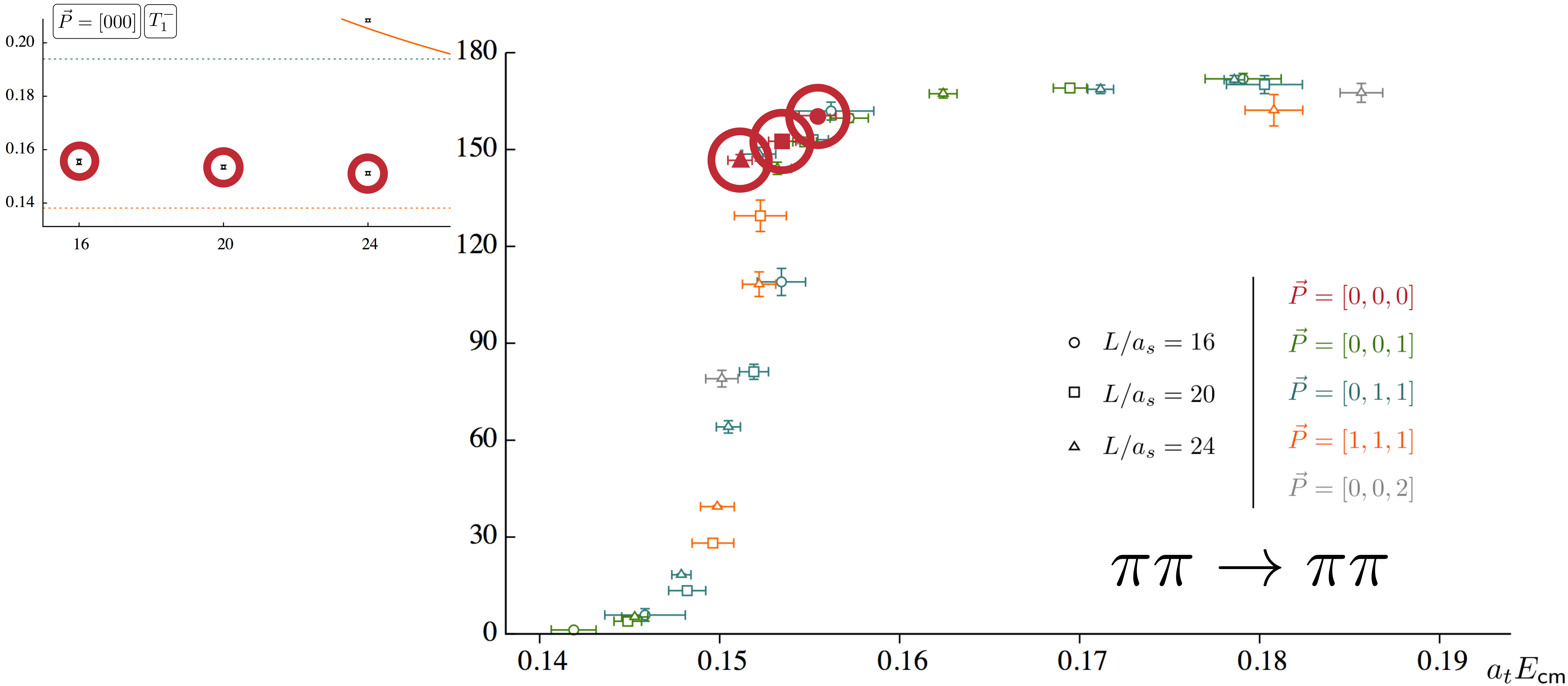

- Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

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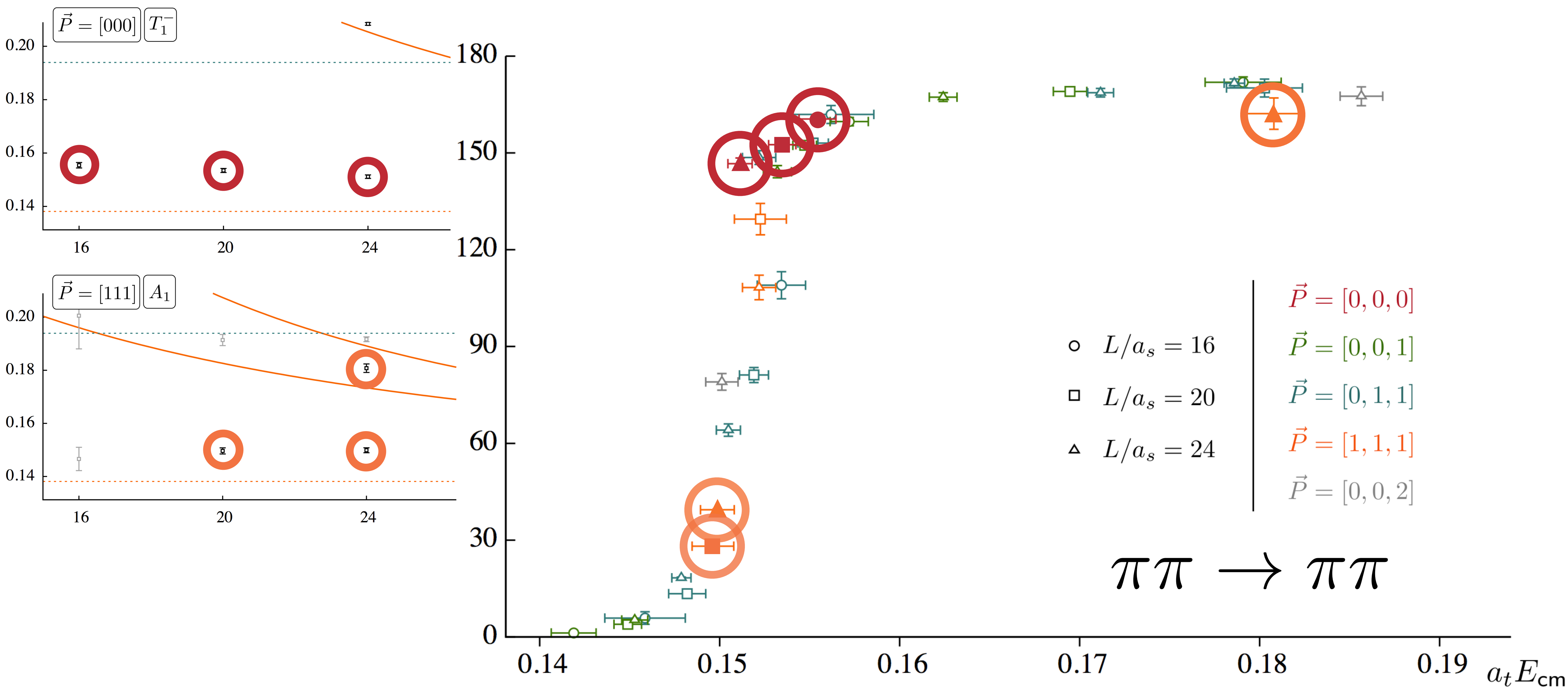
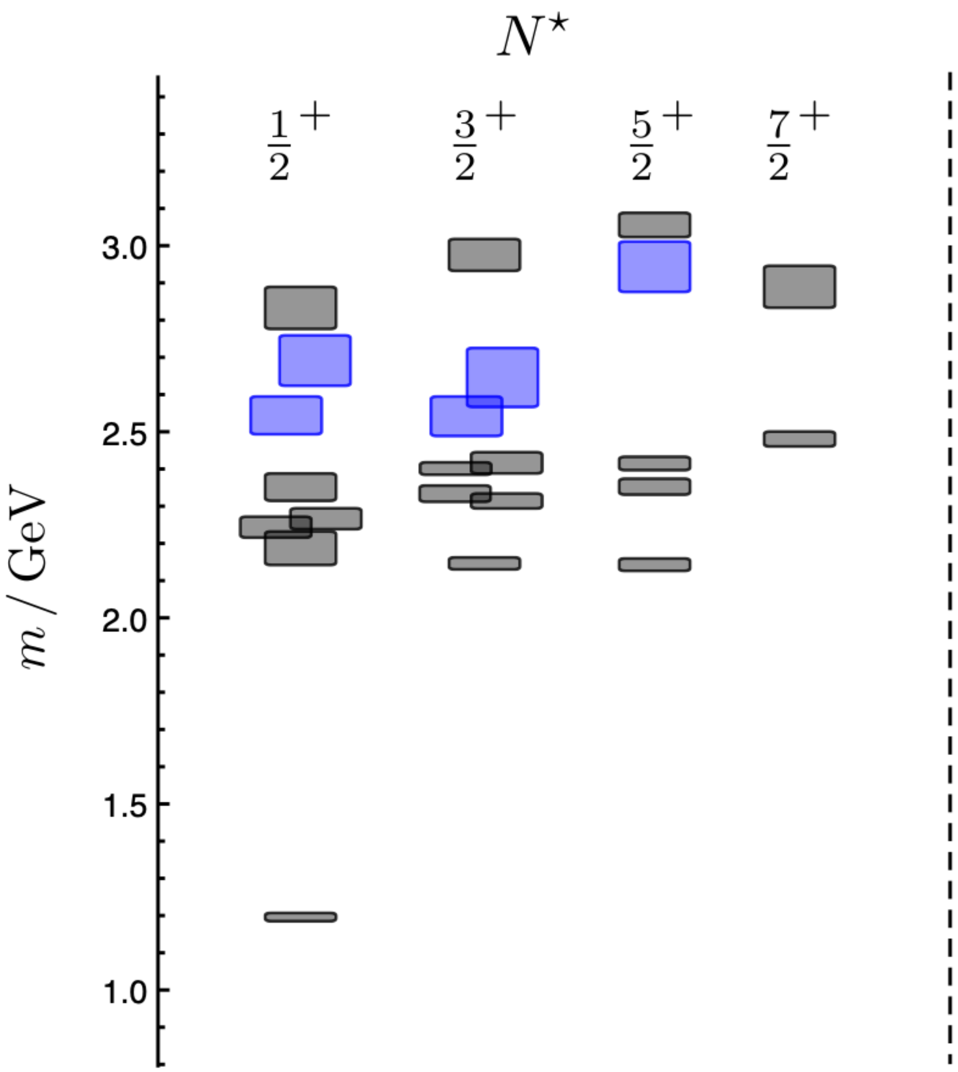


• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

Two types of spectroscopy

Explore the spectrum of compact QCD excited states

(via quark-model inspired local operators)



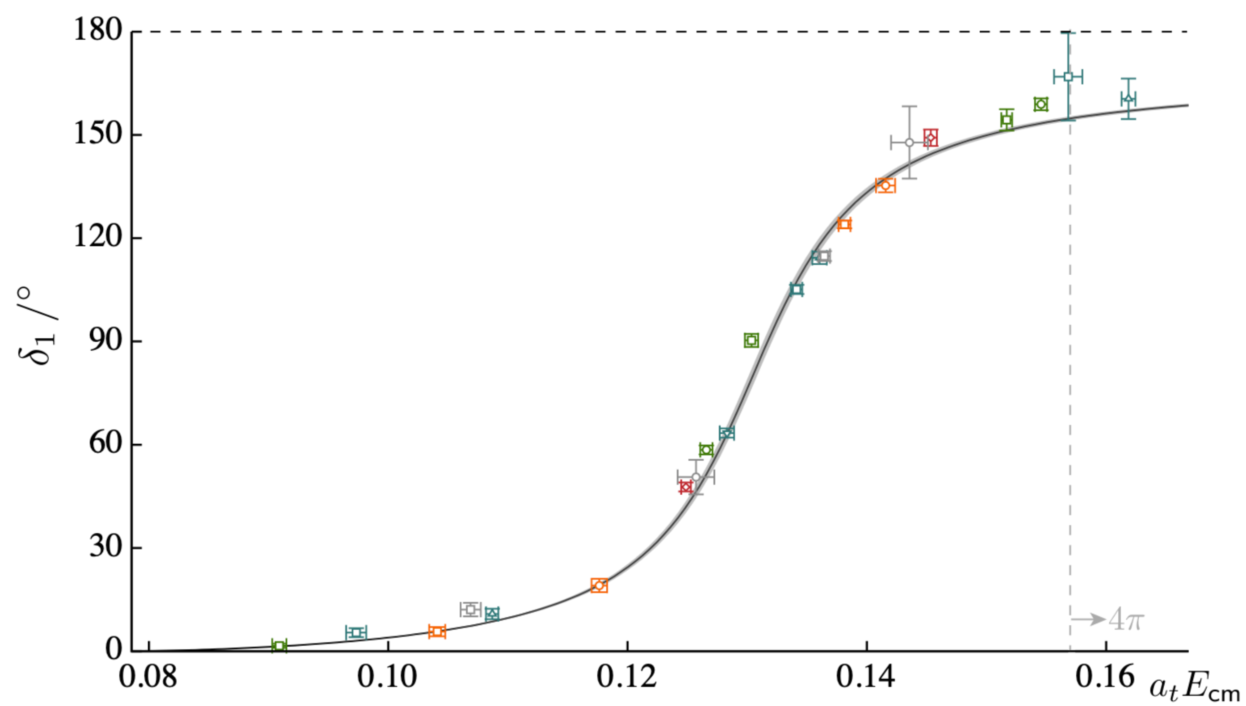
Dudek, Edwards (2012)

Extract the full finite-volume energy spectrum

local operators

+ many multi-hadron operators

$$\pi\pi \to \rho \to \pi\pi$$



Wilson, Briceño, Dudek, Edwards, Thomas (2015)

Two types of spectroscopy

Explore the spectrum of compact QCD excited states

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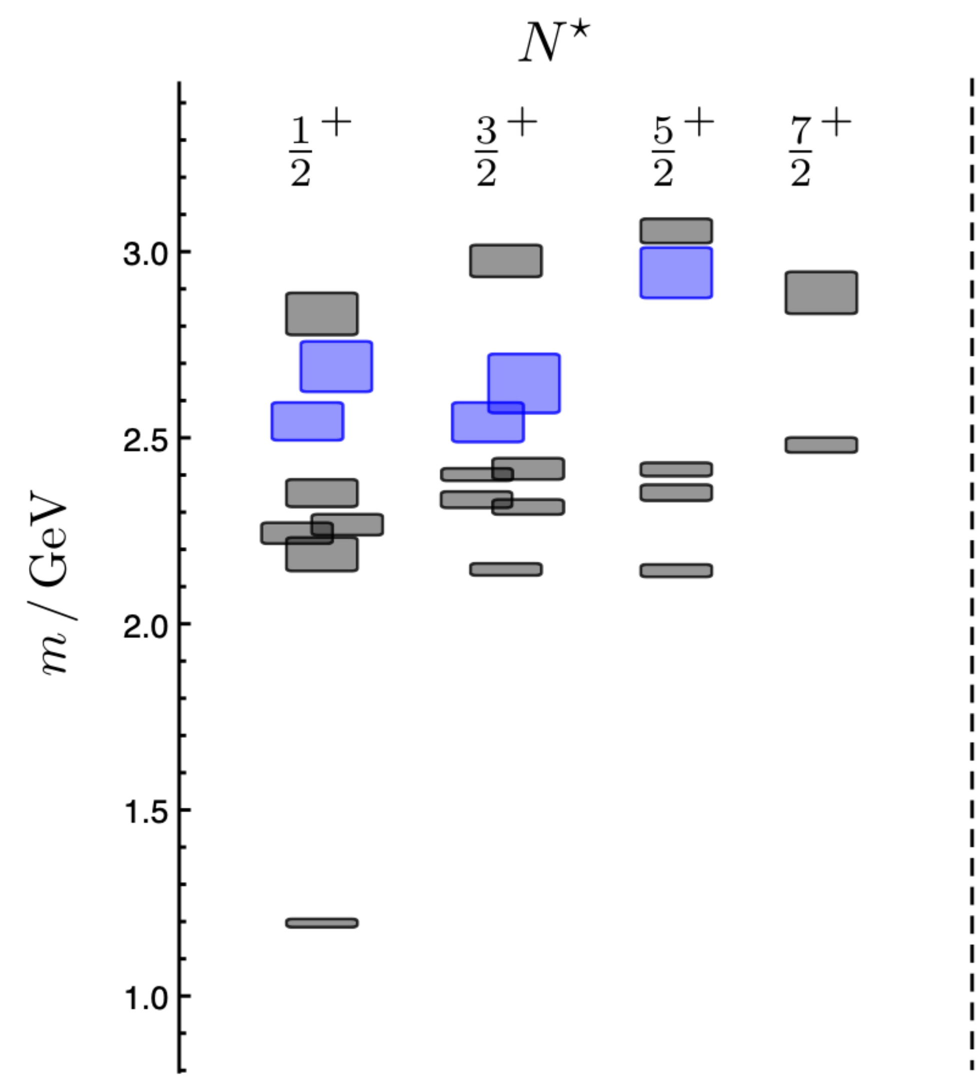
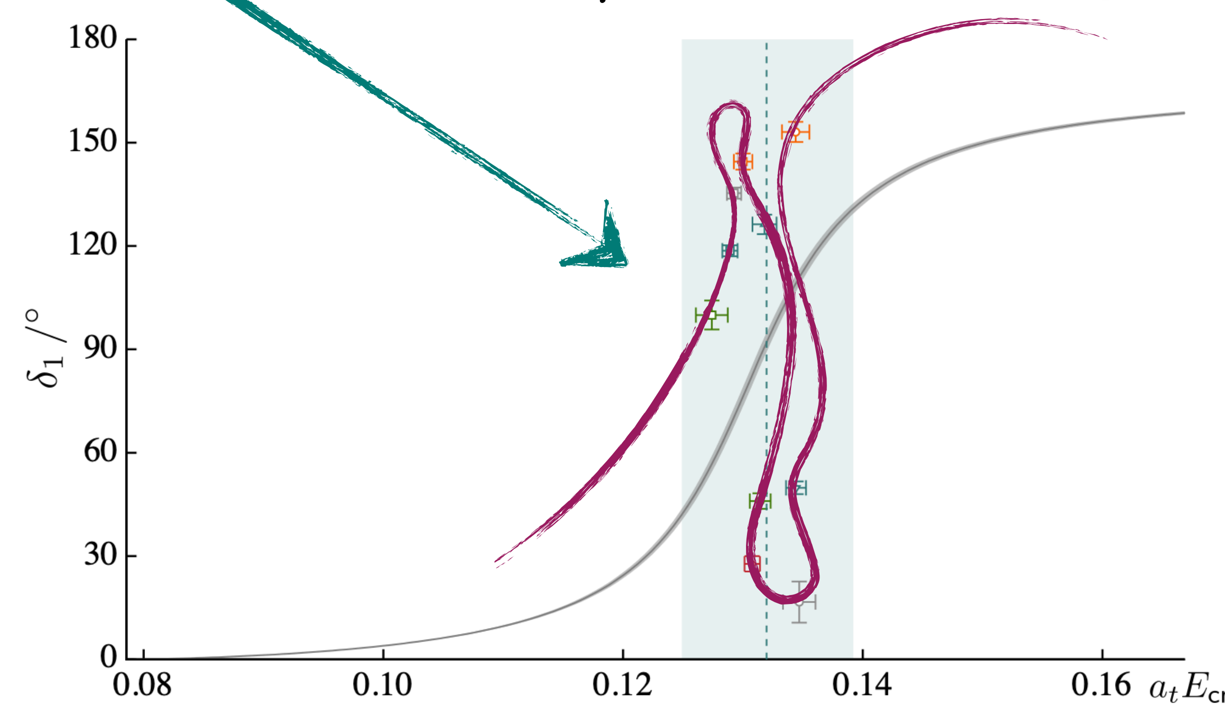


