Hadron and lepton properties from dispersion relations

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International School of Nuclear Physics, 44th Course

From quarks and gluons to hadrons and nuclei

Erice, September 23, 2023

Outline

Part I: lepton properties

- anomalous magnetic moment of the muon $\longrightarrow$ A. Denig's talk
  - ▷ hadronic vacuum polarisation (HVP)
  - ▷ hadronic light-by-light scattering (HLbL)

Part II: hadron properties

- pion–pion (re)scattering and all that
- some illustrative examples

Part III: application

- in-depth analysis: π^0-pole contribution to HLbL

Summary / Outlook

Part I:

lepton properties: $(g-2)_\mu$

Hadronic contributions to $(g-2)_\mu$

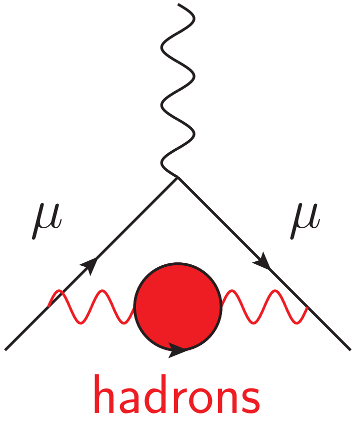
- gyromagnetic ratio: magnetic moment $\leftrightarrow$ spin

$$\vec{\mu} = g\frac{e}{2m}\vec{S} \qquad\qquad g_\mu = 2(1+a_\mu)$$

	$a_\mu\ [10^{-11}]$	$\Delta a_\mu\ [10^{-11}]$
experiment	116 592 059.	22.
QED	116 584 718.931	0.030
electroweak	153.6	1.0
had. VP (LO)	6931.	40.
had. VP (NLO)	–98.3	0.7
had. LbL	92.	19.
total	116 591 810.	43.

BNL E821 2006
+ Fermilab 2021, 2023

Aoyama et al. 2020



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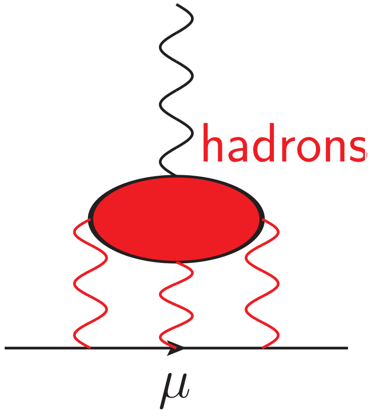
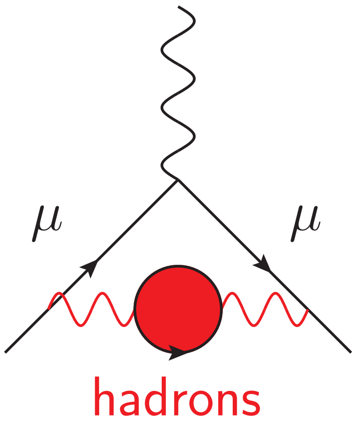
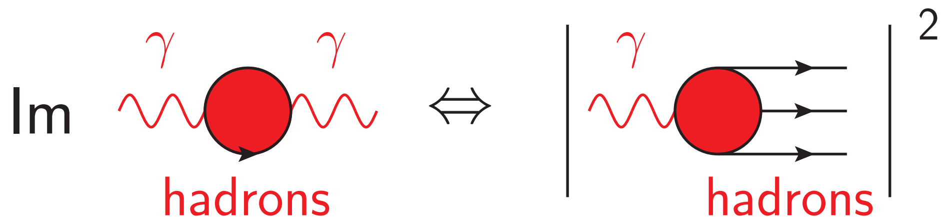


Hadronic vacuum polarisation

- how to control hadronic vacuum polarisation?
- characteristic scale set by muon mass
  $\longrightarrow$ this is not a perturbative QCD problem!
- dispersion relations to the rescue:
  use the optical theorem!





$\propto \sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$

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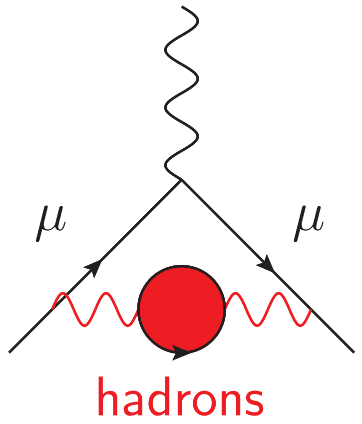


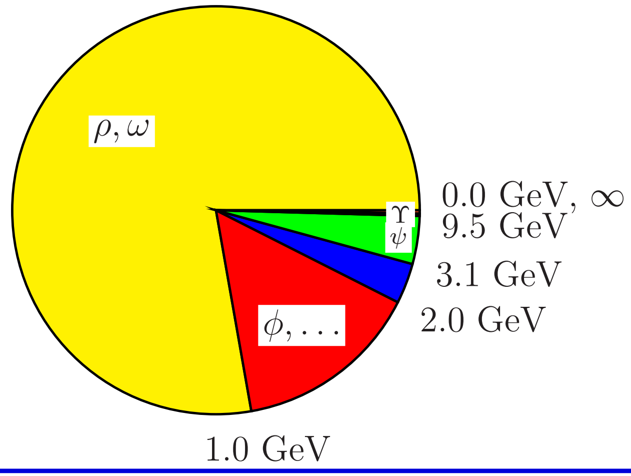
$$a_\mu^{\text{had VP}} \propto \int_{M_{\pi^0}^2}^{\infty} ds\, K(s)\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$$

- $K(s)$: kinematical function, for large s: $\quad K(s) \propto 1/s$,
  $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons}) \propto 1/s$

- more than 75% of $a_\mu^{\text{had VP}}$ given by
  energies $s \leq 1\,\text{GeV}^2$ Jegerlehner, Nyffeler 2009
- well (??) constrained by data

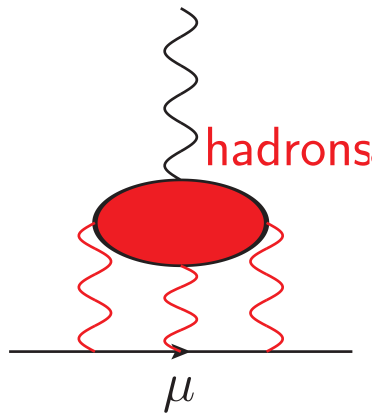
BaBar, BESIII, CMD, KLOE, SND, ...

$\longrightarrow$ *largely* an experimental task

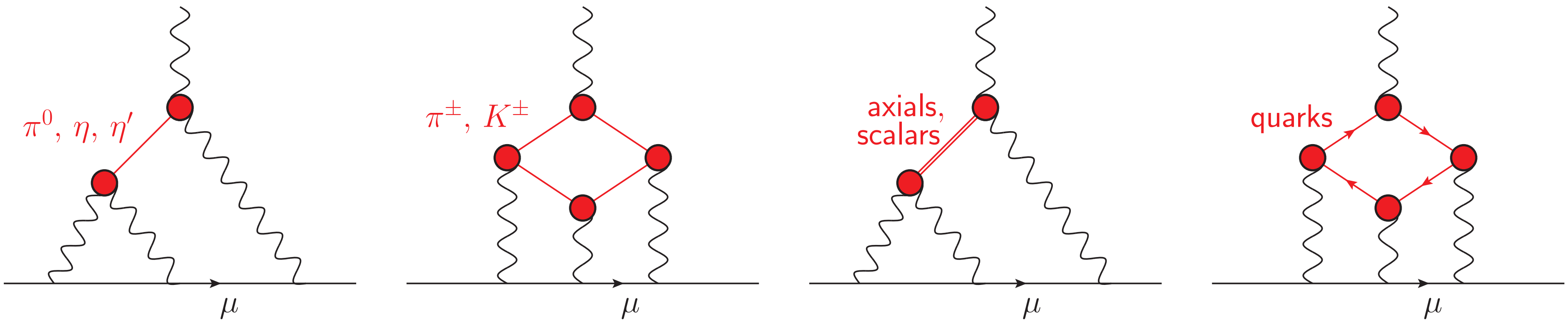


Hadronic light-by-light scattering

- hadronic light-by-light:
  - ▷ subleading in α_{QED}
  - ▷ large relative uncertainty



- different contributions calculated or estimated (in 10^{-11}):



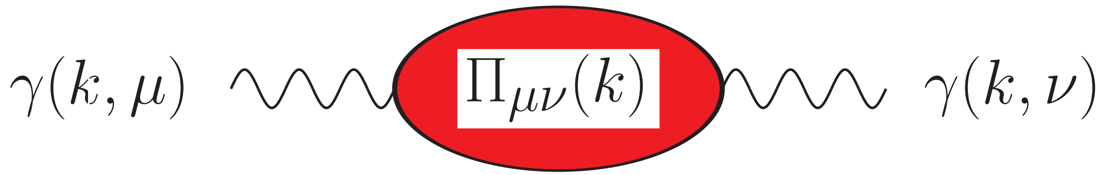
94±4 | –24±1 | 5±7 | 18±10

$\longrightarrow$ increasing systematic control over HLbL using dispersion-theoretical approach

Aoyama et al. 2020

Hadronic vacuum polarisation — why so simple?

- photon two-point function:



  - ▷ one single independent momentum k
  - ▷ symmetric rank-2 tensor: two structures $g_{\mu\nu}$, $k_\mu k_\nu$
  - ▷ scalar invariant can depend on one single invariant k^2

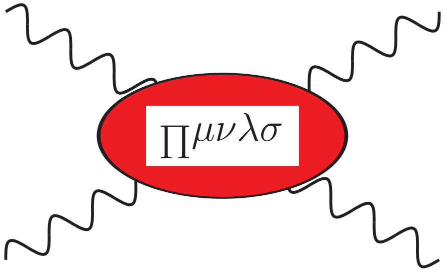
- gauge invariance: $k^\mu \Pi_{\mu\nu}(k) = 0 = k^\nu \Pi_{\mu\nu}(k)$

$$\boxed{\Pi_{\mu\nu}(k) = \big(k^2 g_{\mu\nu} - k_\mu k_\nu\big)\Pi(k^2)}$$

$\longrightarrow$ Lorentz + gauge invariance reduce HVP to one single function of a single variable!

Hadronic light-by-light: dispersive approach

Colangelo, Hoferichter, Procura, Stoffer 2014, 2015



- HLbL tensor $\Pi^{\mu\nu\lambda\sigma}$: Lorentz invariance

  $\longrightarrow$ 138 (136) scalar functions Eichmann et al. 2014
- gauge invariance: Bardeen, Tung 1968; Tarrach 1975

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

$\longrightarrow$ 7 distinct structures, 47 related by crossing

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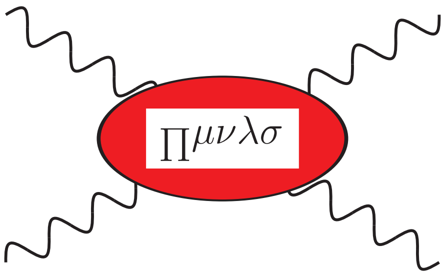

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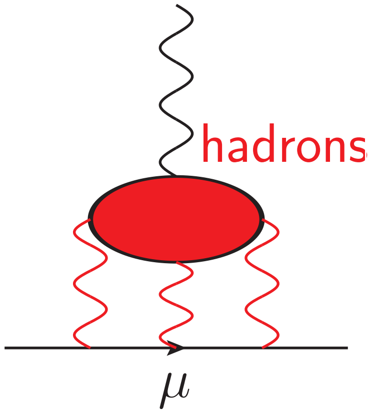


- master formula:

$$a_\mu^{\text{HLbL}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{\sum_{i=1}^{12}\hat{T}_i(q_1,q_2;p)\hat{\Pi}_i(q_1,q_2,-q_1-q_2)}{q_1^2q_2^2(q_1+q_2)^2[(p+q_1)^2-m_\mu^2][(p-q_2)^2-m_\mu^2]}$$