Dudek, Edwards (2012)

local operator spectrum =

not suitable for phase shift extraction

$$\pi\pi \to \rho \to \pi\pi$$

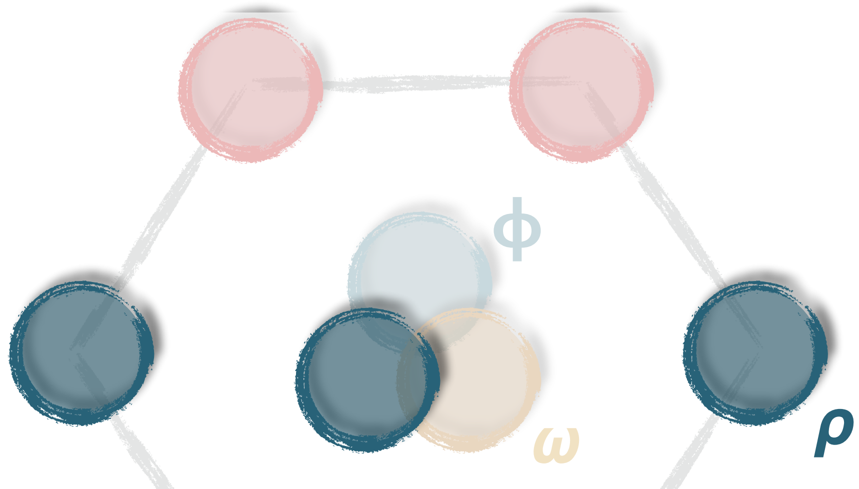


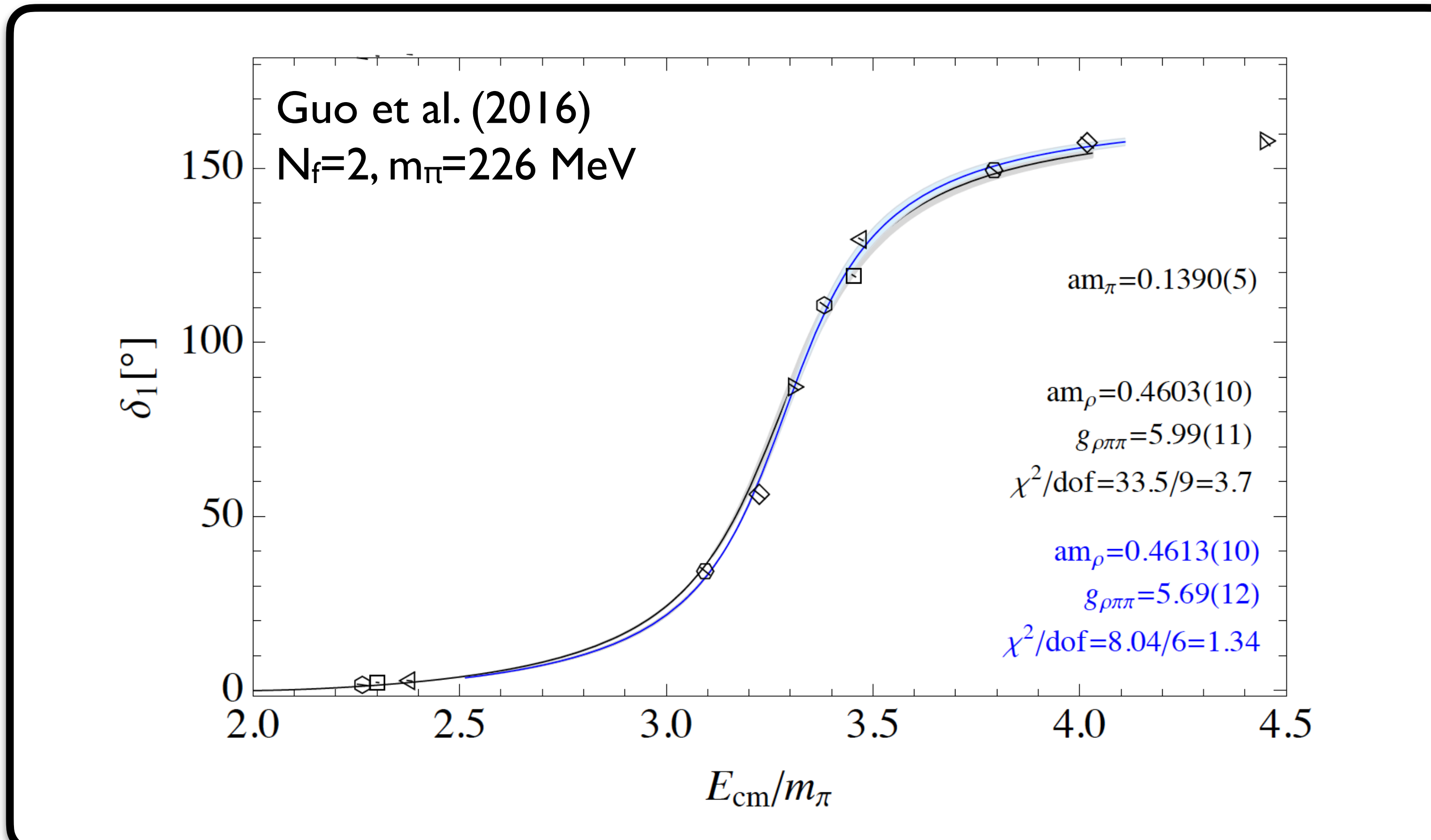
Note: cannot count finite-volume energies to count resonance poles!

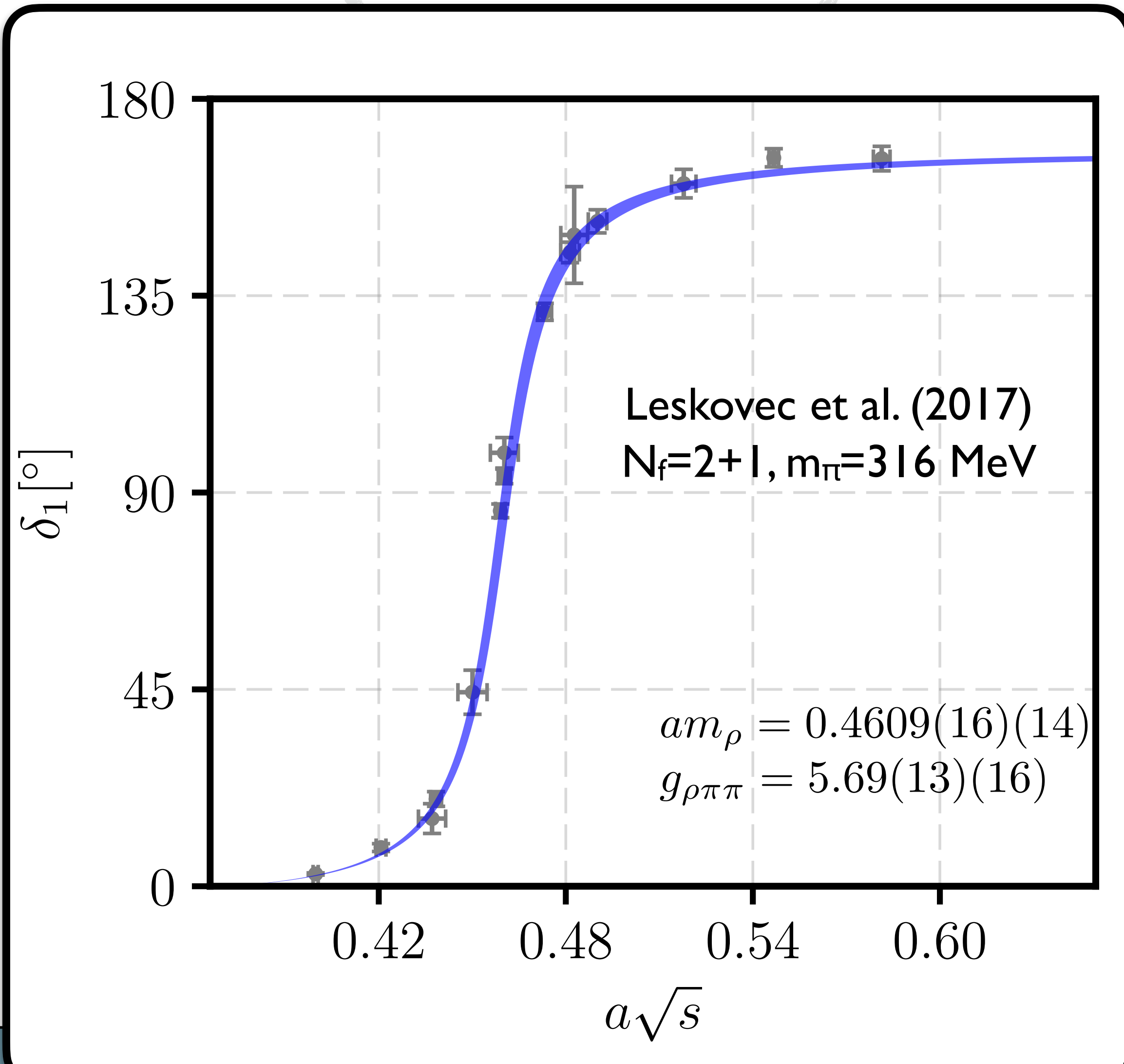
Wilson, Briceño, Dudek, Edwards, Thomas (2015)

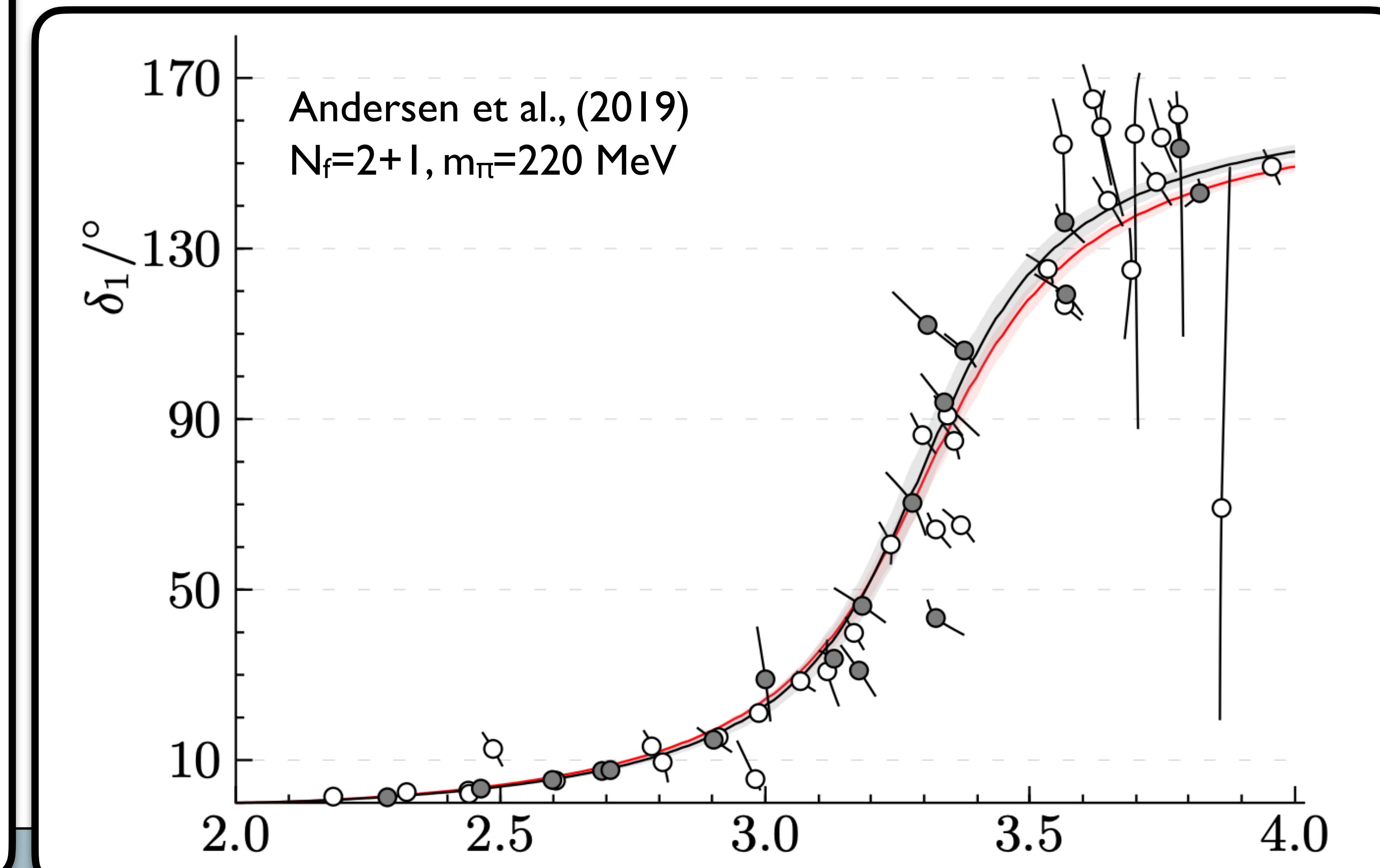
$\rho \to \pi\pi$

$I^G(J^{PC}) = 1^+(1^{--})$

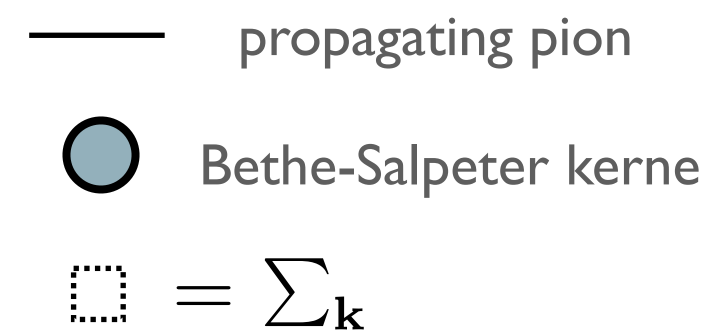




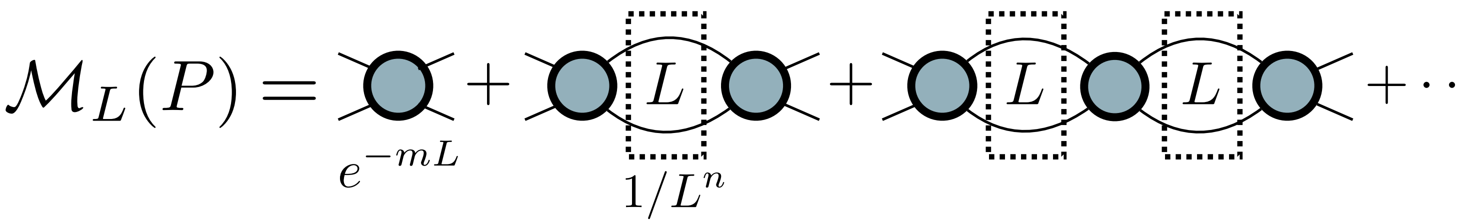




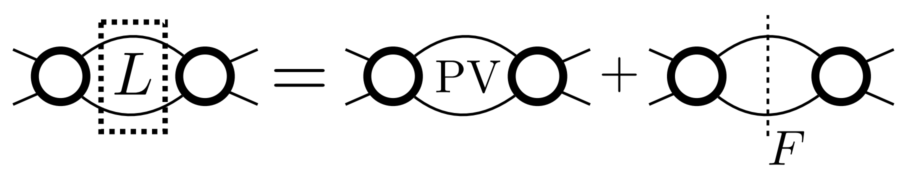
Derivation



- Consider the finite-volume correlator:

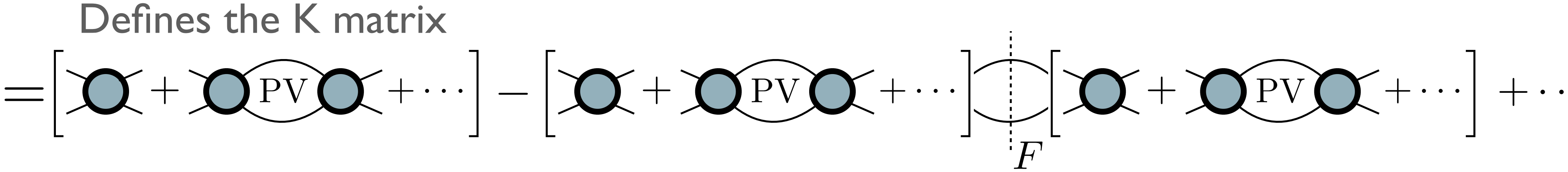


For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

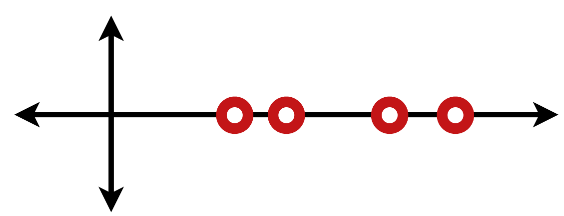


F = matrix of known geometric functions

Defines the K matrix



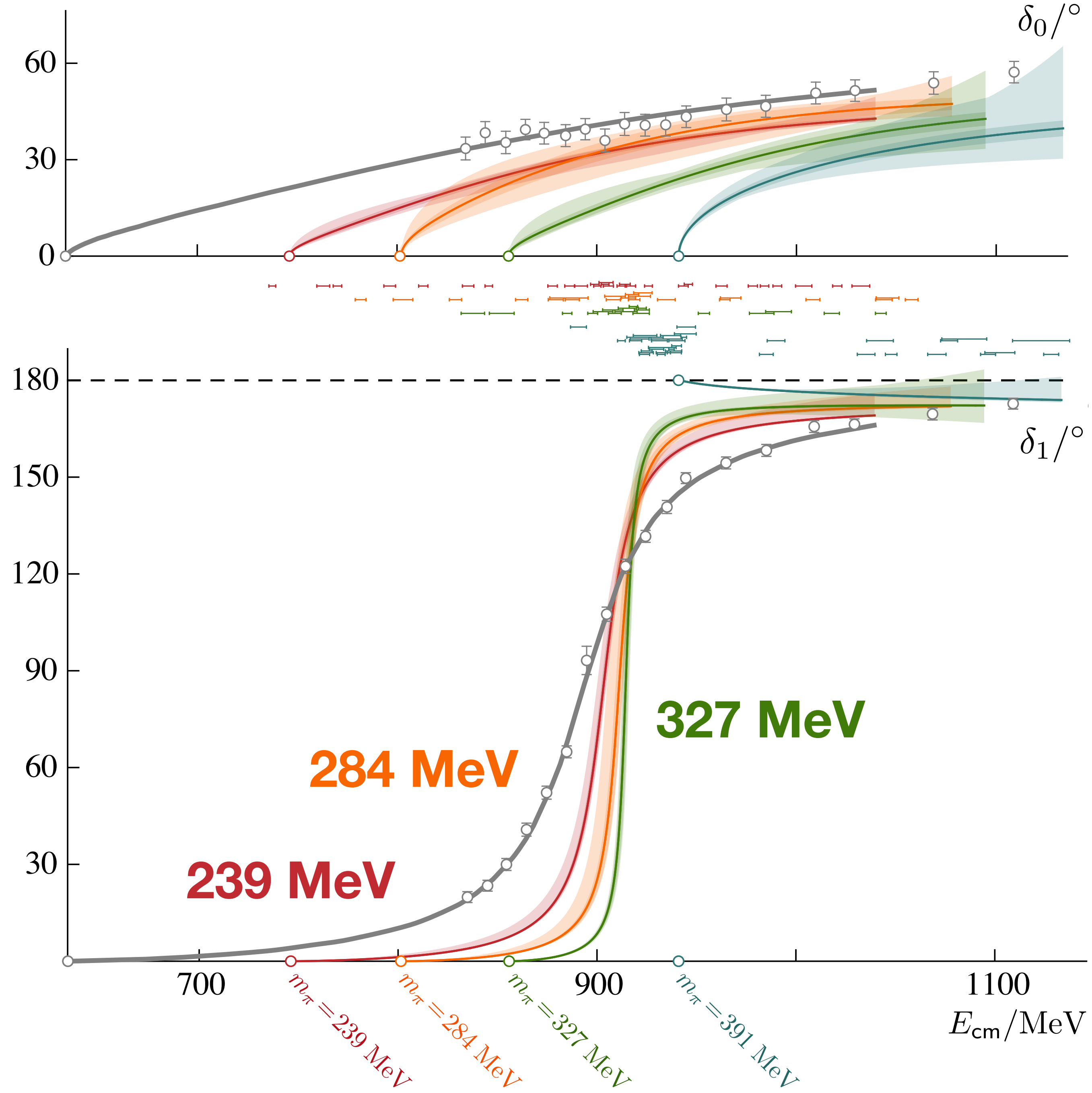
$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$