- $\hat{T}_i$: known kernels

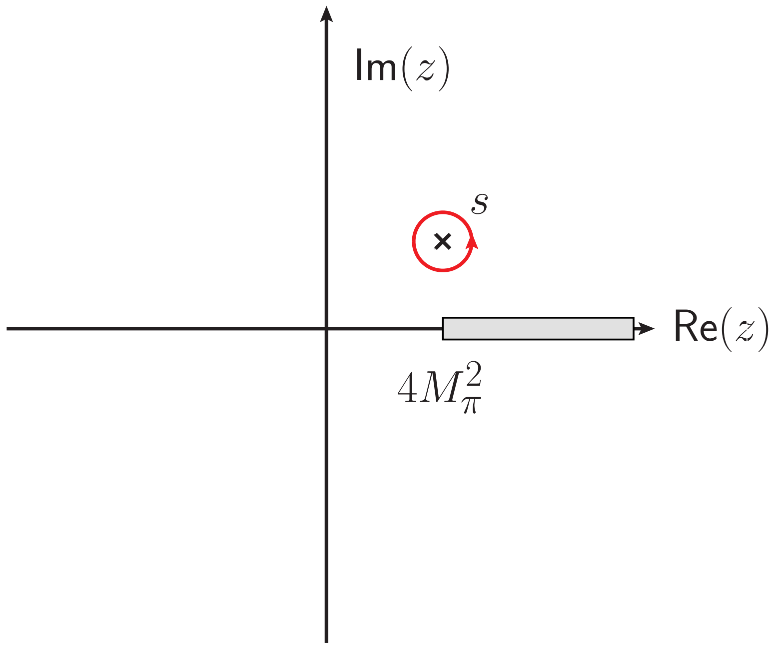
  $\hat{\Pi}_i$: dispersively $\leftrightarrow$ measurable form factors / scatt. amplitudes

Part II:

hadron properties: form factors & (re)scattering

Dispersion relations for pedestrians

$\longrightarrow$ A. Pilloni's talk



analyticity ($\simeq$ causality)

& Cauchy's theorem:

$$T(s) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{T(z)dz}{z-s}$$

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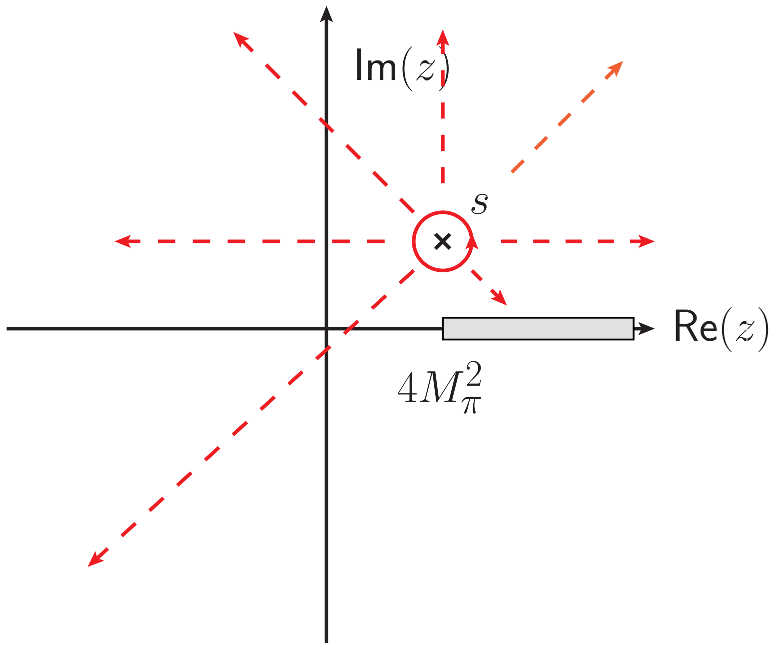

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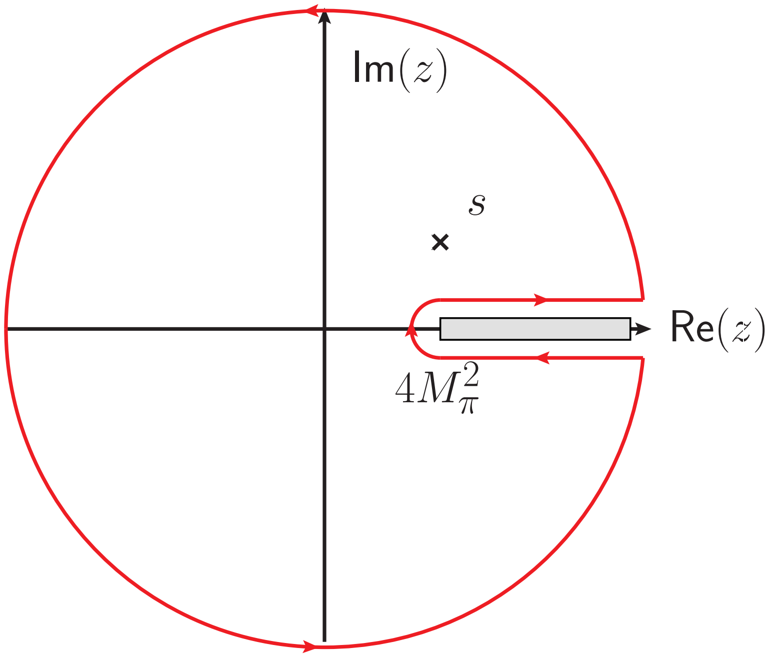

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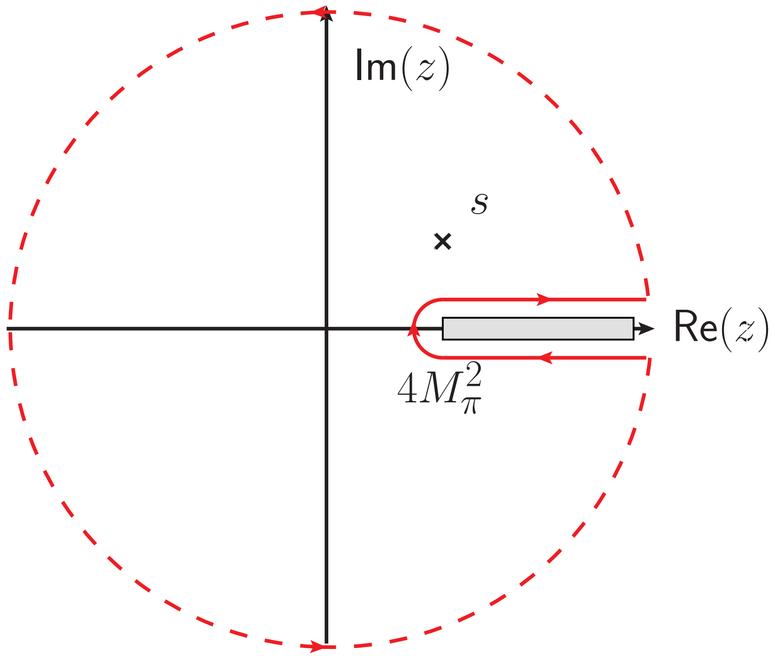