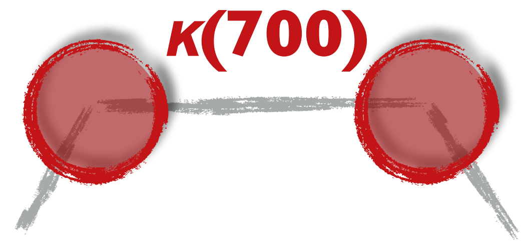


$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

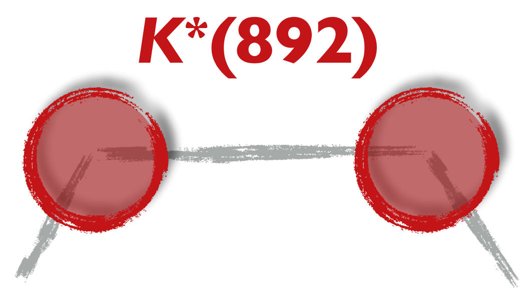
• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$



$$I(J^P) = 1/2(1^-)$$

- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002*

$\kappa, K^* \to K\pi$

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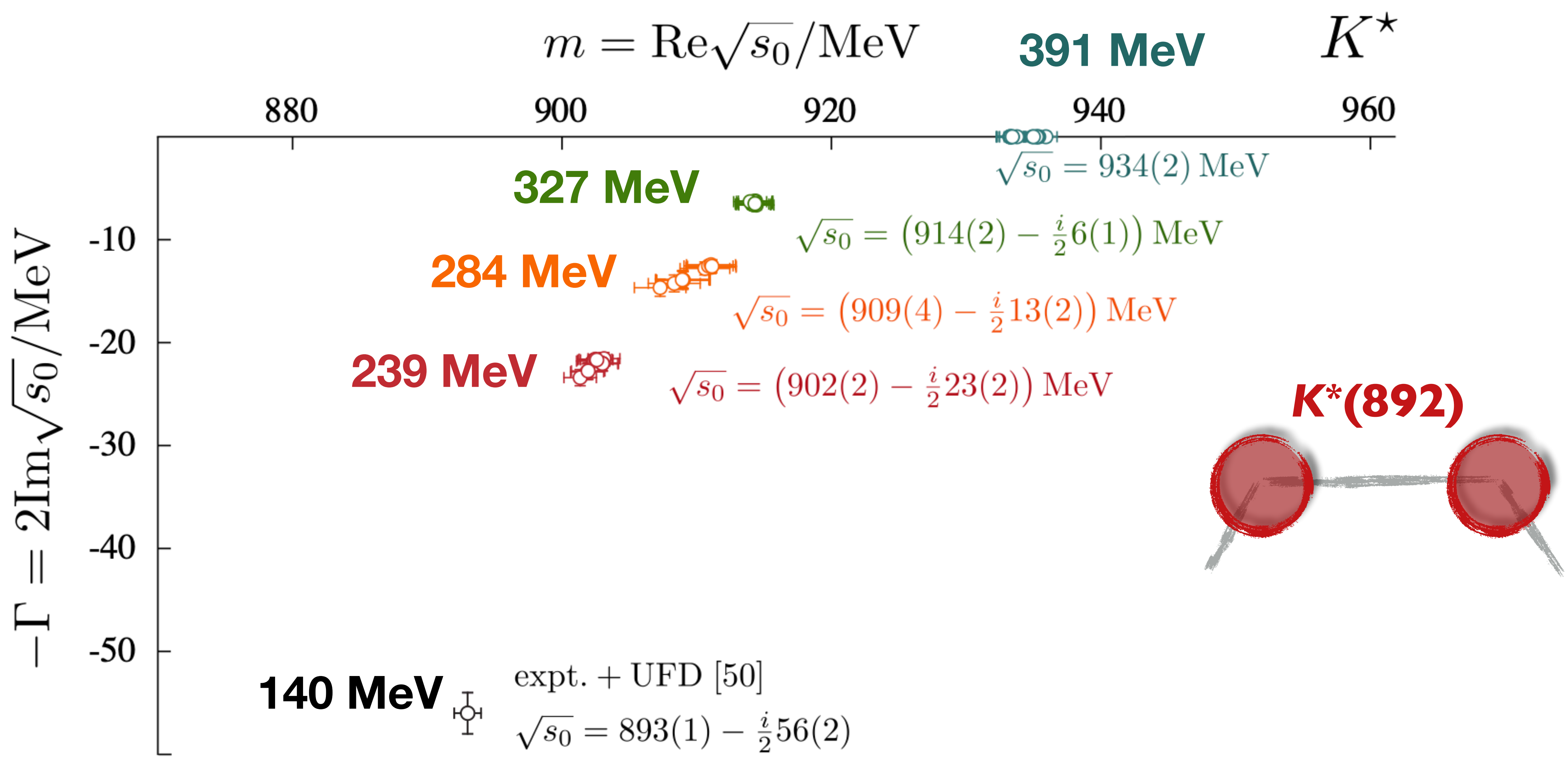
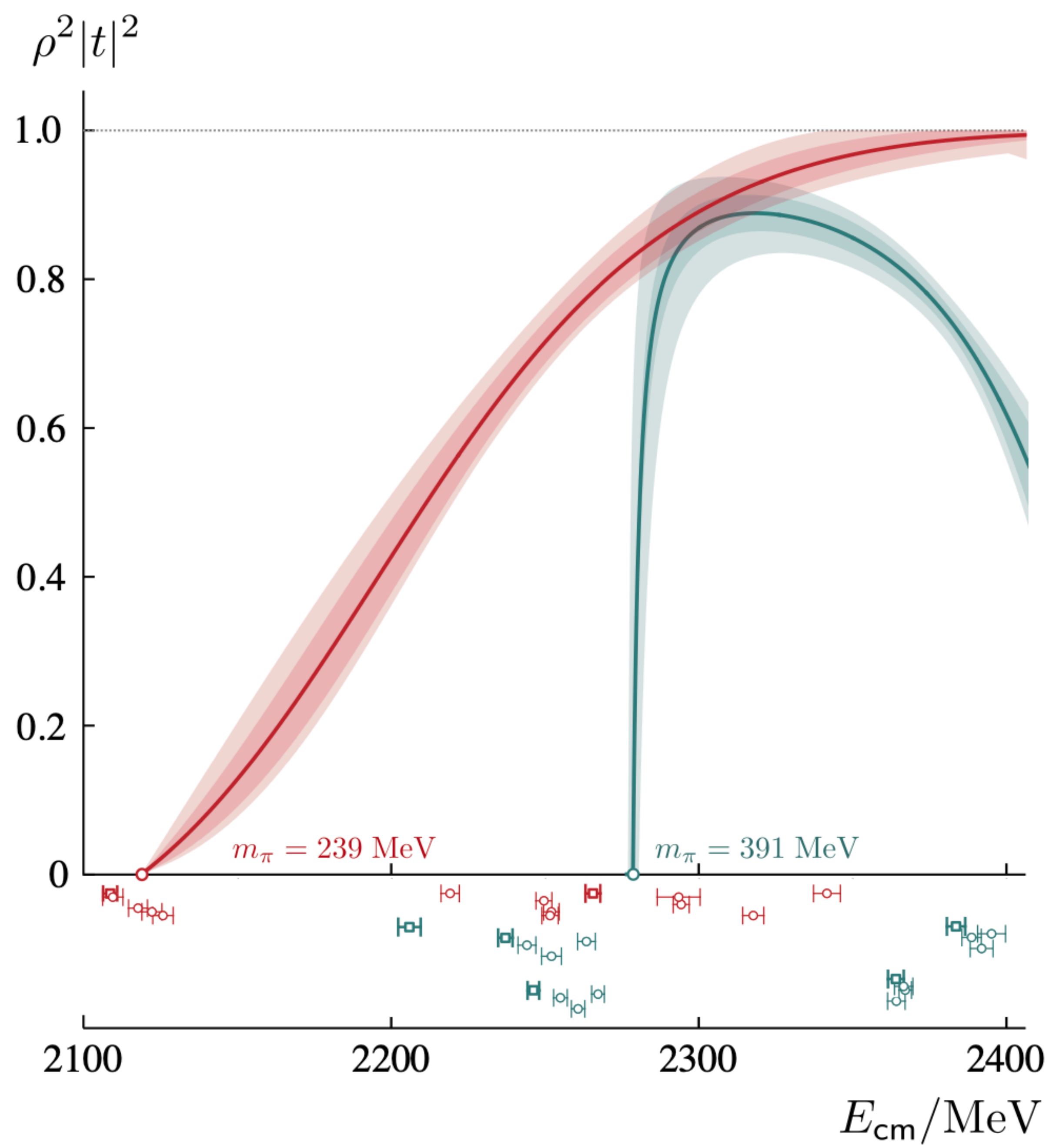
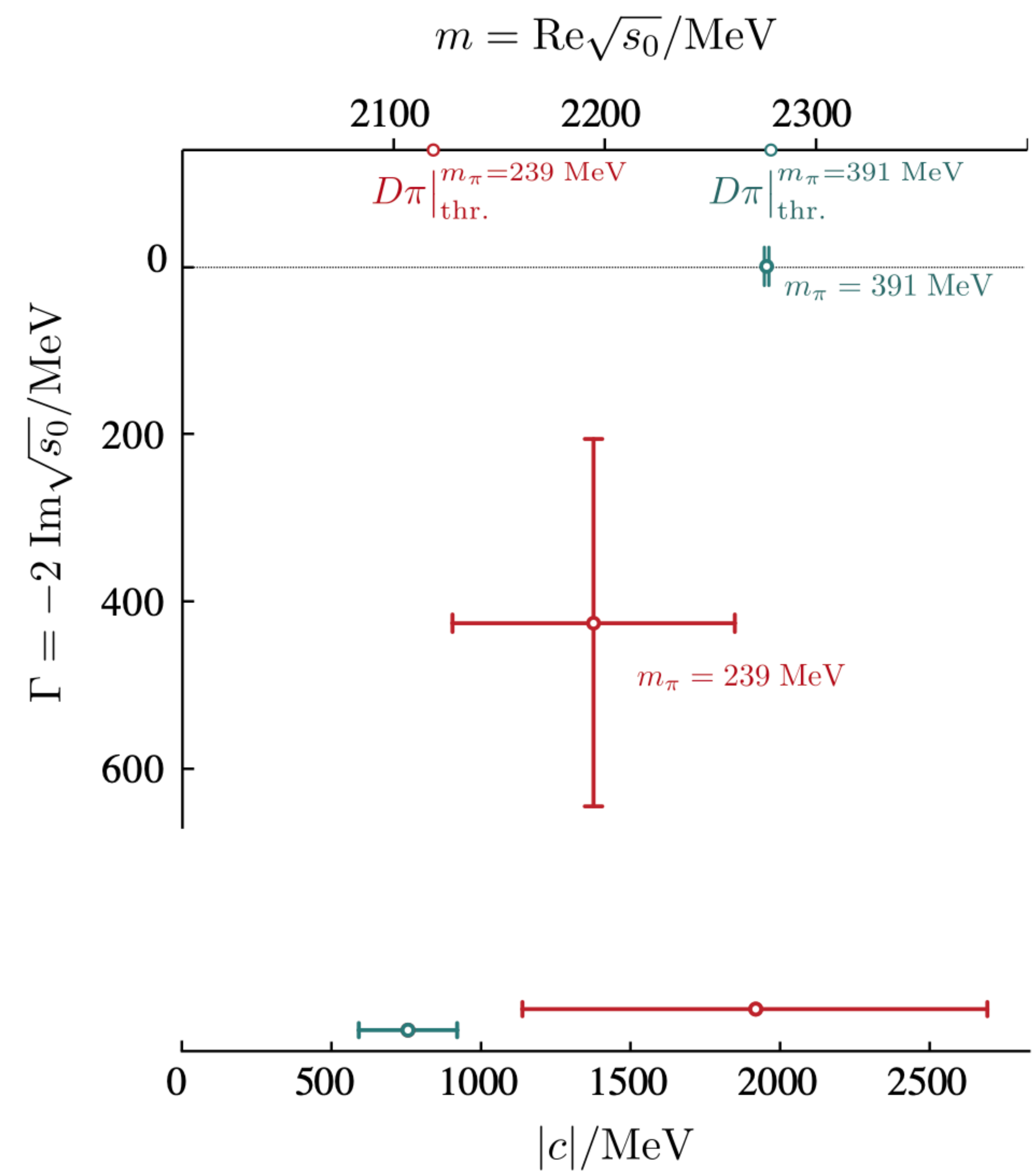


- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002*

$D\pi \to D\pi$, $I = 1/2$



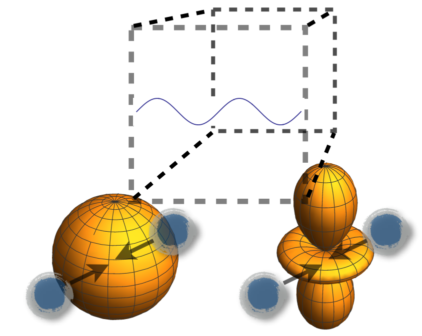


— *Isospin-1/2 Dπ scattering and the lightest D0* resonance from lattice QCD* —
Hadron Spectrum Collaboration — (2021) JHEP 07 (2021) 123

Coupled channels

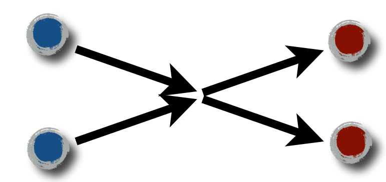
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{K}_{a\to a} & \mathcal{K}_{a\to b} \\ \mathcal{K}_{b\to a} & \mathcal{K}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$



- Workflow…

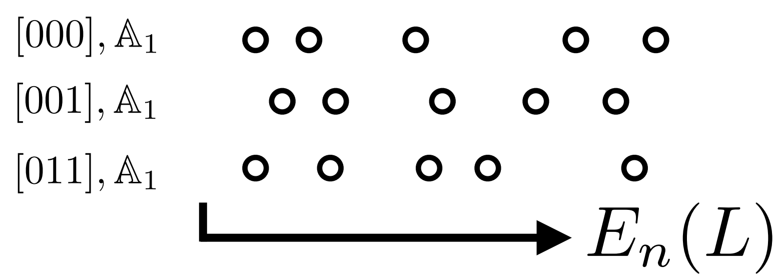
Correlators with a large operator basis
$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→

Reliably extract finite-volume energies
$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→

Vary L and P to recover a dense set of energies



↓

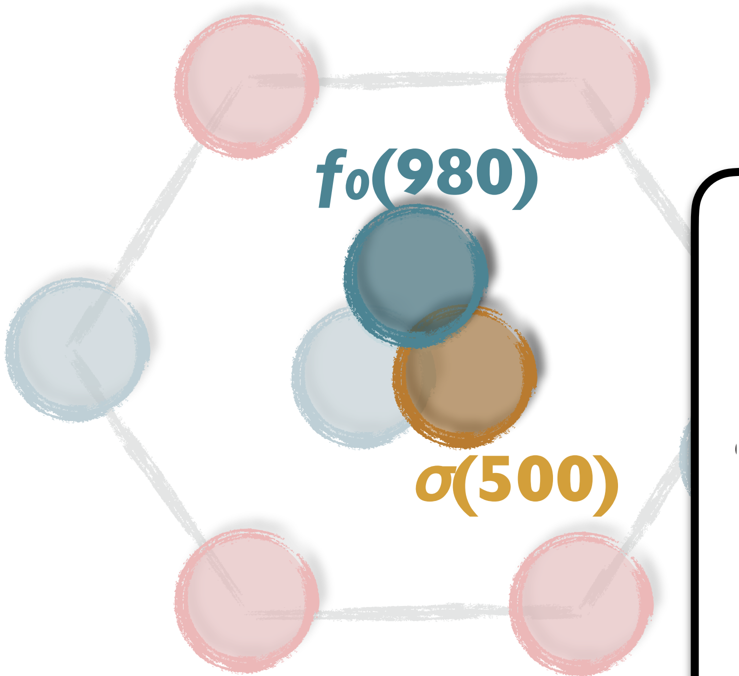
hadspec

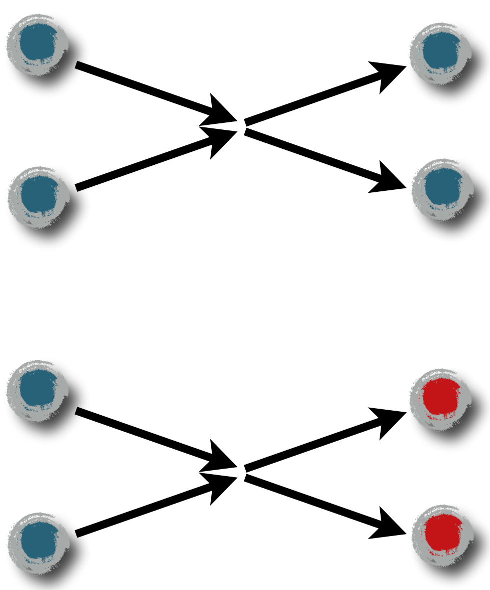
Identify a broad list of K-matrix parametrizations
polynomials and poles | EFT based | dispersion theory based

→

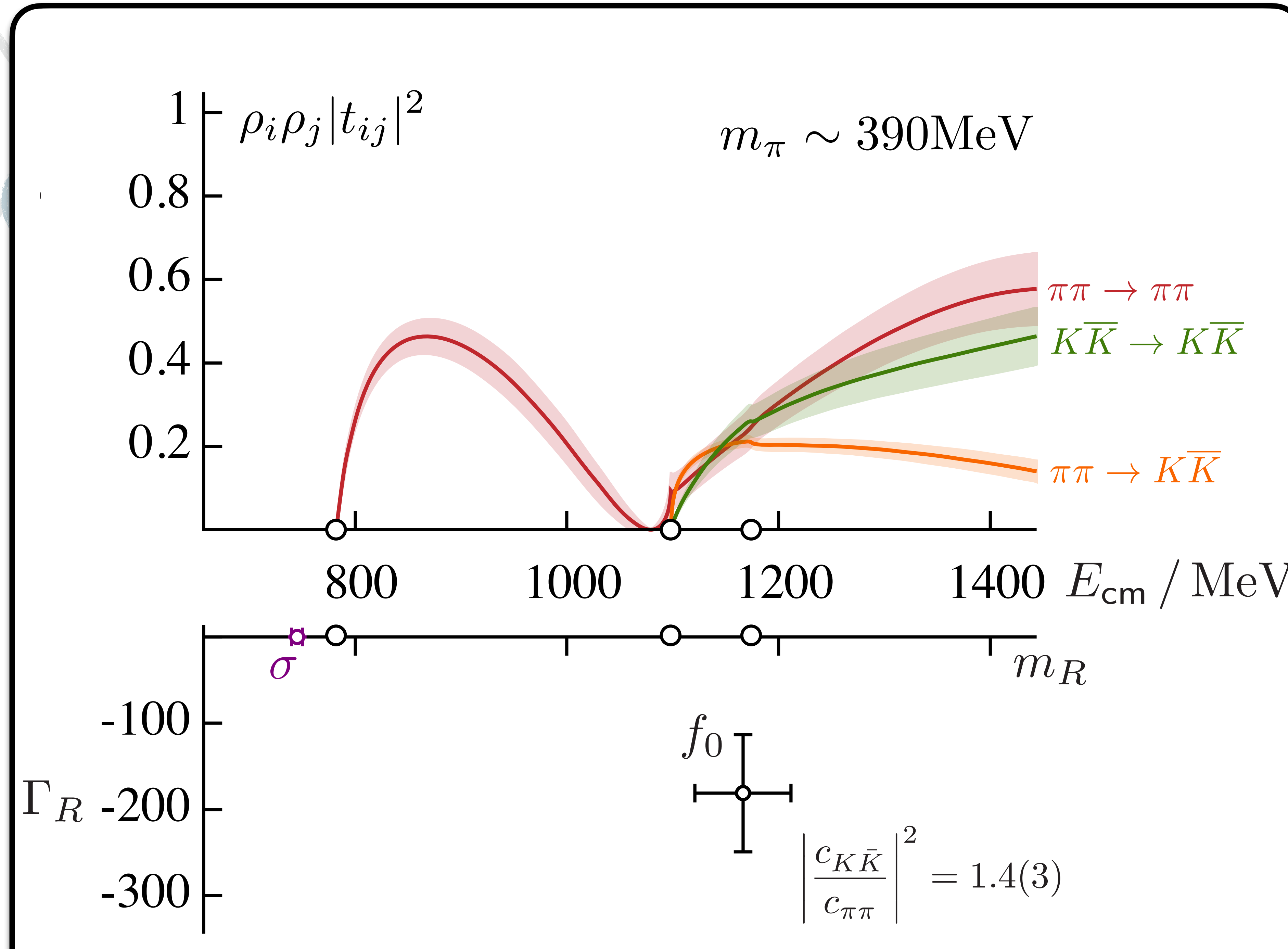
Perform global fits to the finite-volume spectrum

$$I^G(J^{PC}) = 0^+(0^{++})$$





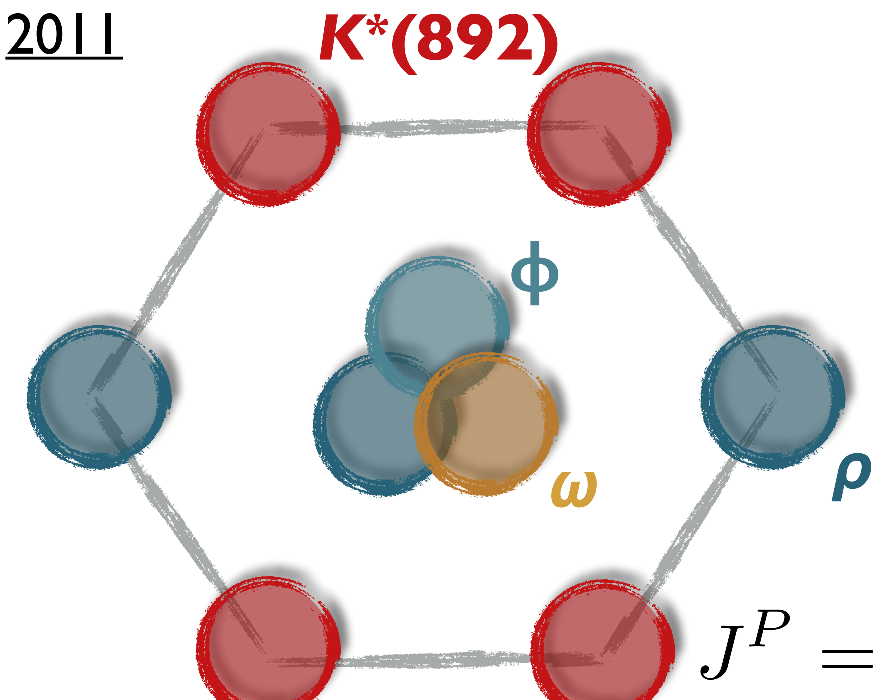
Coupled-channel scattering



Briceño, Dudek, Edwards & Wilson (2017)