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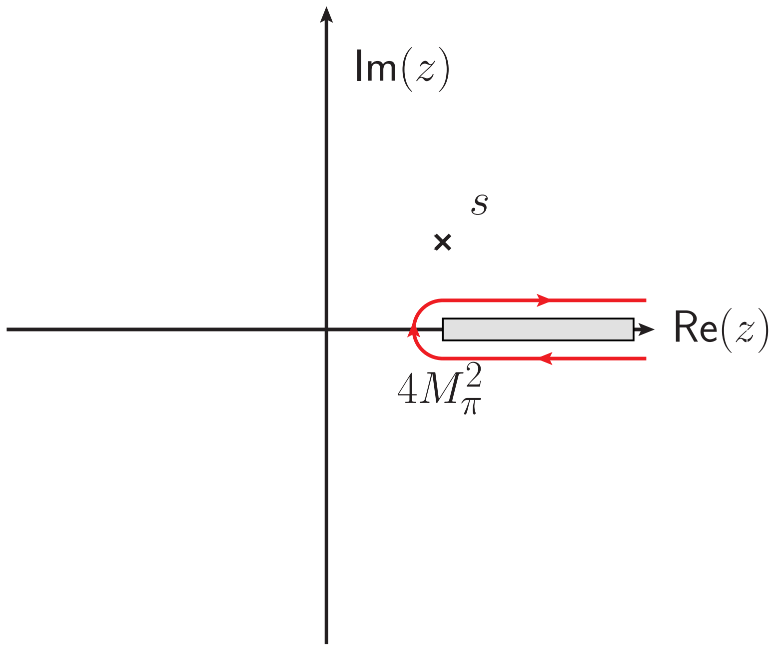

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$$T(s) = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{T(z)dz}{z-s}$$

$$\longrightarrow \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{disc}\,T(z)dz}{z-s}$$

$$= \frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{Im}\,T(z)dz}{z-s}$$

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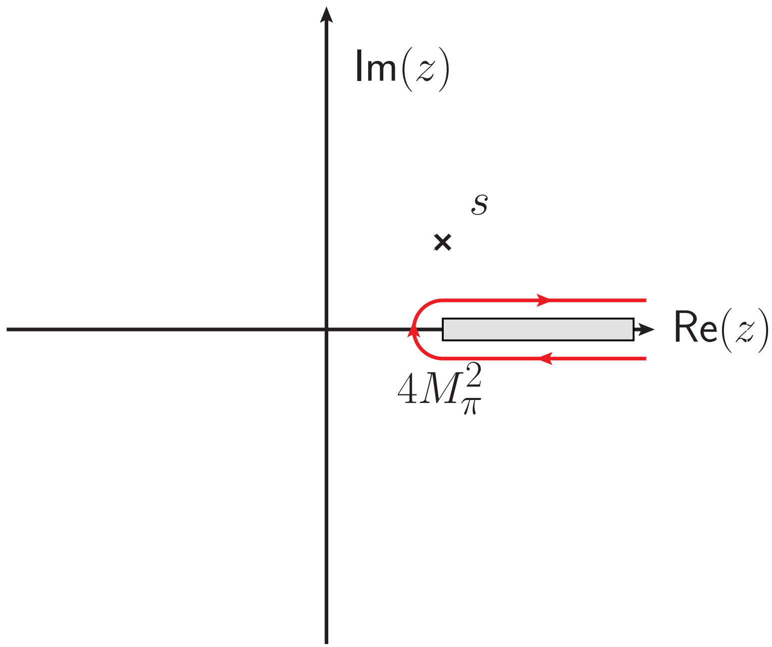

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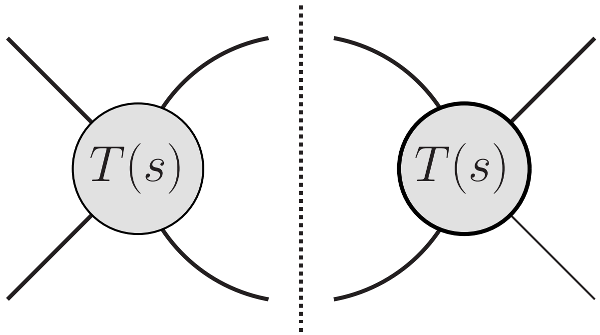
- $\mathrm{disc}\,T(s) = 2i\,\mathrm{Im}\,T(s)$ given by unitarity ($\simeq$ prob. conservation):



e.g. if $T(s)$ is a $\pi\pi$ partial wave $\longrightarrow$

$$\frac{\mathrm{disc}\,T(s)}{2i} = \mathrm{Im}\,T(s) = \frac{2q_\pi}{\sqrt{s}}\theta(s-4M_\pi^2)|T(s)|^2$$

Dispersion relations for pedestrians

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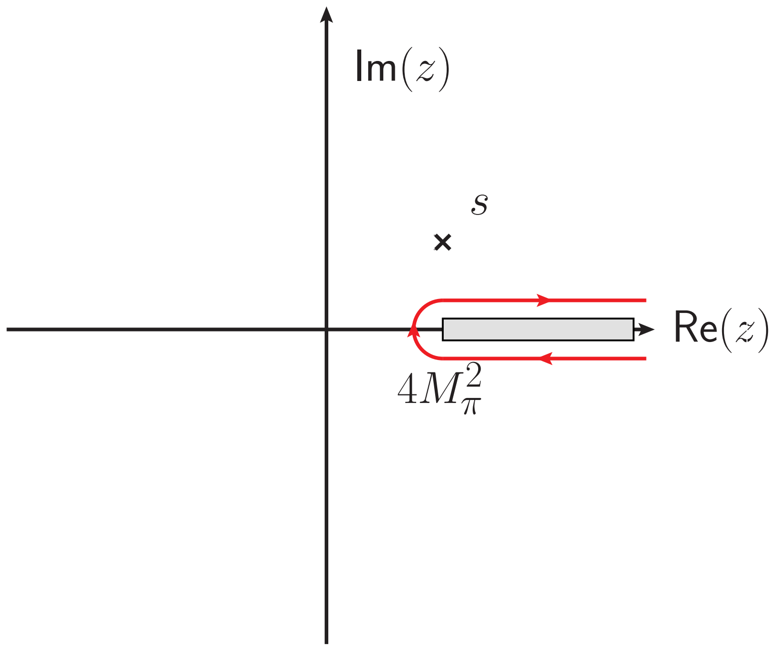

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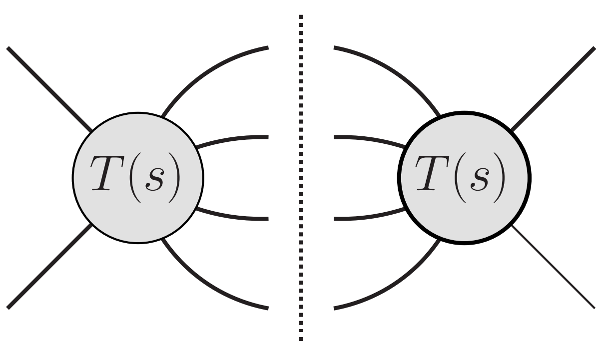
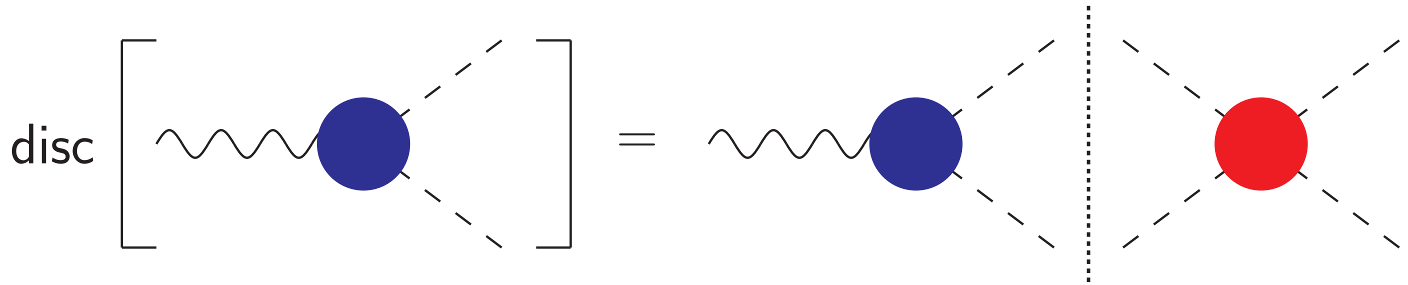


inelastic intermediate states ($K\bar{K}$, 4π) suppressed at low energies

$\longrightarrow$ will often be neglected

Warm up: pion vector form factor



$$\frac{1}{2i}\,\text{disc}\,F_\pi^V(s) = \text{Im}\,F_\pi^V(s) \; = \; F_\pi^V(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_\pi^V(s)$ is just $\delta_1^1(s)$ Watson 1954

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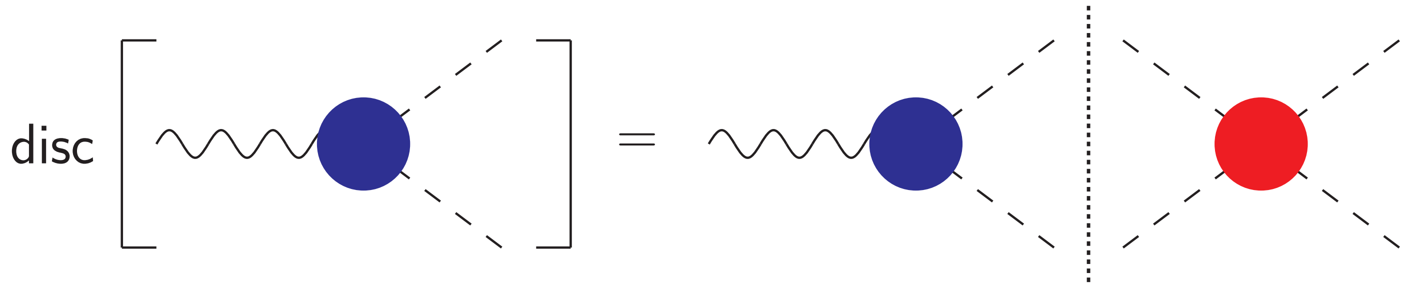

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- solution: $\longrightarrow$ E. Passemar's talk; derivation $\longrightarrow$ spares!

$$F_\pi^V(s) = P(s)\Omega(s)\;, \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds'\,\frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