$\rho \to \pi\pi$

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- HadSpec 2012, 2016
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017
- Andersen et al. 2018
- Fischer et al. 2020
- Erben et al. 2020



$J^P = 1^-$

$\kappa \to K\pi$

$K^* \to K\pi$

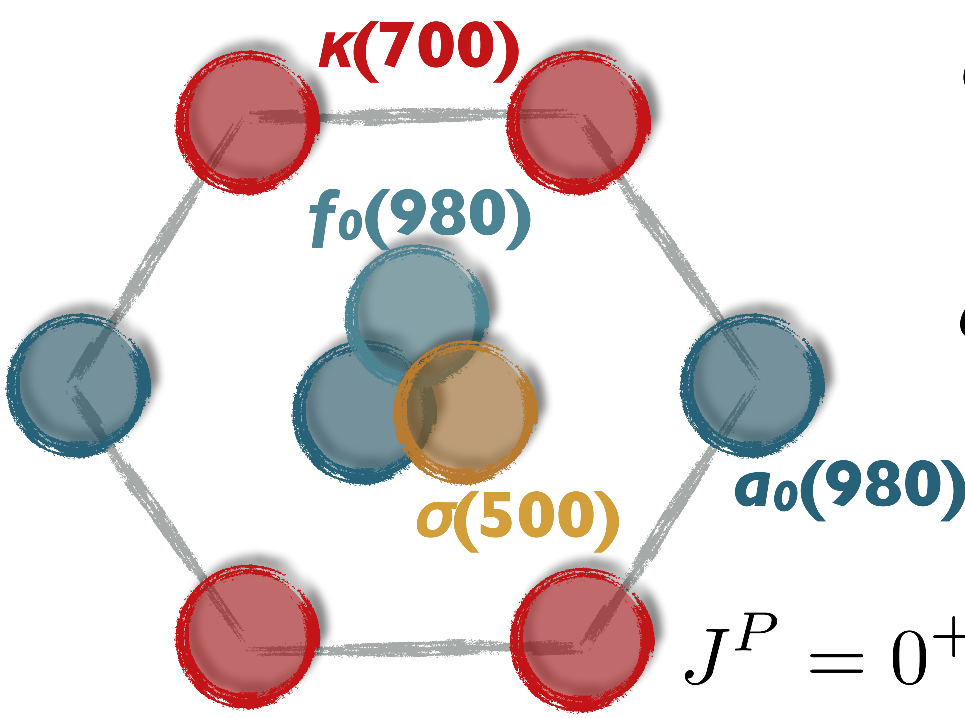
- Lang et al. 2012
- Prelovsek et al. 2013
- Wilson et al. 2015
- RQCD 2015
- Brett et al. 2018
- Wilson et al. 2019
- Rendon et al. 2020

$b_1 \to \pi\omega, \pi\phi$

- Woss et al. 2019

$\sigma \to \pi\pi$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017
- Guo et al. 2018



$J^P = 0^+$

$a_0(980) \to \pi\eta, K\overline{K}$

- Dudek et al. 2016

$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$

- Briceño et al. 2017

See the recent review by Briceño, Dudek and Young

$$\Delta \to N\pi$$

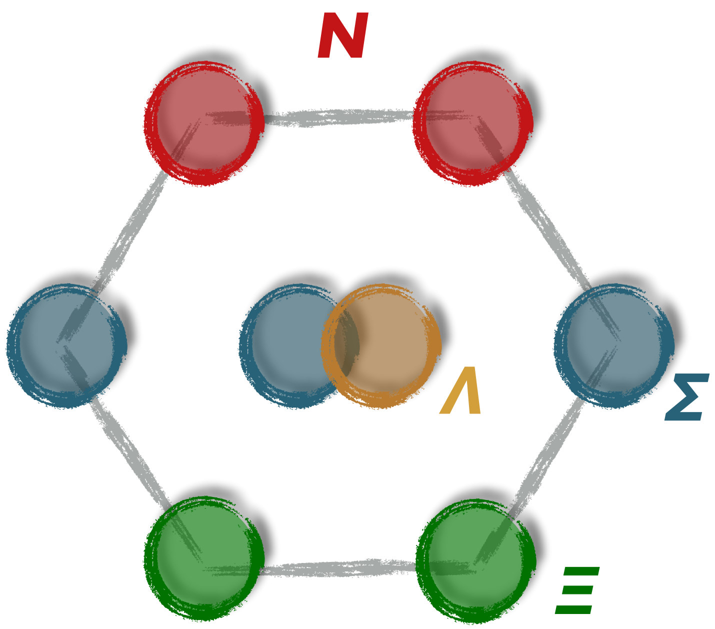
- ❑ Andersen et al. 2018
- ❑ Andersen et al. 2019
- ❑ Silvi et al. 2021
- ❑ Pittler et al. 2021
- ❑ Bulava et al. 2022

$$N^* \to N\pi$$

- ❑ Lang et al. 2017
- ❑ Wu et al. 2017
- ❑ Kiratidis et al. 2017

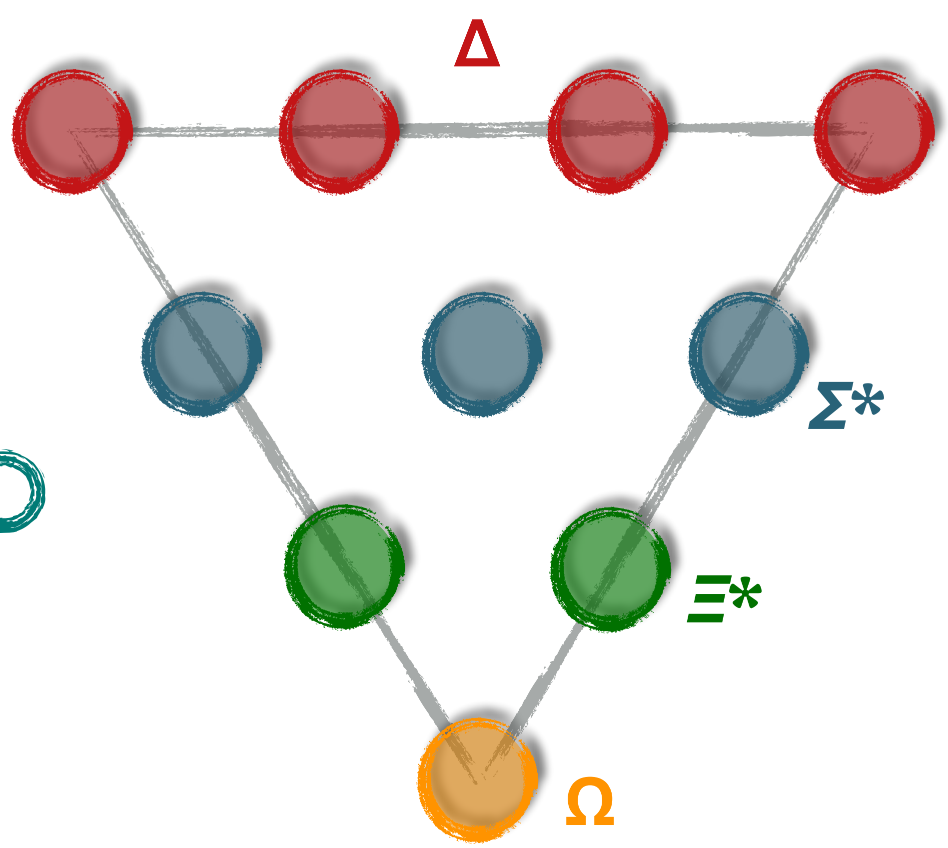
$$\Lambda \to \overline{K}N$$

- ❑ Hall et al. 2015
- ❑ Bulava et al. 2023



(focusing here on studies with scattering states)

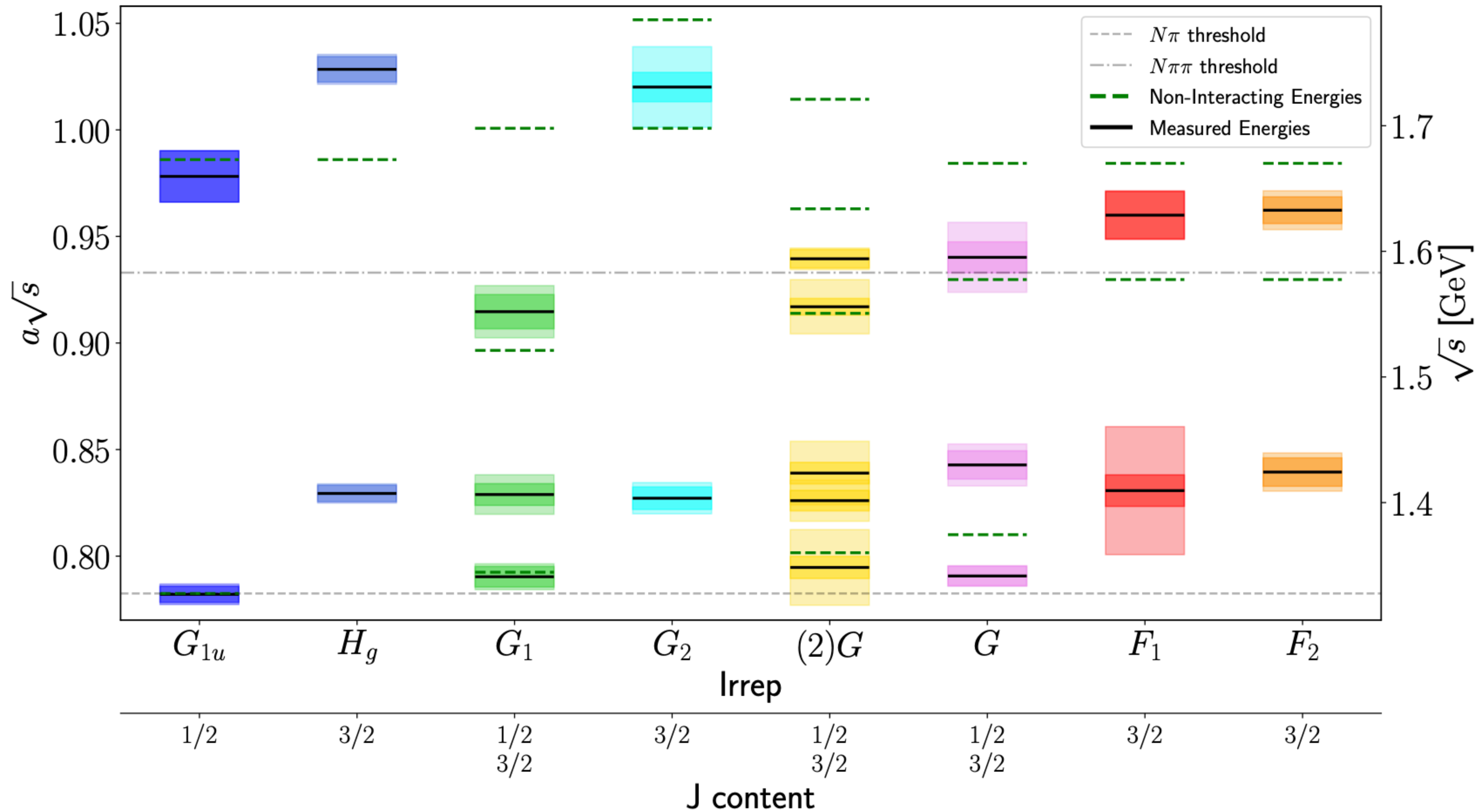
Baryons are difficult!

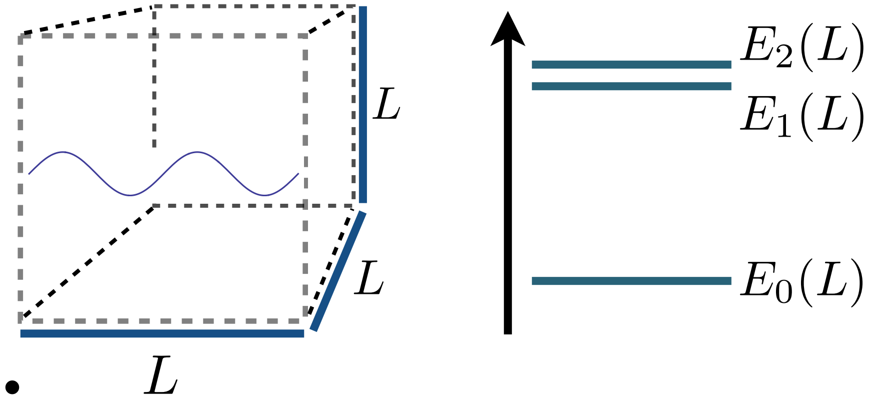


See also…

- ❑ Detmold and Nicholson 2015
- ❑ Wu et al. 2018
- ❑ Xing & Liu, LATT2022 (in prep)

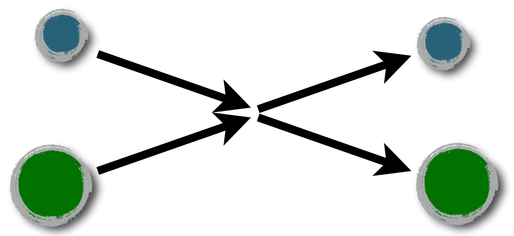
$N\pi$ finite-volume energies $\quad (M_\pi = 255$ MeV$)$

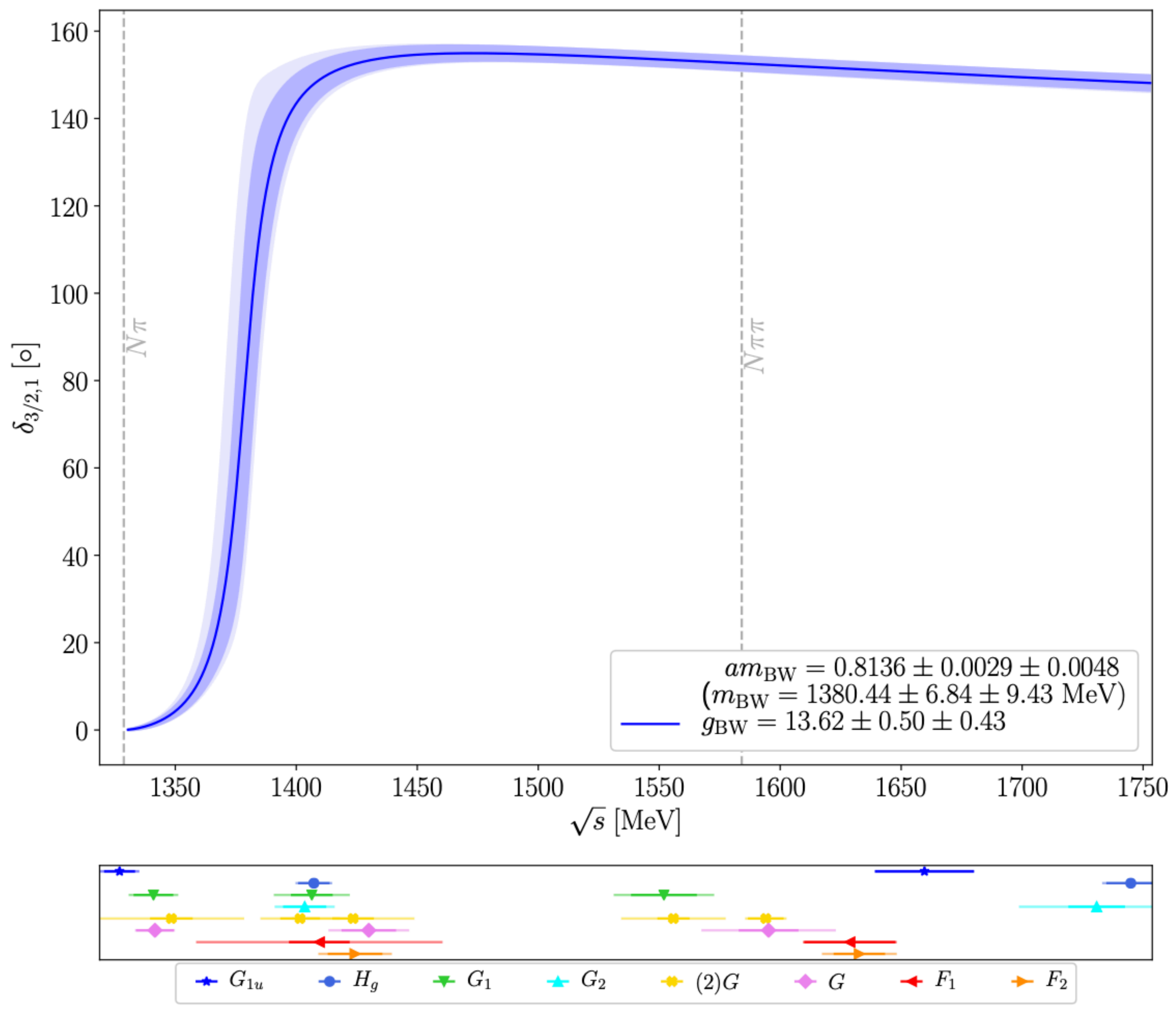




- Silvi et al., PRD 2021, 2101.00689
-

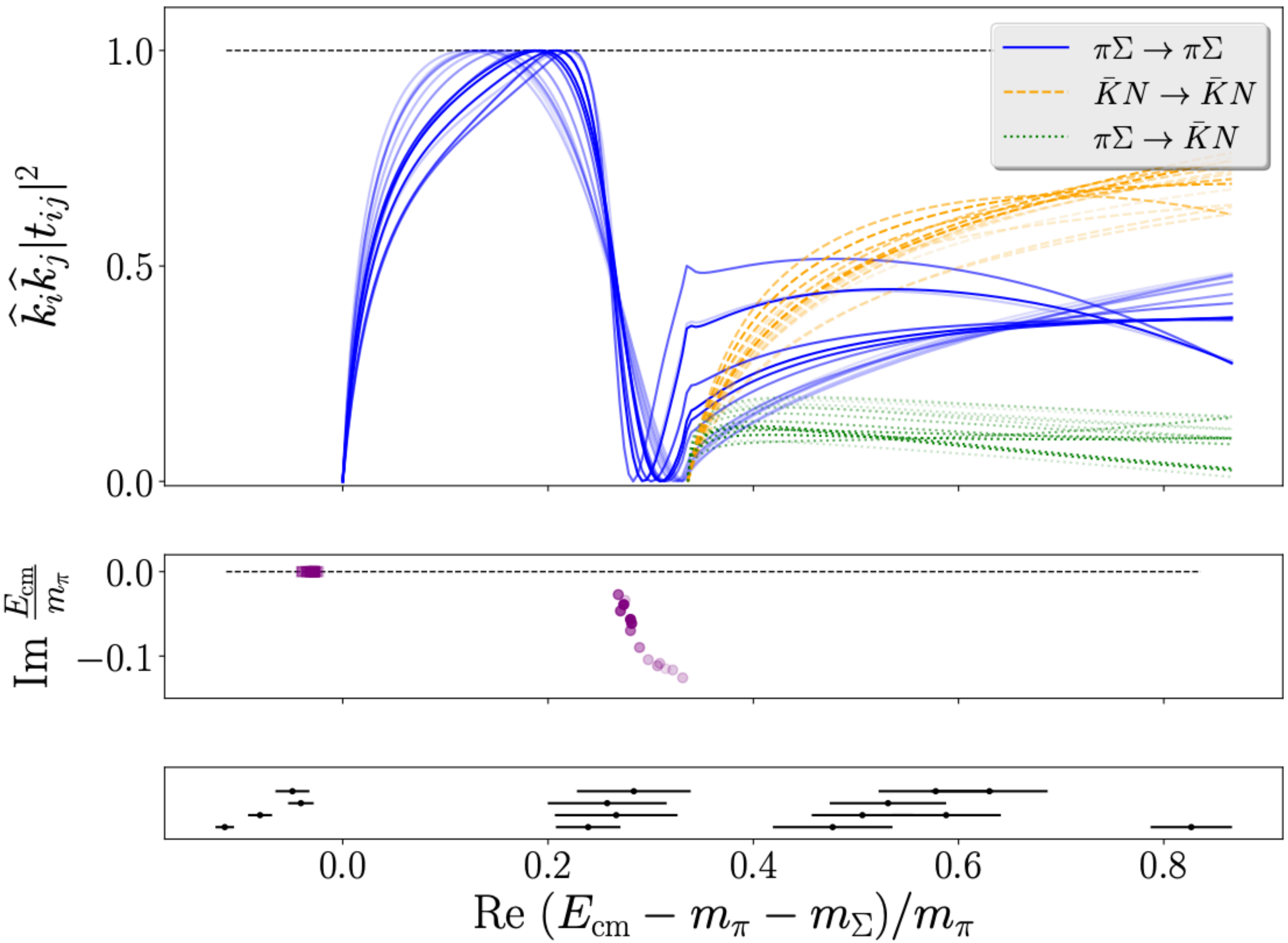
$N\pi \rightarrow \Delta \rightarrow N\pi \quad (M_\pi = 255 \text{ MeV})$





- Silvi et al., PRD 2021, 2101.00689 •

$\Lambda(1405) \to N\overline{K}, \Sigma\pi \quad (M_\pi = 200\ \text{MeV})$



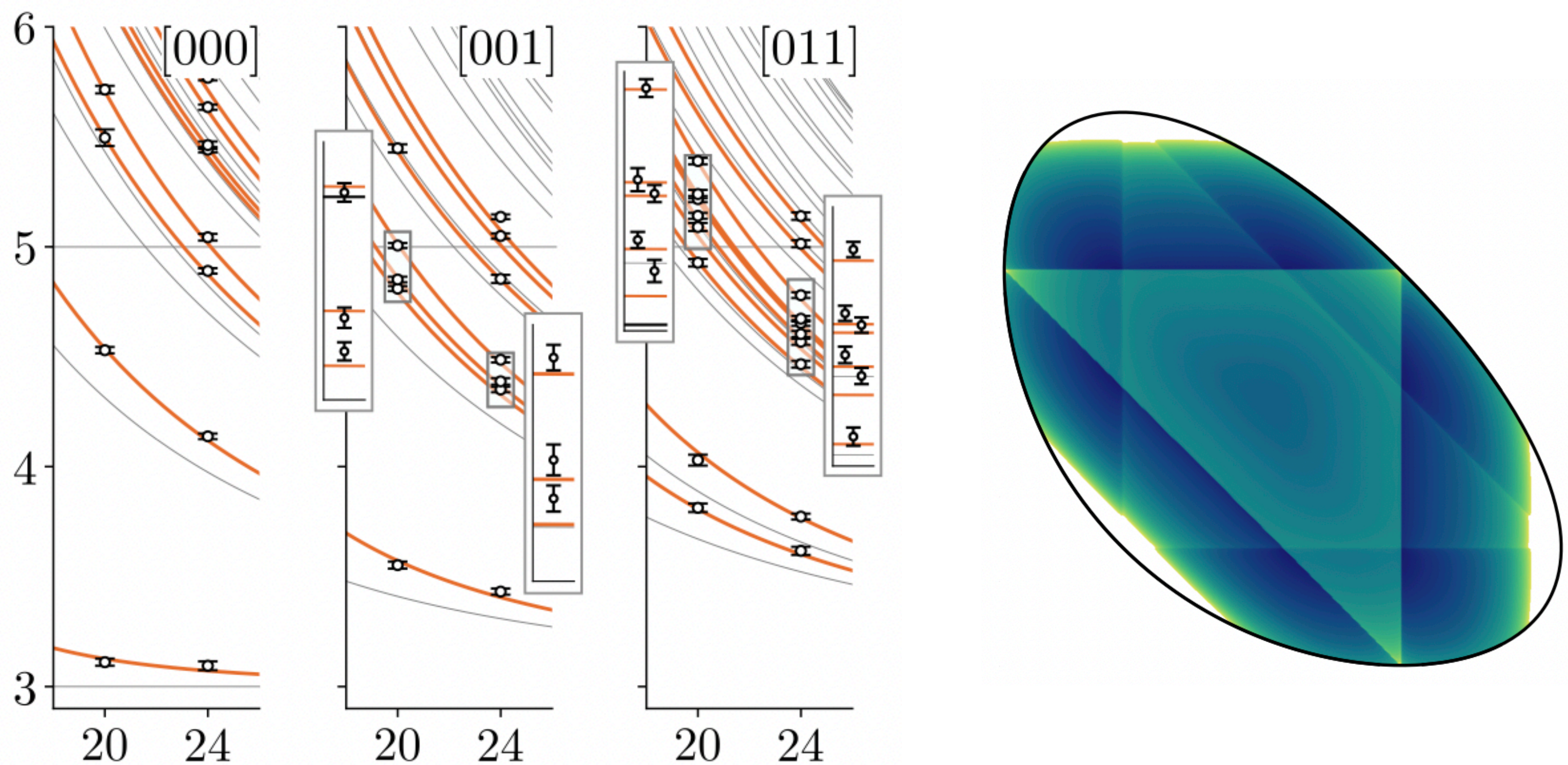
• Bulava et al., *arXiv:2307.13471*, 2023 •

Towards >2 hadrons

- Multiple three-particle finite-volume formalisms developed (so far only spin zero)

  MTH, Sharpe (2014-2016) *See also Döring, Mai, Hammer, Pang, Rusetsky*

- First lattice calculations appearing… e.g. $\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$



  - Extract reliable spectrum
  - Use formalism to fit scheme-dependent K-matrix
  - Solve integral equations to reach physical amplitude

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

- See Blanton et al. 2022 and 2023 for pion and kaon results
- See Sadasivan et al. 2022 for application to $a_1(1260)$

Two strategies…

- ☑ Finite-volume as a tool
  - LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
  - Applicable only in limited energy range for two- and three-hadron states

- ☐ Spectral function method
  - Formally applies for any number of particles / any energy range
  - An answer to the question… "Can't you just analytically continue?"
  - Still important challenges and limitations to consider

Two strategies…

☑ Finite-volume as a tool

- LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
- Applicable only in limited energy range for two- and three-hadron states

☐ Spectral function method

- Formally applies for any number of particles / any energy range
- An answer to the question… "Can't you just analytically continue?"
- Still important challenges and limitations to consider

Correlation functions → observables

- Lattice QCD gives finite-volume Euclidean correlators

$$\langle 0|\, \mathcal{O}_1(0)\, e^{-\hat{H}\tau}\, \mathcal{O}_2(0)\, |0\rangle_L$$

have

- Complete physical information is contained in…

$$\langle 0|\, \mathcal{O}_1(0)\, f(\hat{H})\, \mathcal{O}_2(0)\, |0\rangle_\infty$$

want

- Detailed choice of *f(E)* and operators determines the observable

R-ratio

$$\langle 0|\, j_\mu(0)\, \delta(\hat{H} - \omega)\, j_\mu(0)\, |0\rangle_\infty$$

Meyer • Bailas, Hashimoto, Ishikawa (2020)
Alexandrou et al. (2022)

D-meson total lifetime

$$\langle D|\, \mathcal{H}_W(0)\, \delta(M_D - \hat{H})\, \mathcal{H}_W(0)\, |D\rangle_\infty$$

MTH, Meyer, Robaina (2017)

$\pi\pi \to \pi\pi$ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, \pi(0)\, |\pi\rangle_\infty$$

$j \to \pi\pi$ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, j_\mu(0)\, |0\rangle_\infty$$

Bulava, MTH (2019)

Linear reconstruction

$$\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle = \int d\omega\, e^{-\omega\tau} \, \langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle$$

have — want

$$G(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho(\omega)$$

have — want

- ***Linear, model-independent reconstruction*** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar\omega, \tau) G(\tau) = \sum_\tau \mathcal{K}(\bar\omega, \tau) \int d\omega\, e^{-\omega\tau}\, \rho(\omega) = \int d\omega \left[\sum_\tau \mathcal{K}(\bar\omega, \tau)\, e^{-\omega\tau} \right] \rho(\omega)$$

$$= \int d\omega\, \widehat{\delta}_\Delta(\bar\omega, \omega)\, \rho(\omega)$$

δ is exactly known

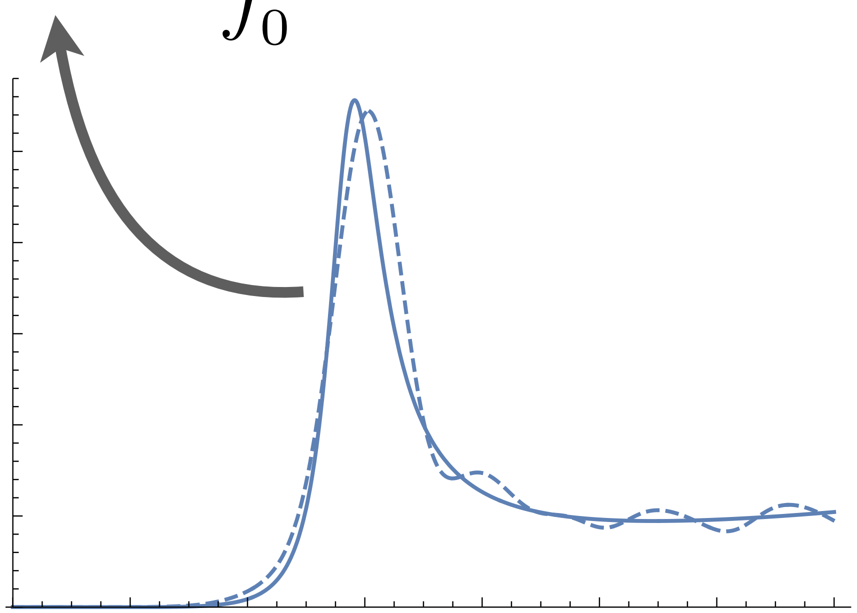
- Non-linear (***not discussed here…***)
  - Maximum Entropy Method (MEM)
  - Direct fits
  - Neural networks

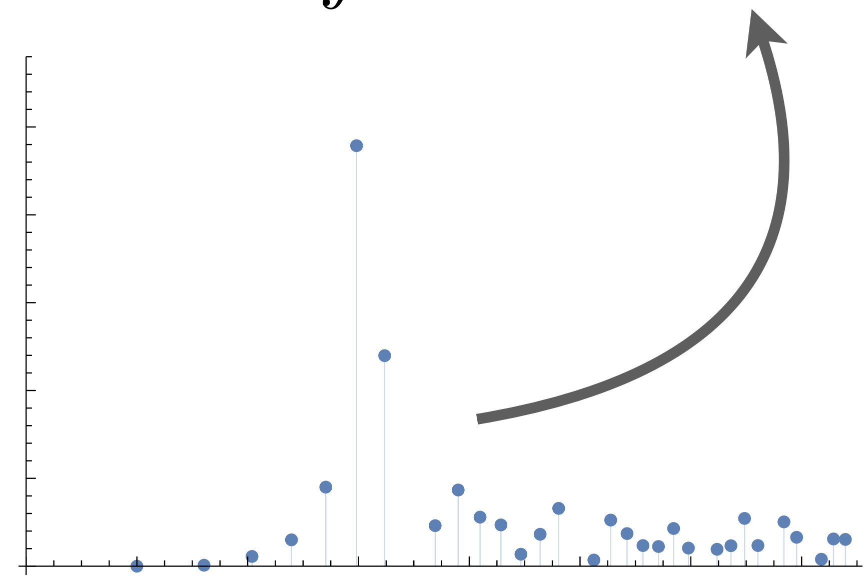
See multiple ECT and CERN workshops, work by Aarts, Allton, Amato, Brandt, Burnier, Del Debbio, Francis, Giudice, Hands, Harris, Hashimoto, Jäger, Karpie, Liu, Meyer, Monahan, Orginos, Robaina, Rothkopf, Ryan, …*

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^{\infty} d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$$

$$G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- ☐ Any reconstructed spectral function that ≠ forest of deltas…

contains implicit smearing (or else $L \to \infty$)

We require…

$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

MTH, Meyer, Robaina (2017)

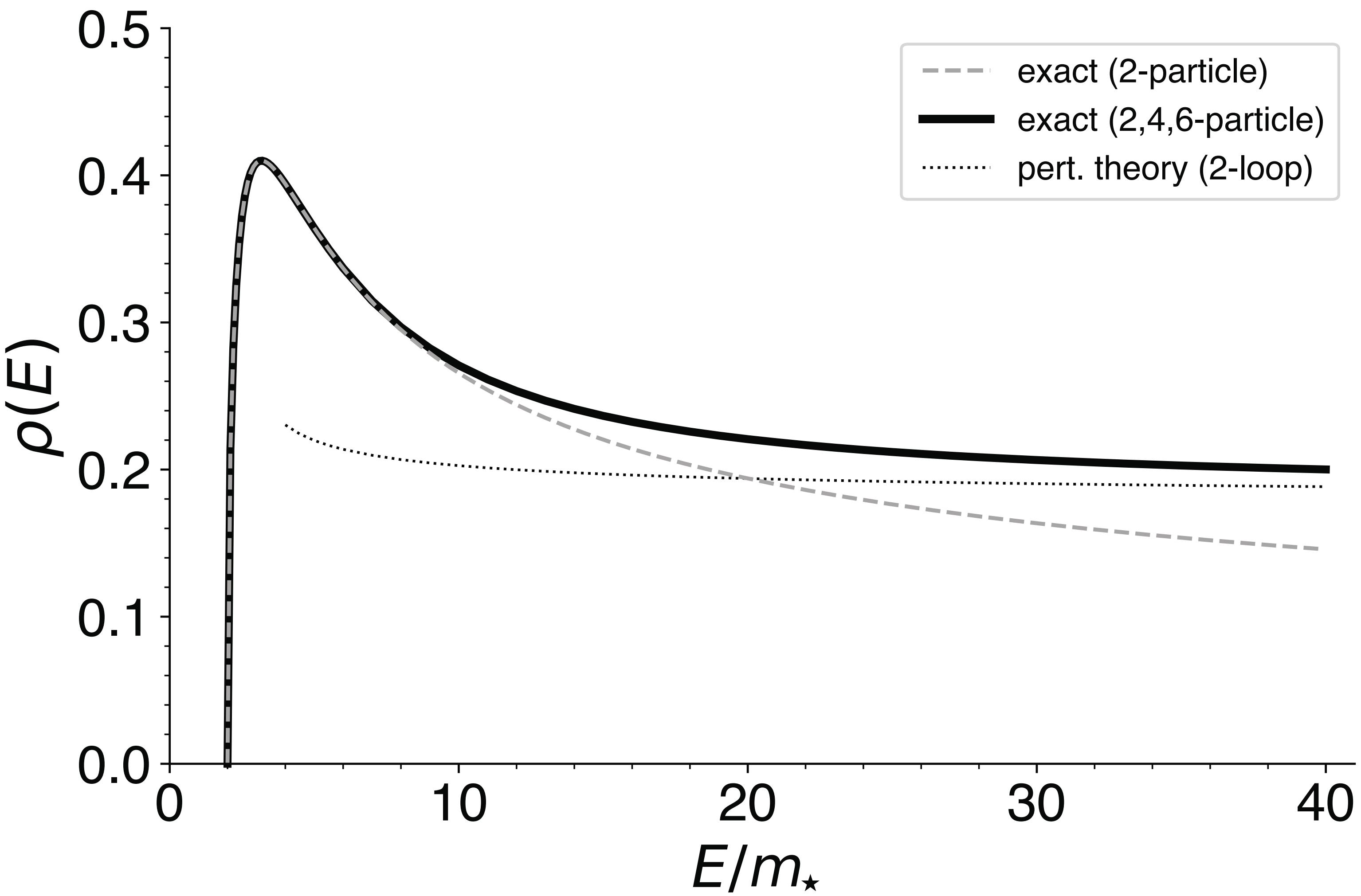
1+1 O(3) Model

$$S[\sigma] = \frac{1}{2g^2} \int d^2x\, \partial_\mu \sigma(x) \cdot \partial_\mu \sigma(x)$$

- *Integrable* theory with some nice similarities to QCD
  - Asymptotically free
  - Dynamically generated mass gap
  - Iso-spin like symmetry
  - Conserved iso-vector vector current

$$j^c_\mu(x) = \frac{1}{g^2} \epsilon^{abc} \sigma^a(x) \partial_\mu \sigma^b(x)$$

conserved current



$$\rho(E) = 2\pi \, \langle \Omega | \hat{j}^a_1(0) \, \delta^2(\hat{P} - p) \, \hat{j}^a_1(0) | \Omega \rangle$$

spectral function

$$\rho^{(2)}(E) = \frac{3\pi^3}{8\theta^2} \, \frac{\theta^2 + \pi^2}{\theta^2 + 4\pi^2} \, \tanh^3 \frac{\theta}{2}$$

$$\theta = 2 \cosh^{-1} \frac{E}{2m}$$

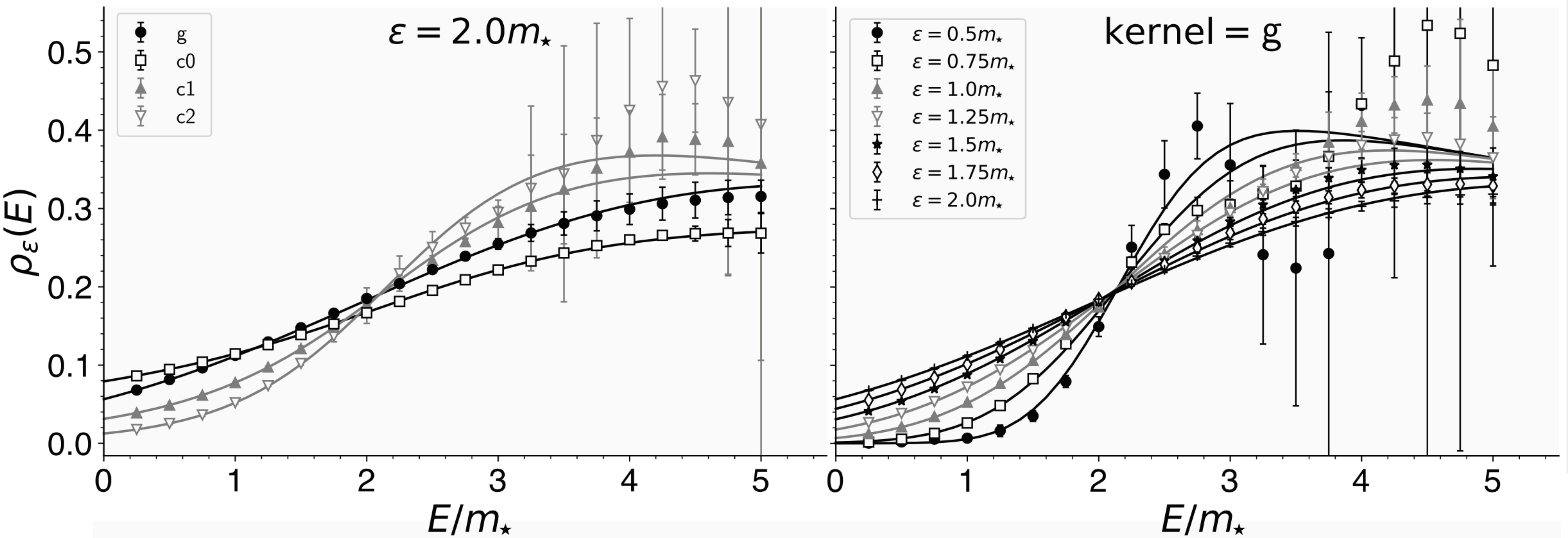
Smeared spectral function vs analytic result