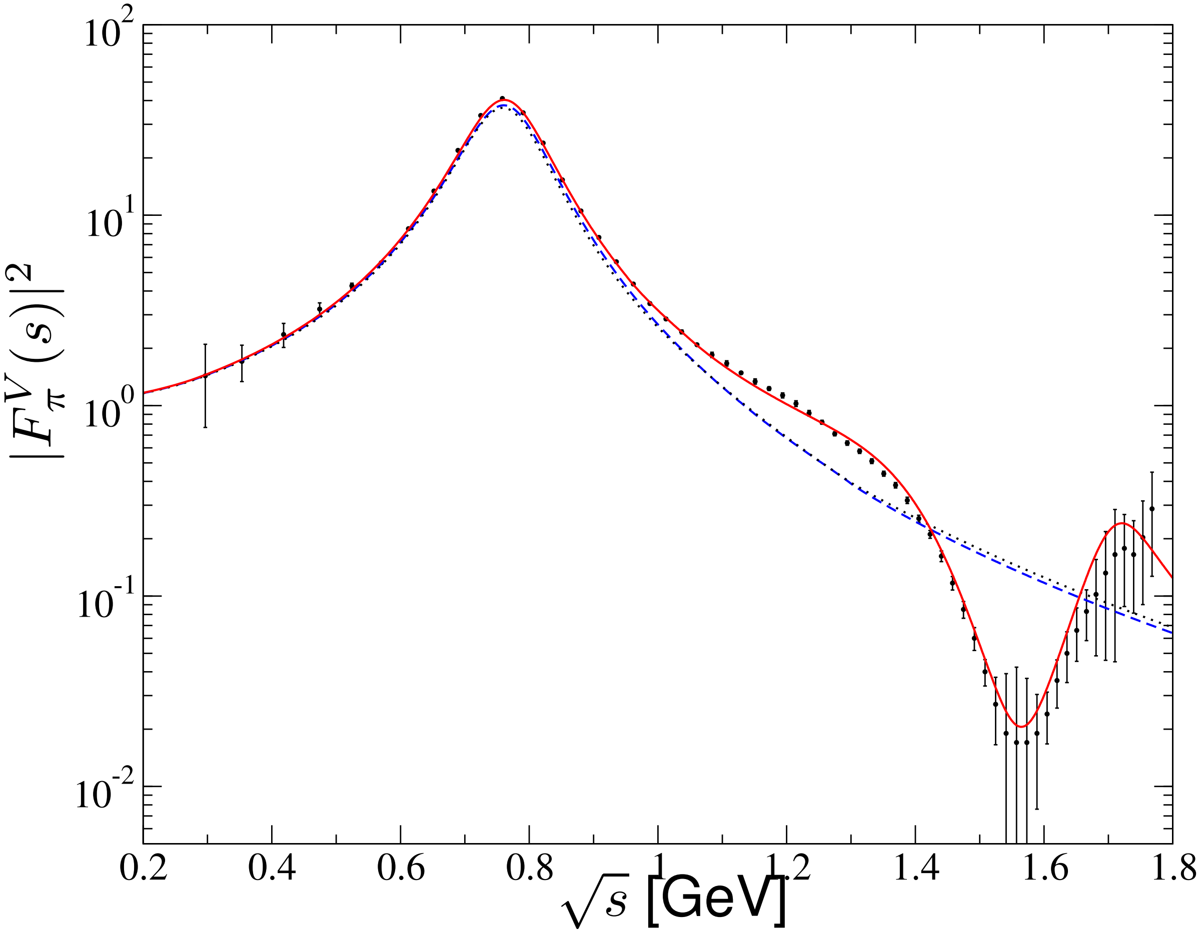
$P(s)$ polynomial, $\Omega(s)$ Omnès function Omnès 1958

  - ▷ $\pi\pi$ phase shifts from Roy equations
    Ananthanarayan et al. 2001, García-Martín et al. 2011
  - ▷ $P(0) = 1$ from symmetries (gauge invariance)

- below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\,\text{GeV}^{-2}s)\Omega(s)$
  slope due to inelastic resonances ρ', ρ''... Hanhart 2012

Pion vector form factor vs. Omnès representation

Data on pion form factor in $\tau^- \to \pi^-\pi^0\nu_\tau$ Belle 2008



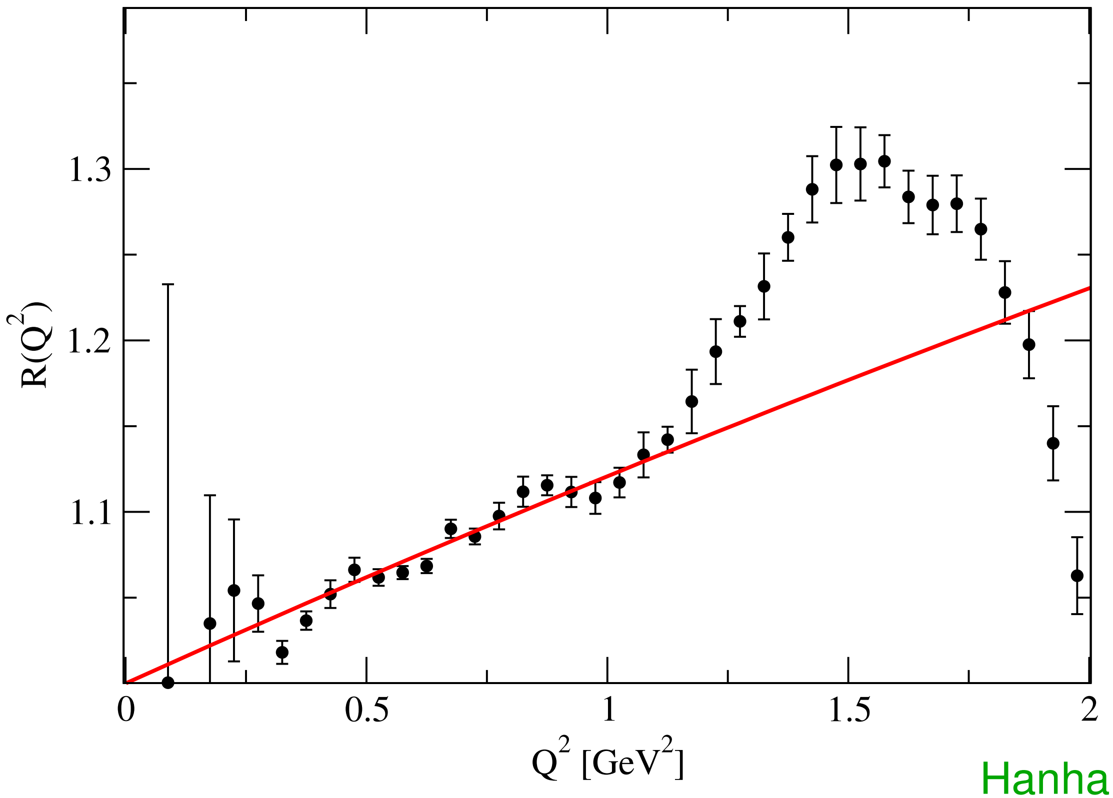
$\pi\pi$ P-wave phase shift / effective form factor phase incl. ρ', ρ''