- Construct different smearings of $\rho(\omega)$

$$\rho_\epsilon^\lambda(E) = \int_0^\infty d\omega\, \delta_\epsilon^\lambda(E,\omega)\, \rho(\omega)$$

$$\delta_\epsilon^{\rm g}(x) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\rm c0}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

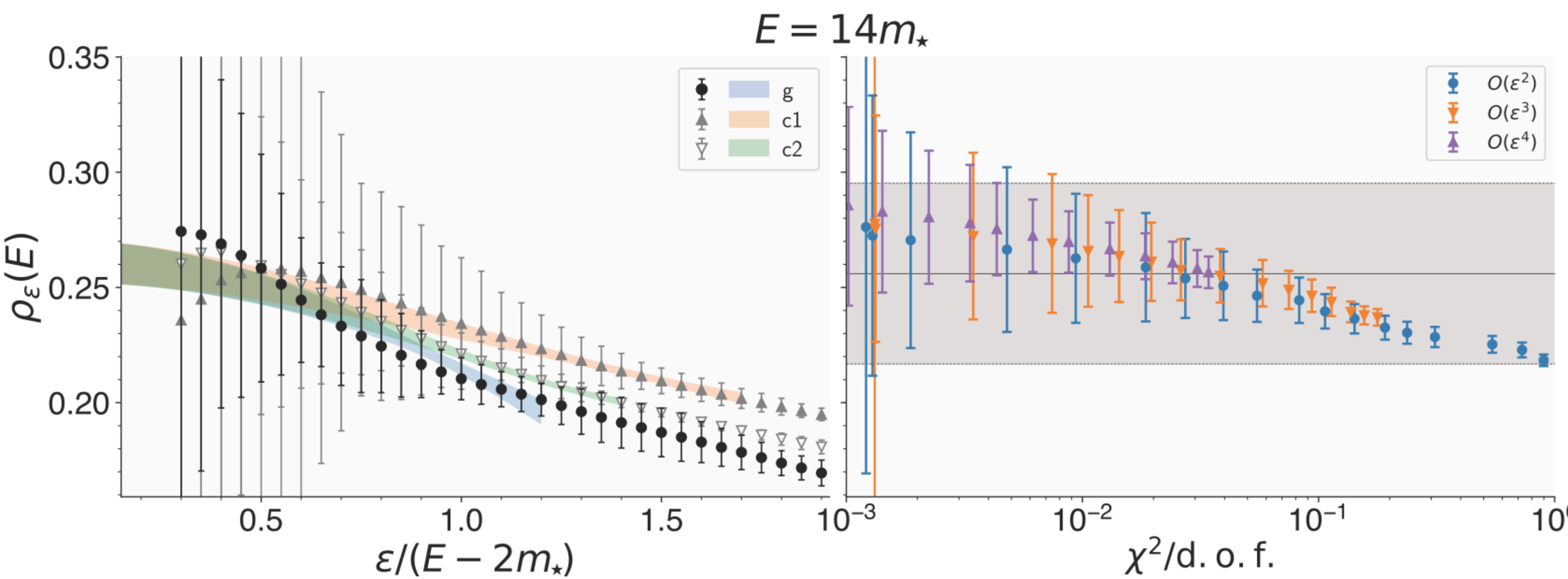
$$\delta_\epsilon^{\rm c1}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\rm c2}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3}.$$



Bulava, MTH, Hansen, Patella, Tantalo (2021)

Extrapolation

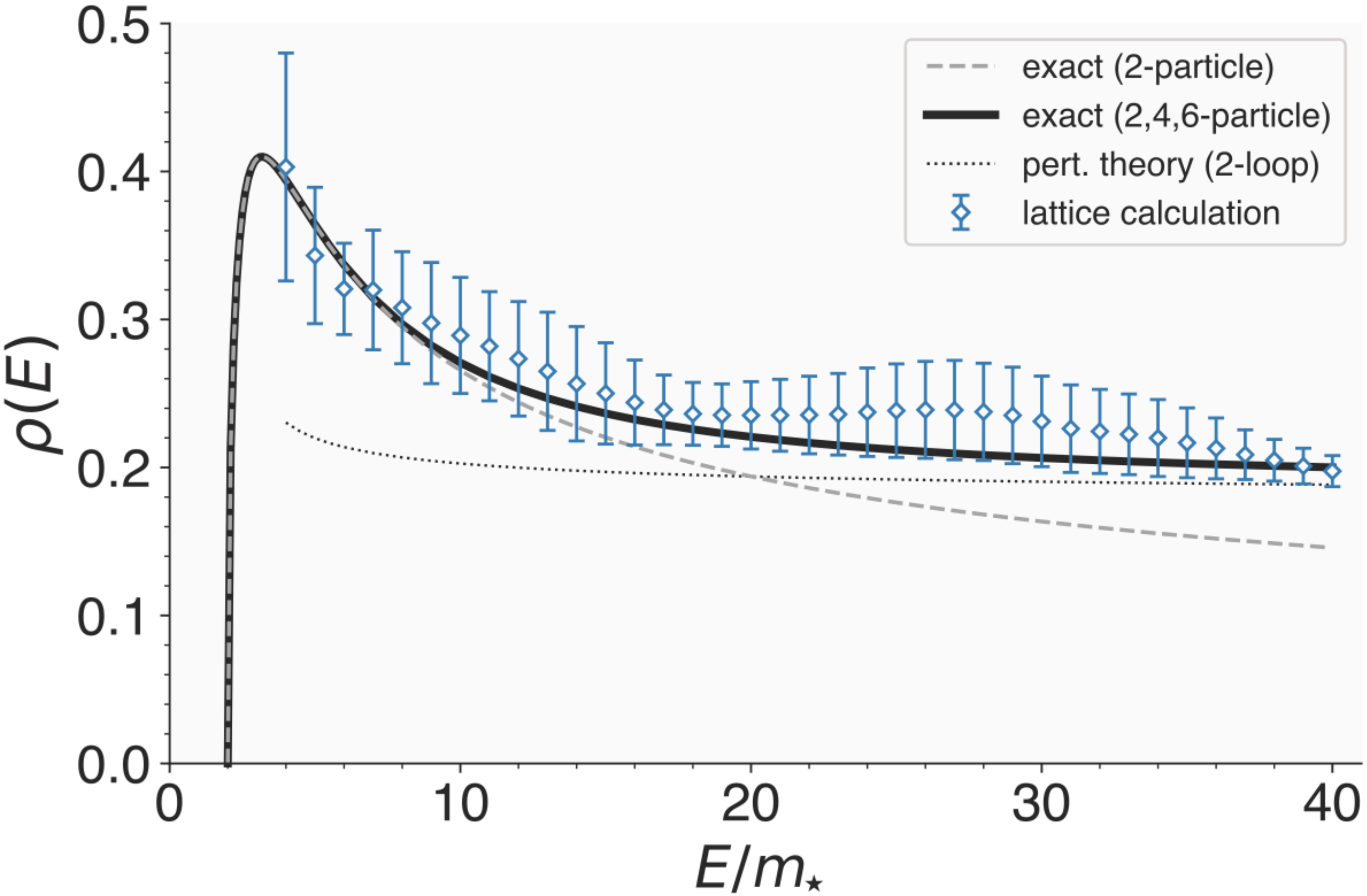
- Targeting $\rho(E)$ for $E = 14m$ here



- Use known relations between different smearing kernels

Bulava, MTH, Hansen, Patella, Tantalo (2021)

Result



Bulava, MTH, Hansen, Patella, Tantalo (2021)

Many QCD applications already published… see work by A. Barone, S. Hashimoto, A. Jüttner, T. Kaneko, R. Kellermann, R. Frezzotti, G. Gagliardi, V. Lubicz, F. Sanfilippo, S. Simula …

Not discussed here

- Transition and decay amplitudes ($K \to \pi\pi, \quad p\gamma^\star \to p\pi$)

  finite-volume methods exist, but restricted to two- and three-hadron domain

- Left-hand branch cuts

  finite-volume methods break on left-hand cuts (e.g. T_{cc}^{+}) → ***extensions to appear***

  See, e.g. Raposo and Hansen 2023

- Using effective field theory with lattice scattering calculations

  See, e.g. Mai et al., Towards a theory of hadron resonances, 2022

- Hadronic structure calculations (*form factors, distribution functions*)

- Many other exciting channels

  See also talks from Sinead Ryan and Sasa Prelovsek

Conclusions

- LQCD is in the era of ‘rigorous resonance spectroscopy’
- The finite-volume = ***a useful tool***
- Challenges and progress

  formal analysis was technical → *ground work is now set*

  scattering demands high precision excited states → *advanced algorithms make this possible*

  many calculations at unphysical quark masses → *physical-mass scattering now appearing*

  → *varying masses probes resonance structure*

- Next steps…

  complete 3-particle formalism → *extend to N-particle formalism*

  extend studies involving an external current

  push more channels into the precision regime

- ‘Spectral methods’ offer an alternative, *need further exploration*

Thanks!

Funded by UKRI Future Leaders Fellowship

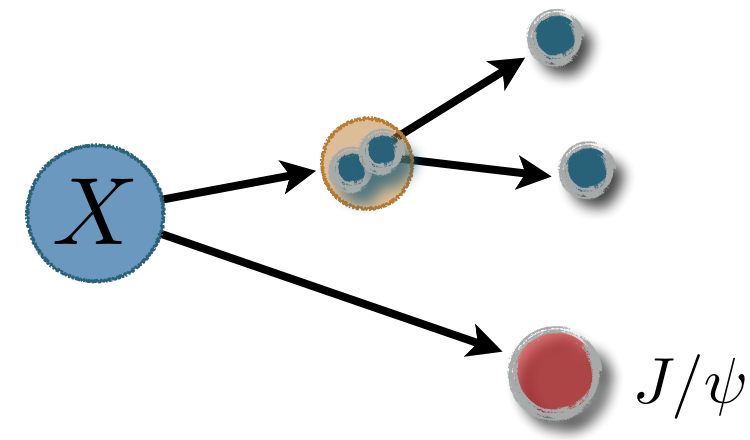
UK Research and Innovation

Back-up slides

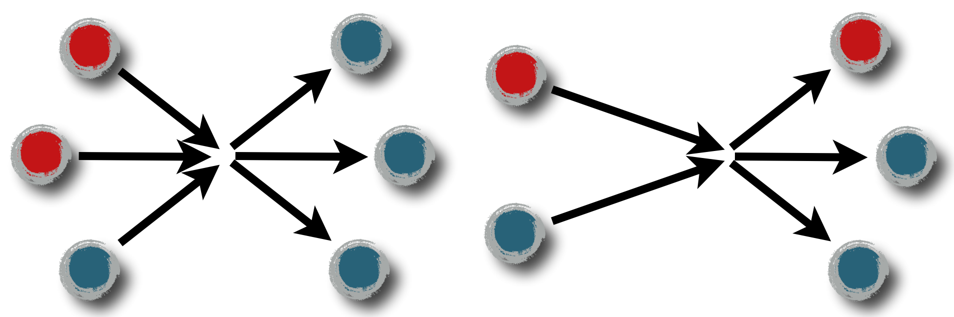
3-particle amplitudes

2-to-2 only samples $J^P \quad 0^+ \quad 1^- \quad 2^+ \cdots$

many interesting resonances have significant 3-body decays



Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

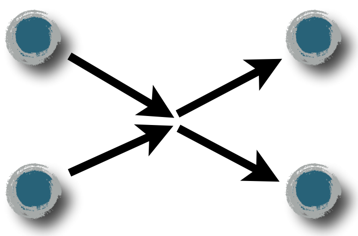
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\ a_1(1420) \to \pi\pi\pi \qquad X(3872) \to J/\psi\pi\pi \qquad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom

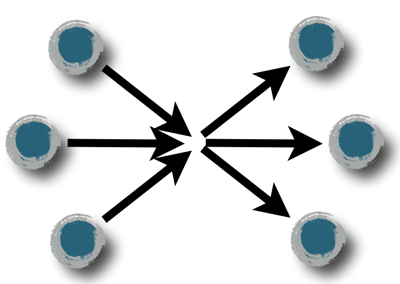


12 momentum
components

-10 Poincaré generators

2 degrees of freedom

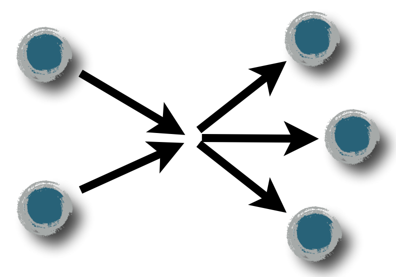
$$\vec{p}_1 + \vec{p}_2 \;\rightarrow\; \vec{p}_3 + \vec{p}_4 \quad \longrightarrow \quad \text{Mandelstam } s, t$$



18 momentum
components

-10 Poincaré generators

8 degrees of freedom



15 momentum
components

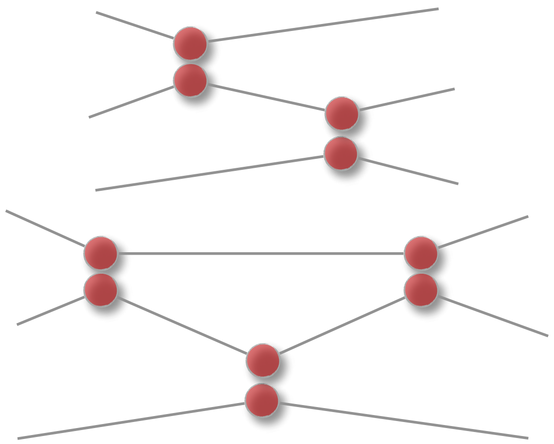
-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin
AND
ROBERT SUGAR
Physics Department, Columbia University, New York, New York
AND
GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received 31 January 1966)

$$b=\frac{(m_1+m_3)(m_2+m_3)}{m_1 m_2}$$

It follows that if

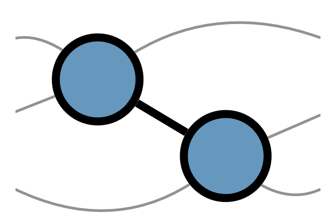
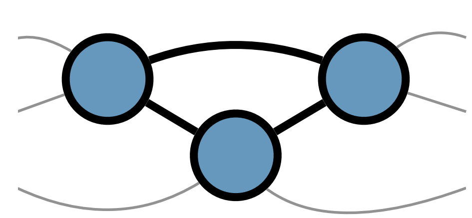
$$b^{n-2}(b-1)>1\,, \qquad \text{(IV.18)}$$

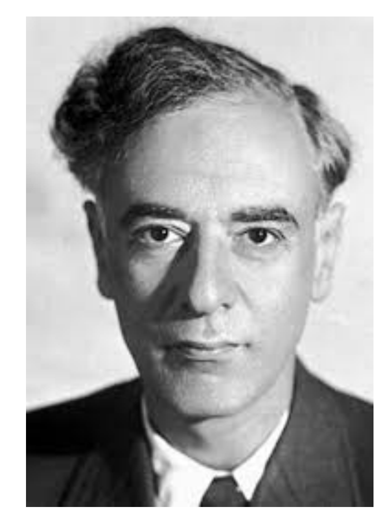
then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

$b < 2$
5 collisions possible
$\pi K K$

- Correspond to Landau singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator $=$  $+$  $+ \cdots$

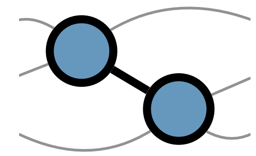
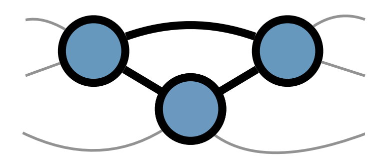


complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles

Two key observations

- Intermediate $K_{\text{df},3}$ removes singularities

$\mathcal{K}_{\text{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+\cdots$

same degrees of freedom as *M₃*　　smooth real function　　relation to *M₃* = known

- $K_{\text{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\text{df},3}(p_3,p_2,p_1;k_3,k_2,k_1) = \mathcal{K}_{\text{df},3}^{\text{iso},0} + \mathcal{K}_{\text{df},3}^{\text{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

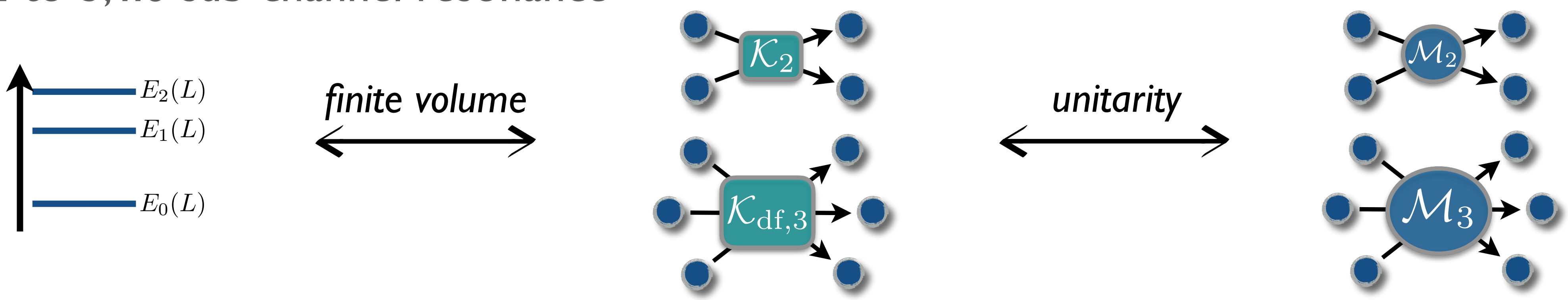
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

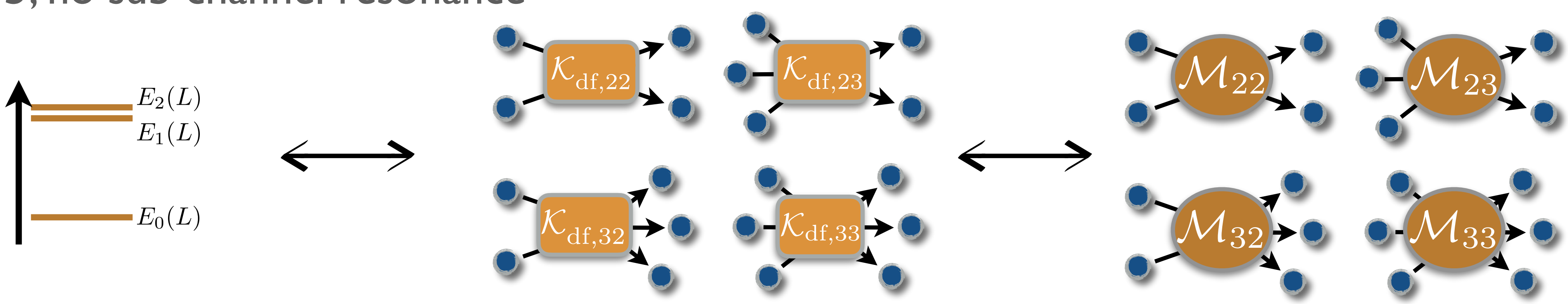
- General relation between *energies* and *two-and-three scalar scattering*

No **2-to-3,** ***no*** **sub-channel resonance**



• MTH, Sharpe (2014, 2015) •

2-to-3**, no sub-channel resonance**



• Briceño, MTH, Sharpe (2017) •

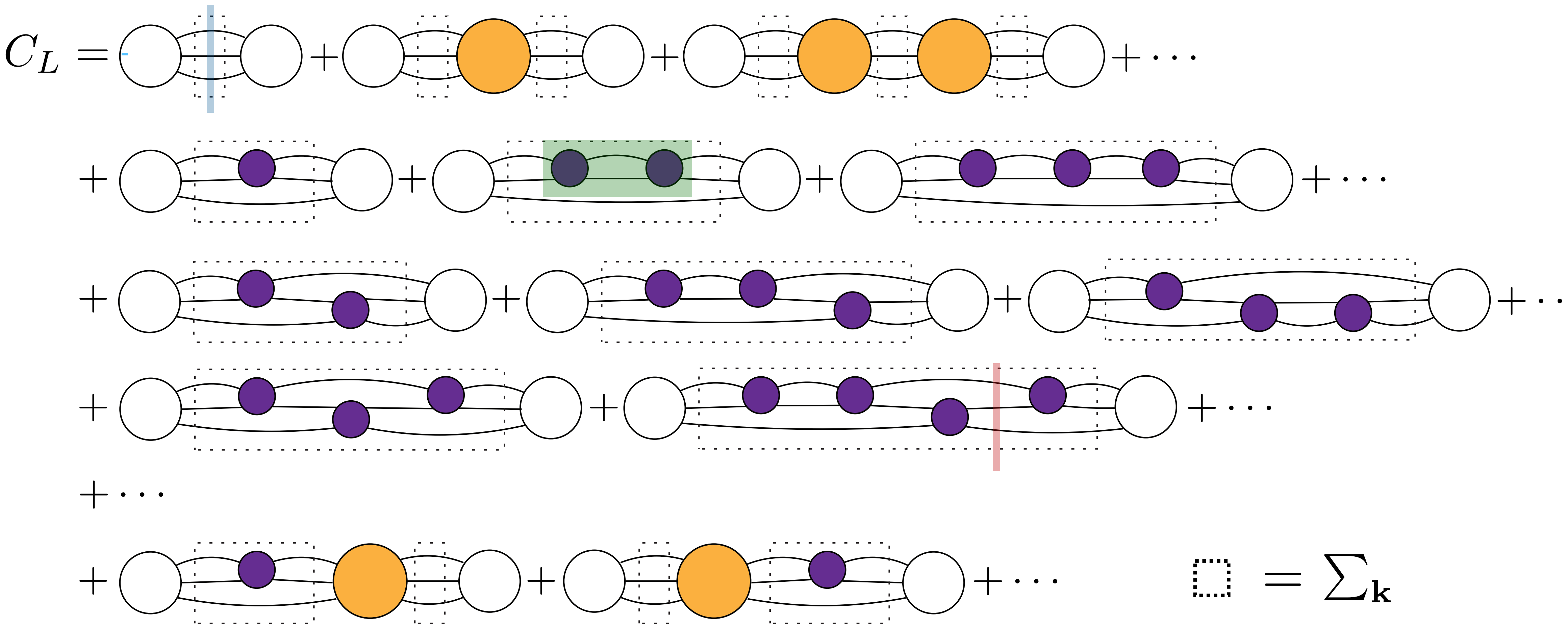
Including sub-channel resonances + *different isospins* + *non-degenerate*

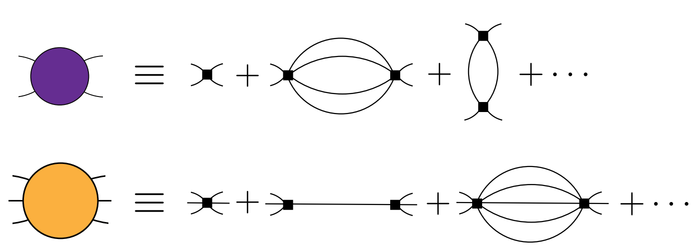
$$\pi\pi\pi \longrightarrow \rho\pi \longrightarrow \omega \longrightarrow \rho\pi \longrightarrow \pi\pi\pi$$

• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

3-particle derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***





kernels have suppressed *L* dependence

lines = fully dressed hadrons

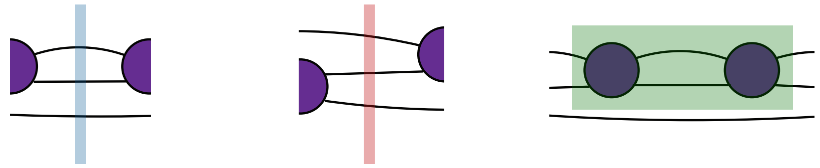
• MTH, Sharpe (2014) •

General relation

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(s) + F_3(P,L|\mathcal{K}_2)\right] = 0$$

$F_3(P,L|\mathcal{K}_2) \equiv$ Matrix of functions depending on kinematics + two-particle dynamics

$$F_3 \equiv \frac{1}{3}F + F\mathcal{K}_2 \frac{1}{1-(F+G)\mathcal{K}_2} F$$



Holds only for three-particle energies

Neglects e^{-mL}

- MTH, Sharpe (2014-2016) • *See also Döring, Mai, Hammer, Pang, Rusetsky* •

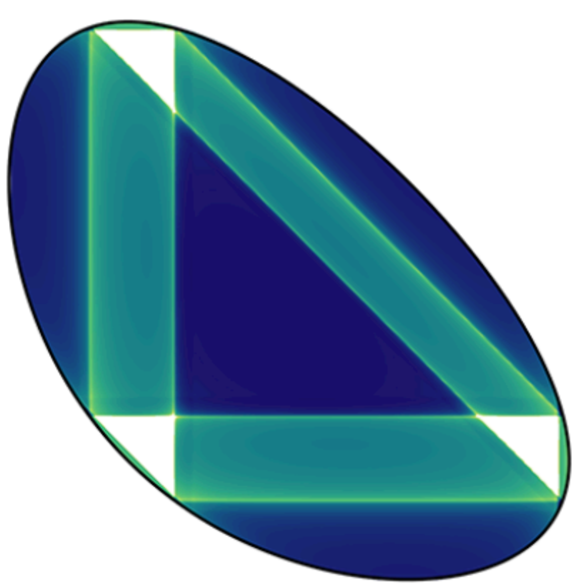
Review: *Lattice QCD and Three-particle Decays of Resonances*
MTH and Sharpe, *1901.00483*

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen,[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,‖]

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*

Phys. Rev. Lett. **126**, 012001 (2021)

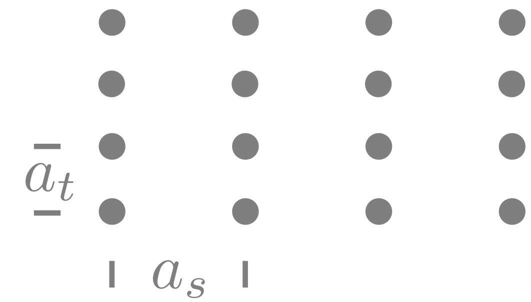
$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

lattice details

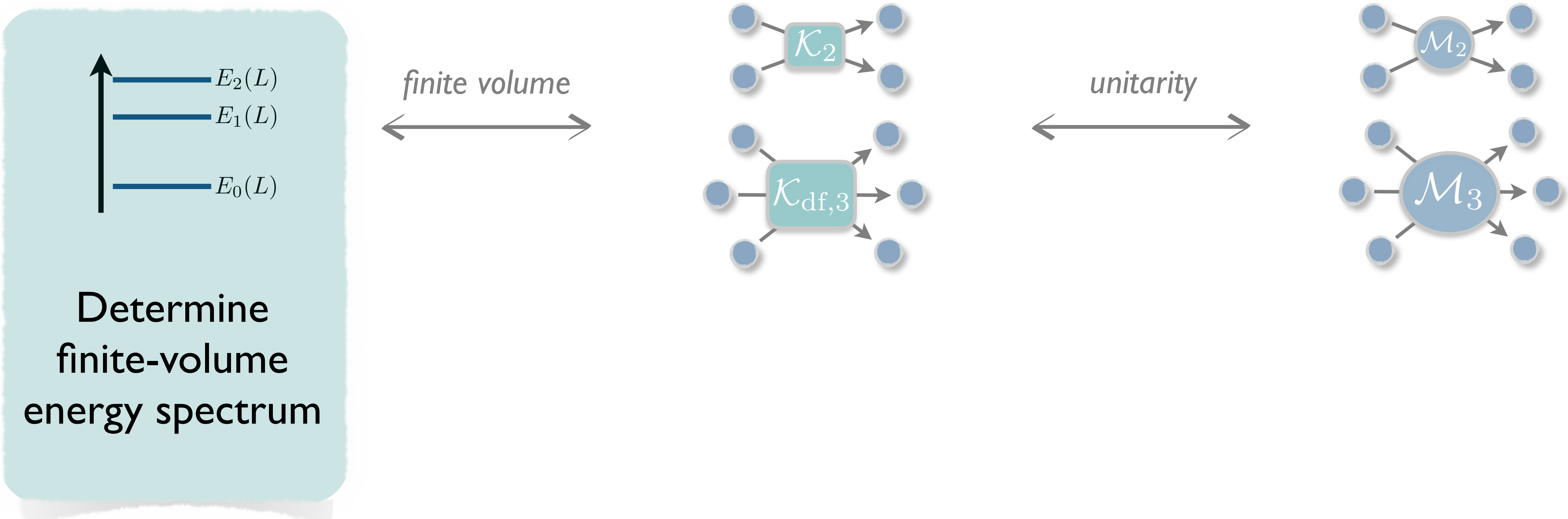
$N_f = 2+1 \qquad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

$L_s/a_s = 20, 24$

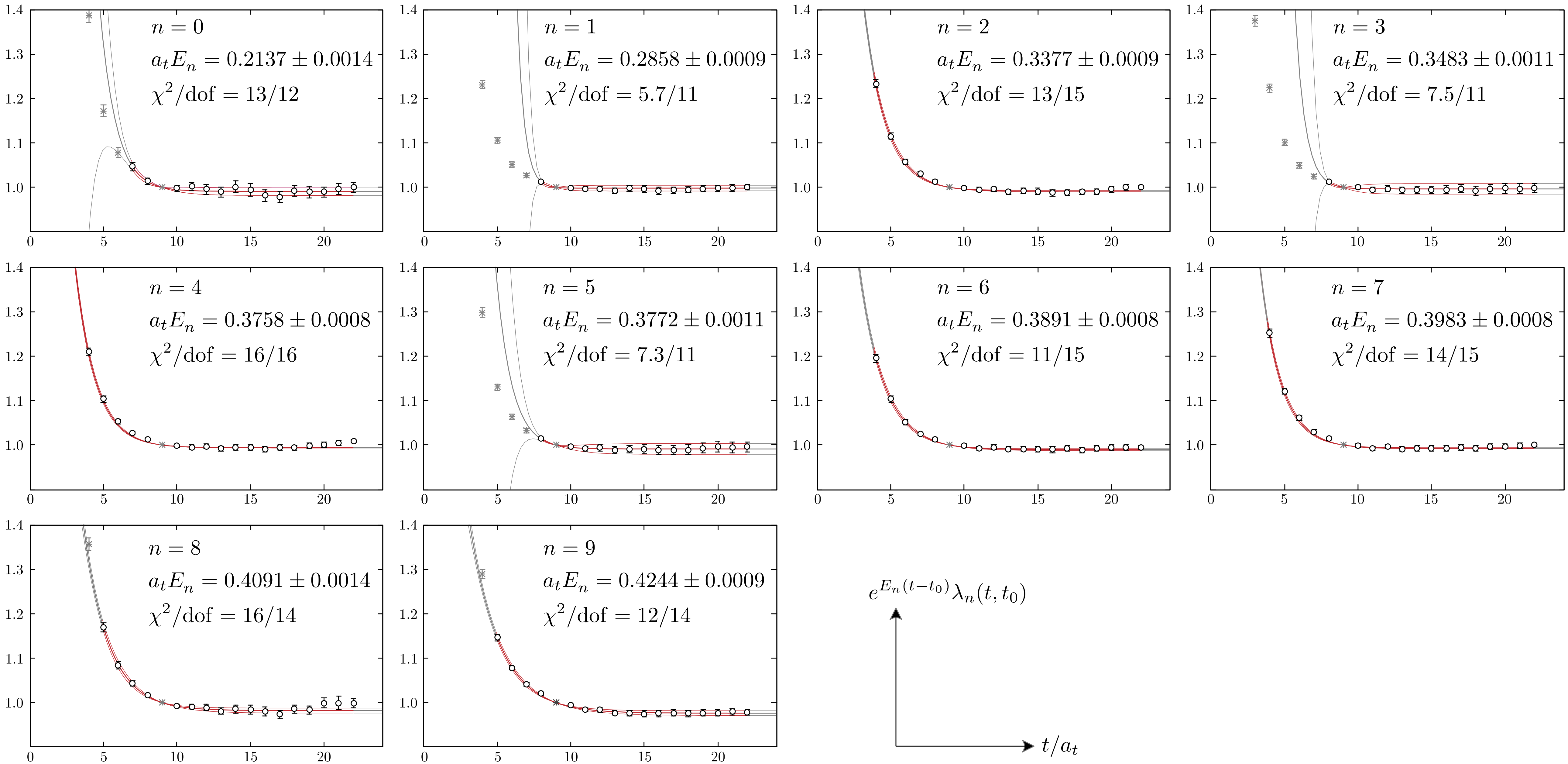


- Workflow outline



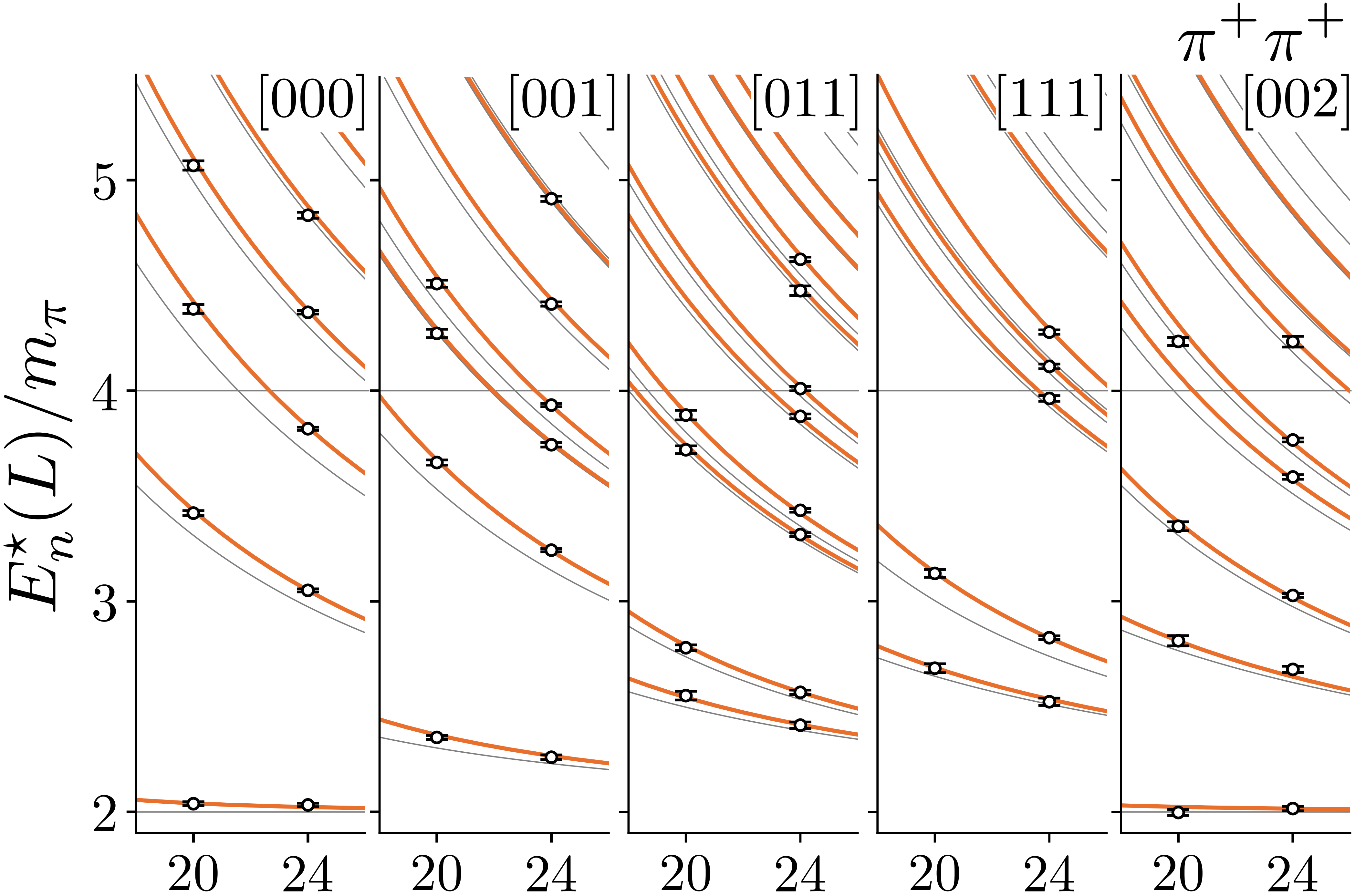
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_s = 24$



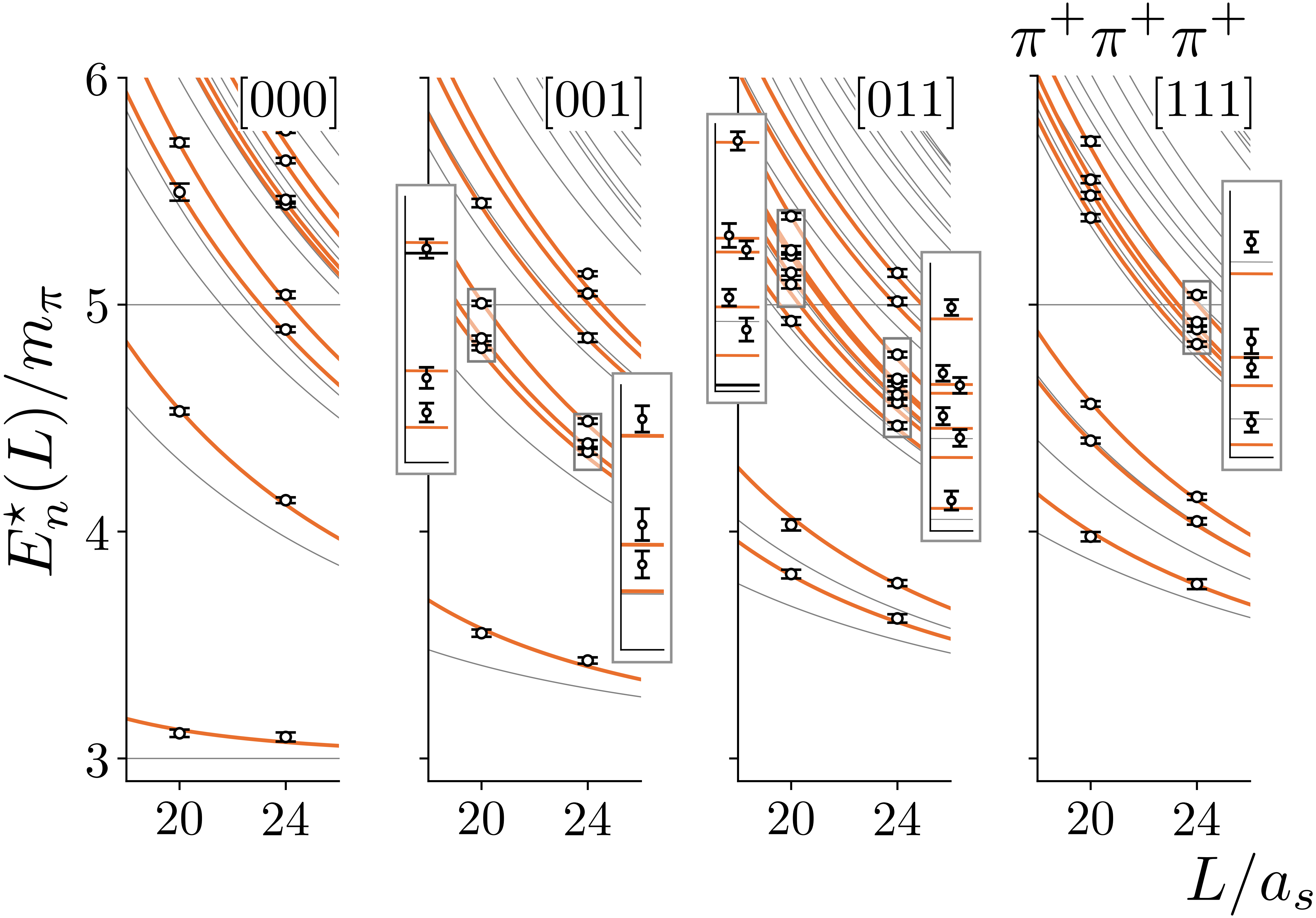
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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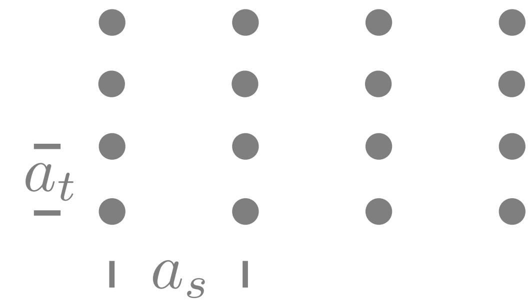
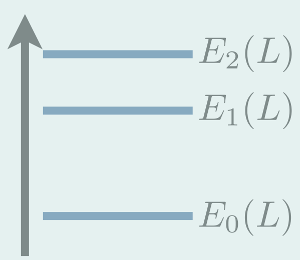