Schneider et al. 2012

Pion vector form factor vs. Omnès representation

Data on pion form factor in $\tau^- \to \pi^- \pi^0 \nu_\tau$ Belle 2008

- divide $\tau^- \to \pi^- \pi^0 \nu_\tau$ form factor by Omnès function:



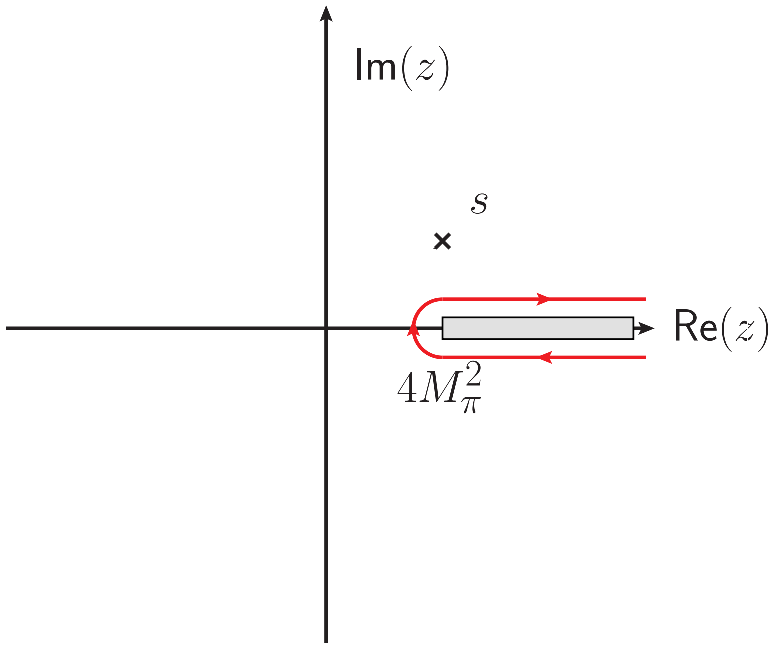
Hanhart et al. 2013

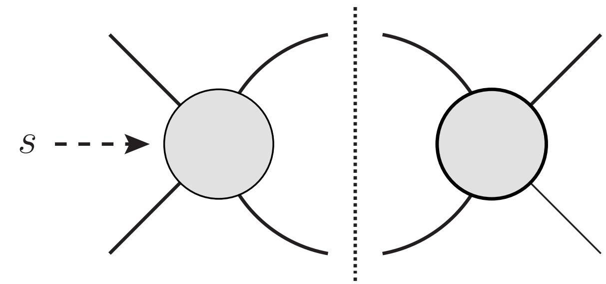
$\longrightarrow$ linear below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\,\text{GeV}^{-2} s)\Omega(s)$

$\longrightarrow$ above: inelastic resonances ρ', ρ''...

What are left-hand cuts?

Example: pion–pion scattering





- right-hand cut due to unitarity: $s \geq 4M_\pi^2$

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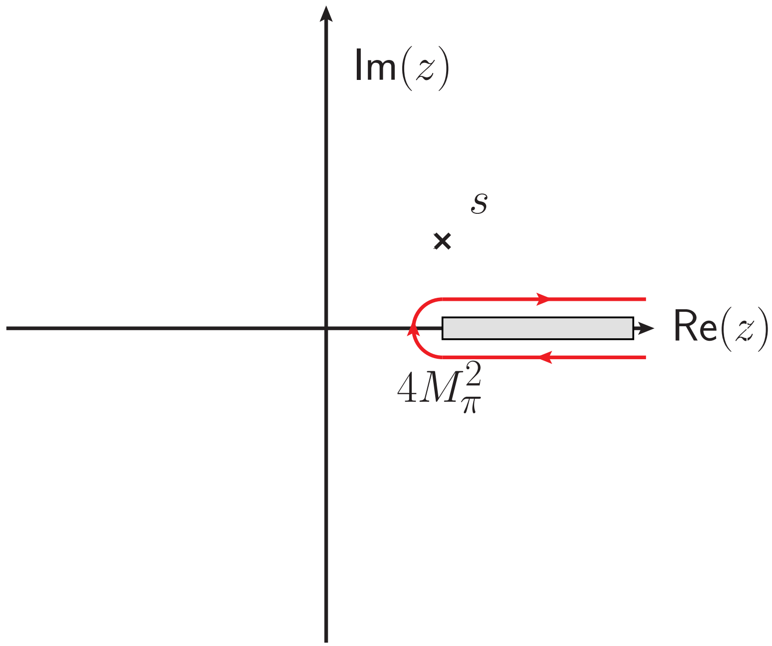

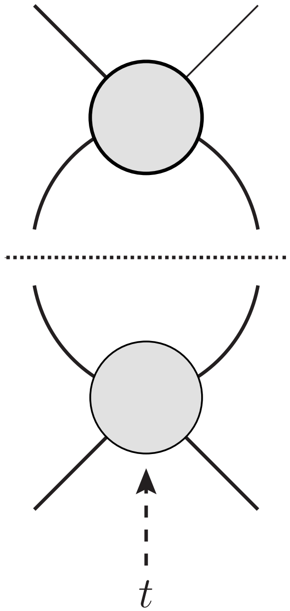

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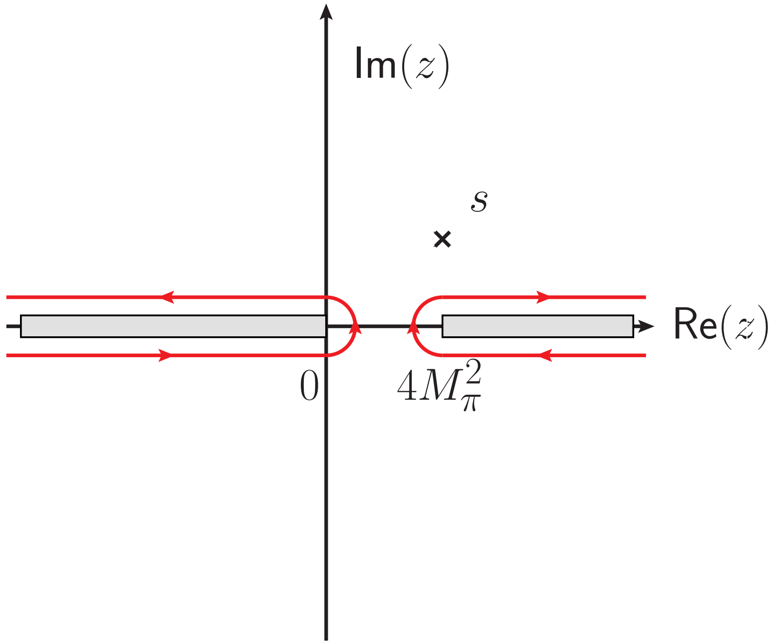