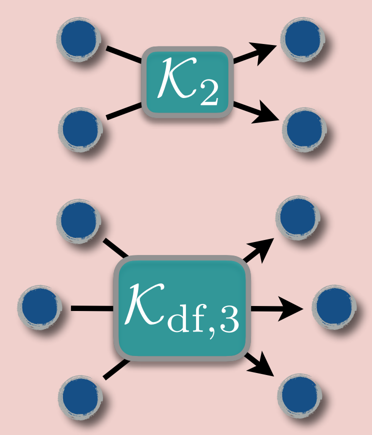
☐ Workflow outline



Determine finite-volume energy spectrum

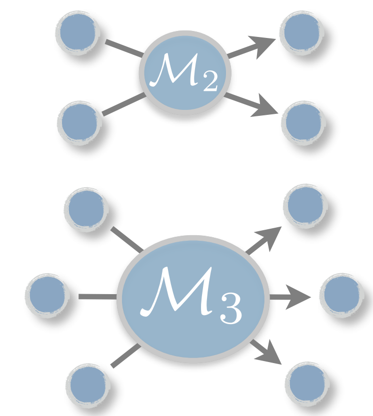
✓

finite volume ⟷



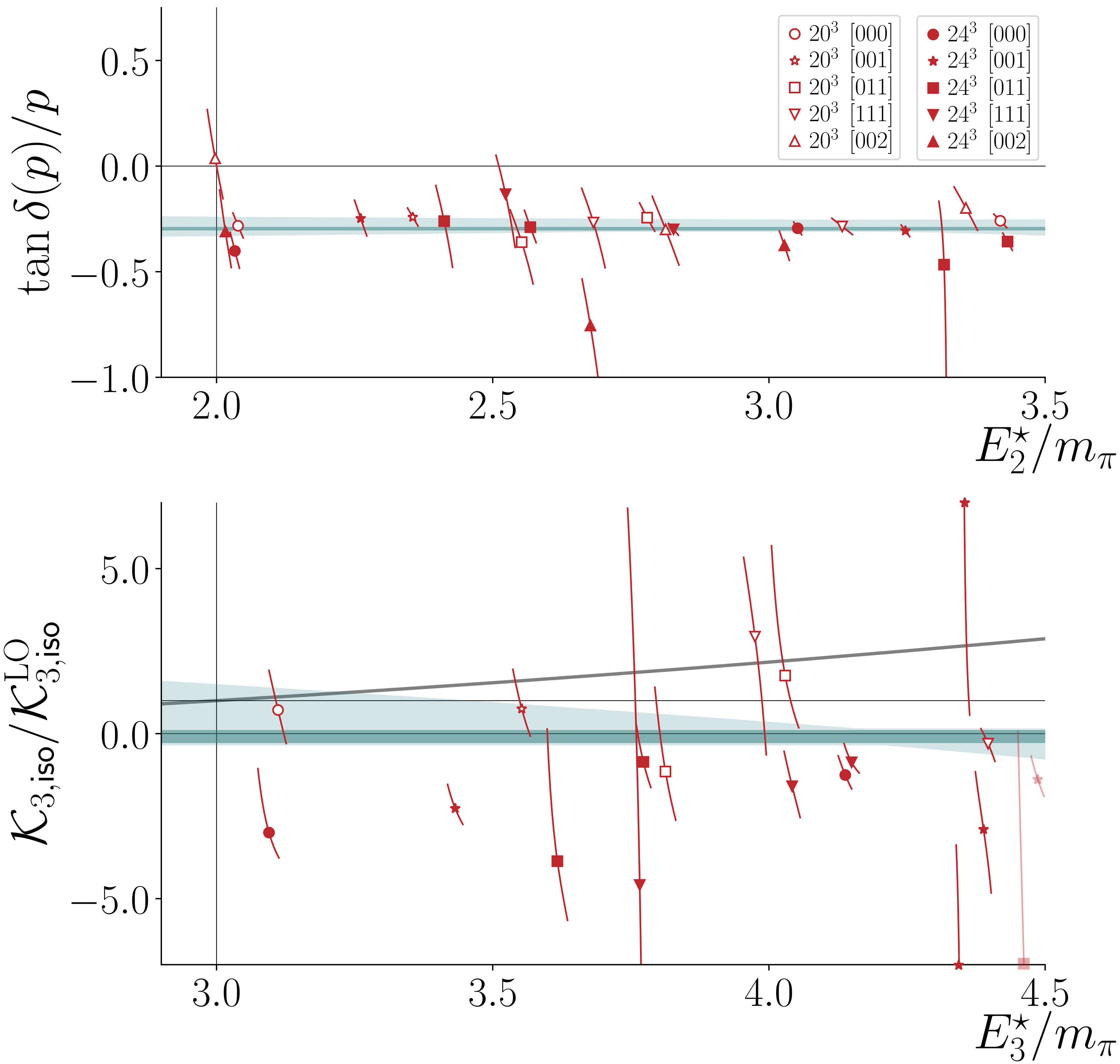
Fit to constrain two- and three-body K matrices

unitarity ⟷



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\text{df},3}$ depending only on $E_{\text{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

$K_{\text{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

$N_f = 2+1 \qquad a_s/a_t = 3.444(6)$

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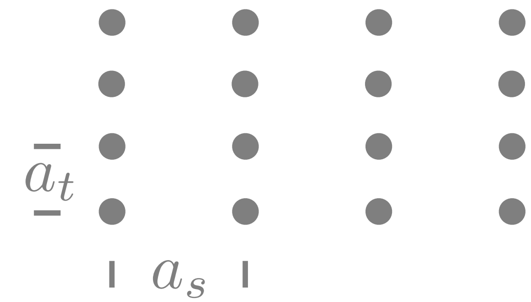
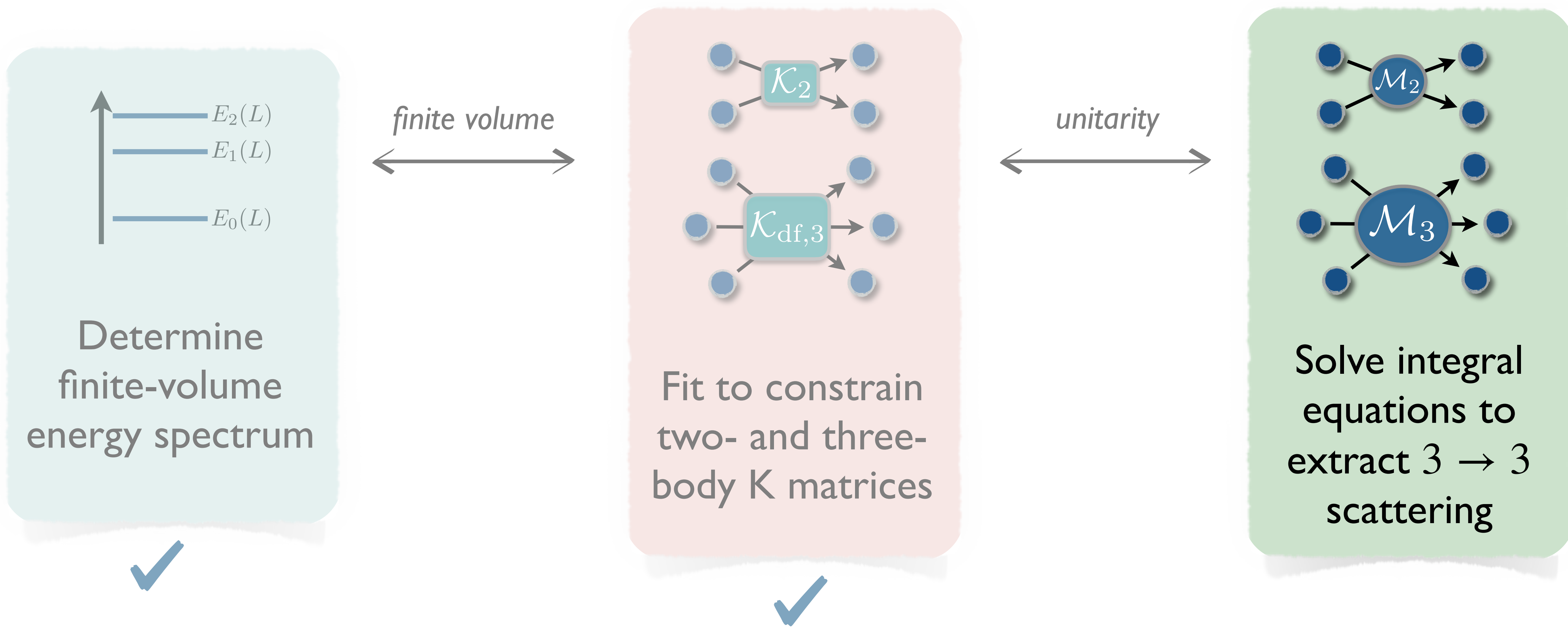


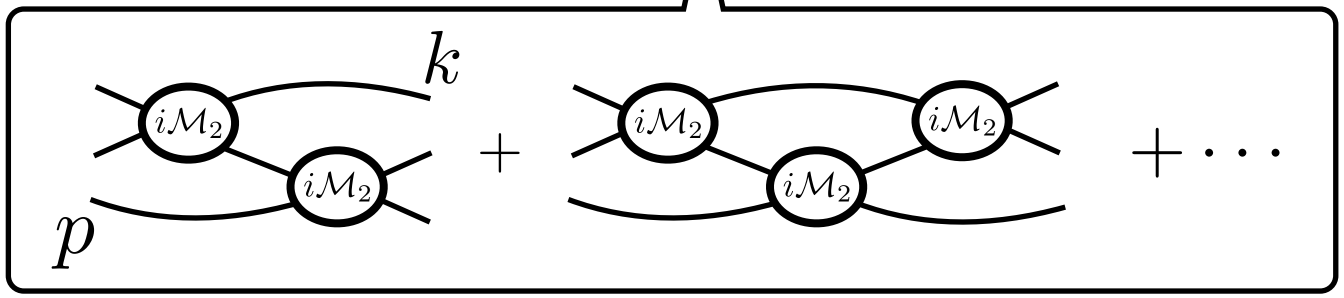
☐ Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star,\boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star,\boldsymbol{k})$$



Vanishes for $K_{\text{df},3} = 0$

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{P}\cdot\boldsymbol{D}(N,\epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\boldsymbol{D}_{pk}(N,\epsilon)$$

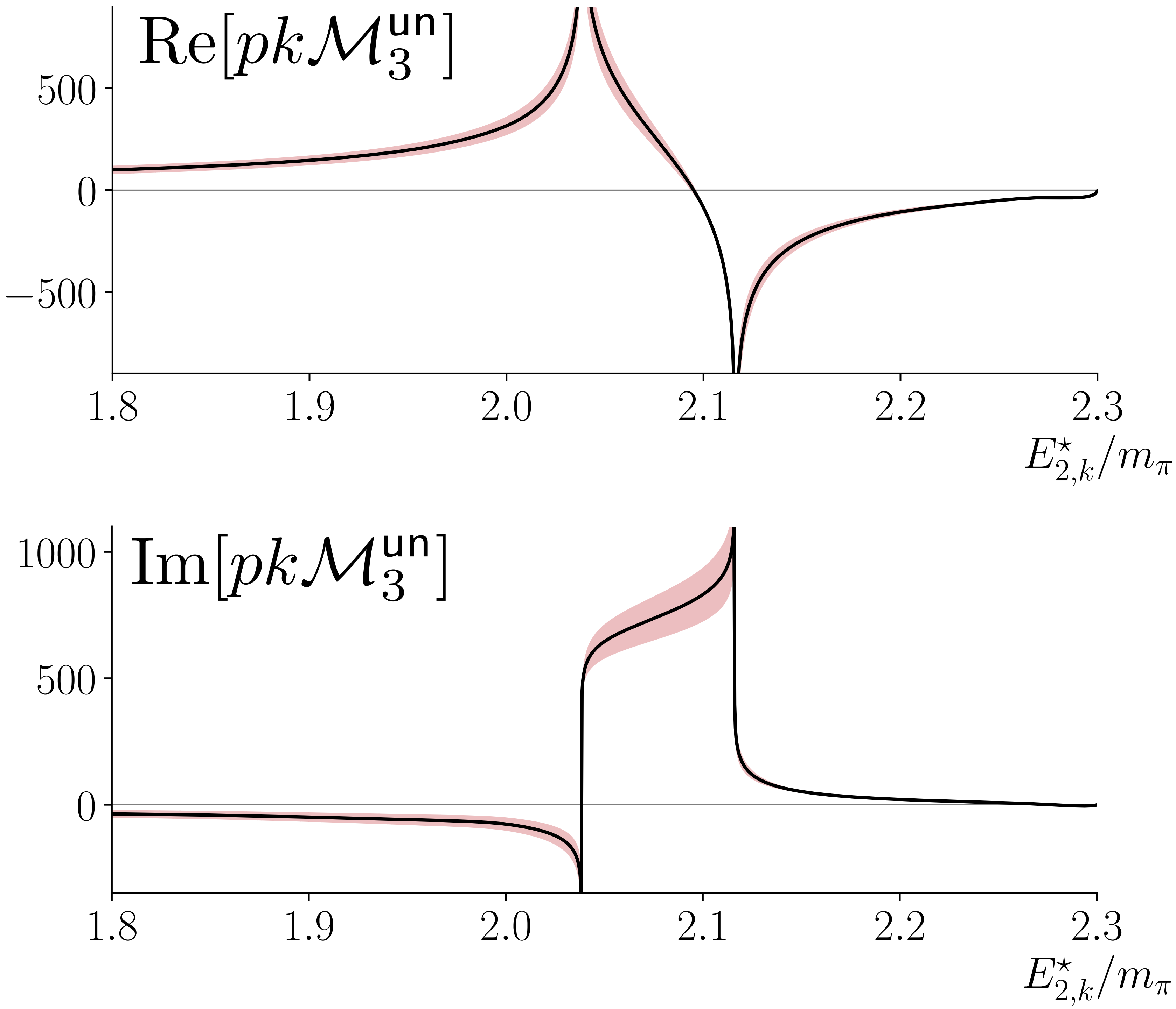
- See also…

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,[1,2,*] Raúl A. Briceño,[1,2,†] Sebastian M. Dawid,[3,4,‡] Md Habib E Islam,[2,§] and Connor McCarty[5,¶]

arXiv: 2010.09820

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

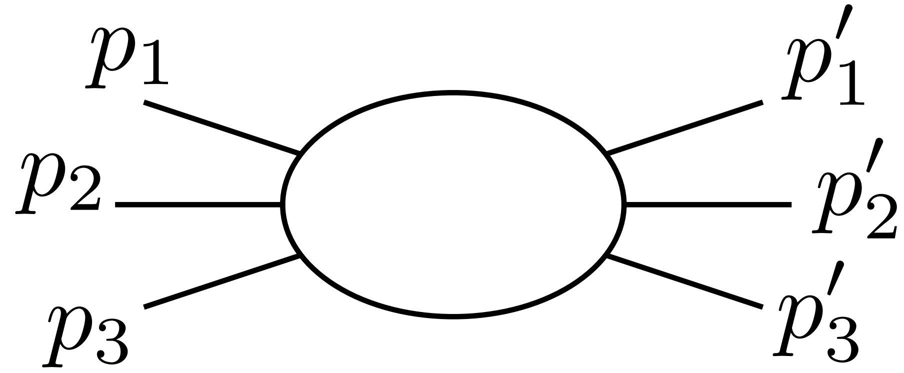
Plot at fixed $E_3^\star$ and p

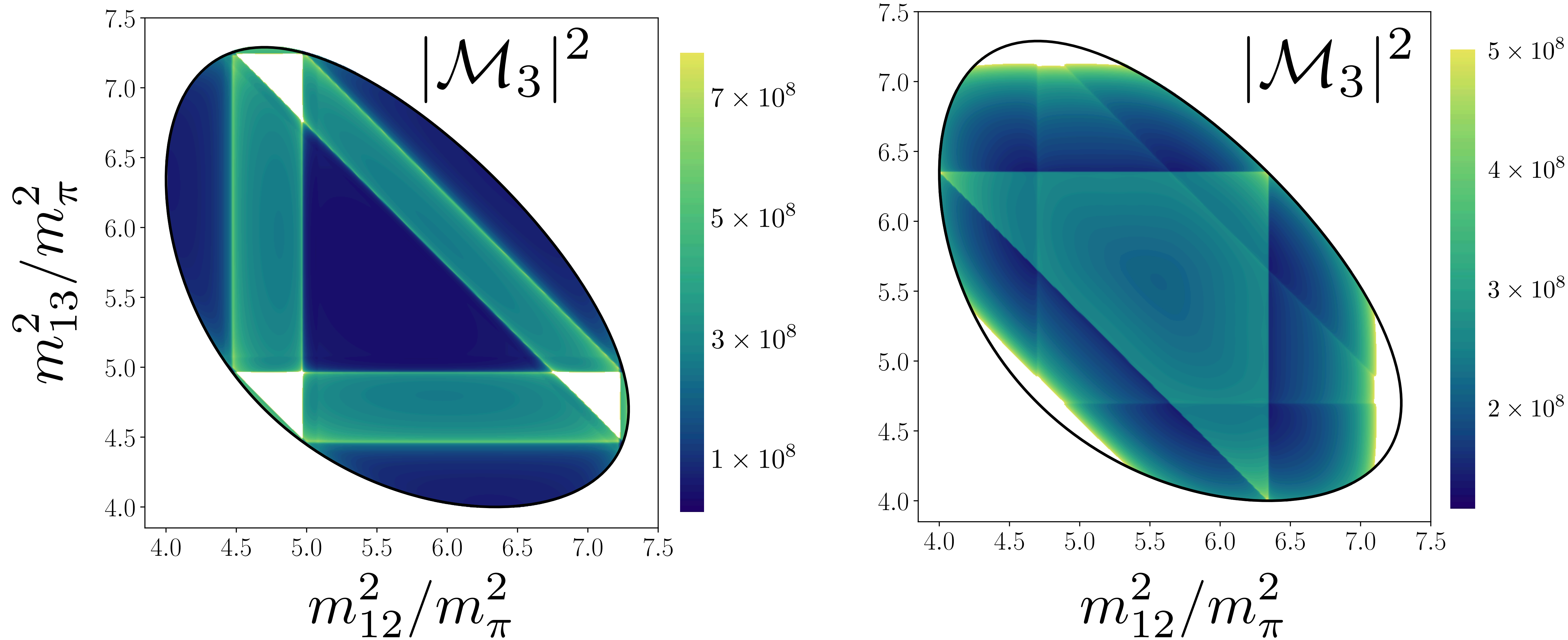
Both two- and three-body uncertainties estimated

Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j \in \{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p'_i, p_j)$$





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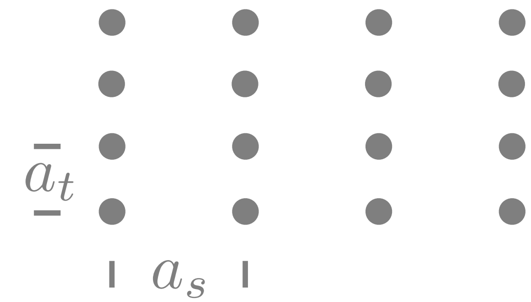
$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

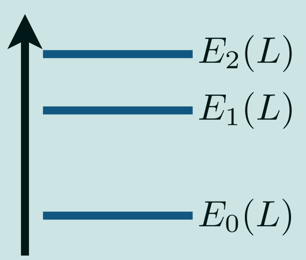
$N_f = 2+1 \qquad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

$L_s/a_s = 20, 24$



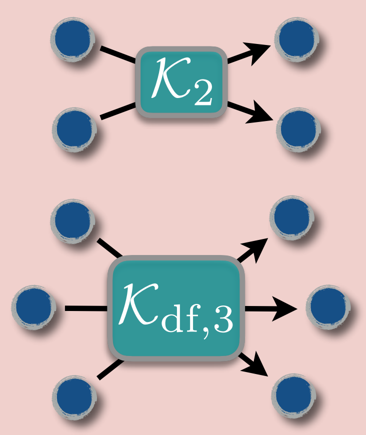
☐ Workflow outline



Determine finite-volume energy spectrum

✓

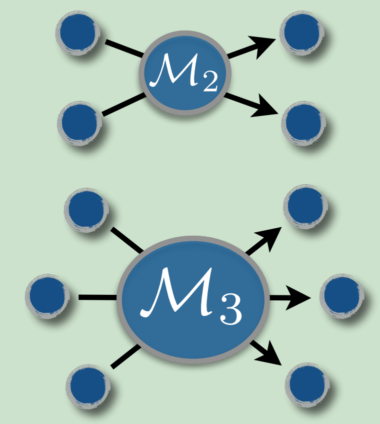
finite volume ⟷



Fit to constrain two- and three-body K matrices

✓

unitarity ⟷



Solve integral equations to extract $3 \to 3$ scattering

✓

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001