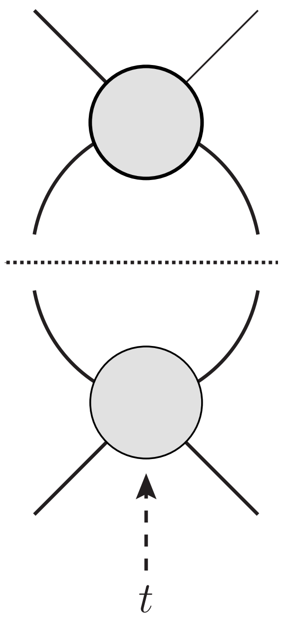

- right-hand cut due to unitarity: $s \geq 4M_\pi^2$
- crossing symmetry: cuts also for $t,\ u \geq 4M_\pi^2$
- partial-wave projection: $T(s,t) = 32\pi \sum_i T_i(s) P_i(\cos\theta)$

$$t(s,\cos\theta) = \frac{1-\cos\theta}{2}(4M_\pi^2 - s)$$

$\longrightarrow$ cut for $t \geq 4M_\pi^2$ becomes cut for $s \leq 0$ in partial wave

$\pi\pi$ scattering constrained by analyticity and unitarity

Roy equations = coupled system of partial-wave dispersion relations
+ crossing symmetry + unitarity

- twice-subtracted fixed-t dispersion relation:

$$T(s,t) = c(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \left\{ \frac{s^2}{s'^2(s'-s)} + \frac{u^2}{s'^2(s'-u)} \right\} \mathrm{Im}T(s',t)$$

- subtraction function $c(t)$ determined from crossing symmetry

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- subtraction function $c(t)$ determined from crossing symmetry
- project onto partial waves $t_J^I(s)$ (angular momentum J, isospin I)
$\longrightarrow$ coupled system of partial-wave integral equations

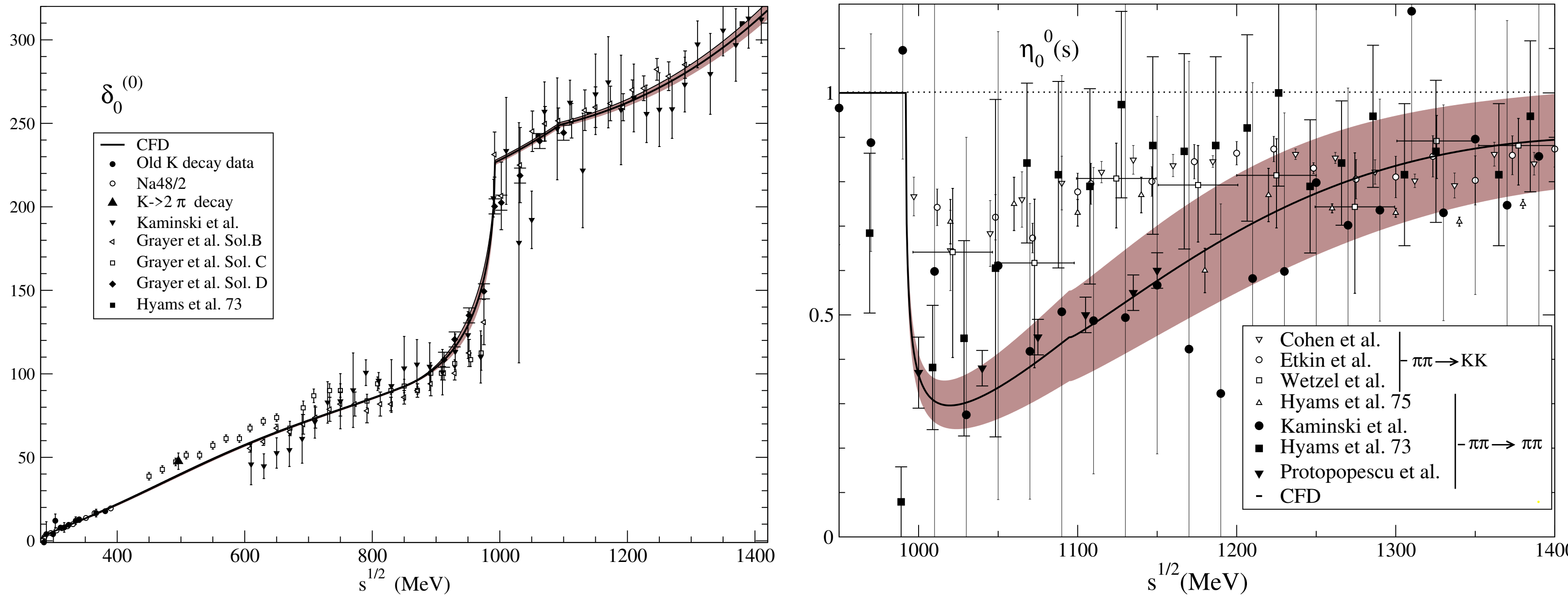
$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s') \mathrm{Im} t_{J'}^{I'}(s')$$

Roy 1971

- subtraction polynomial $k_J^I(s)$: $\pi\pi$ scattering lengths
can be matched to chiral perturbation theory   Colangelo et al. 2001
- kernel functions $K_{JJ'}^{II'}(s,s')$ known analytically

$\pi\pi$ scattering constrained by analyticity and unitarity

- elastic unitarity $\longrightarrow$ coupled integral equations for phase shifts
- modern precision analyses:
  - ▷ $\pi\pi$ scattering — Ananthanarayan et al. 2001, García-Martín et al. 2011
  - ▷ πK scattering — Büttiker et al. 2004, Peláez, Rodas 2020
- example: $\pi\pi$ $I = 0$ S-wave phase shift & inelasticity

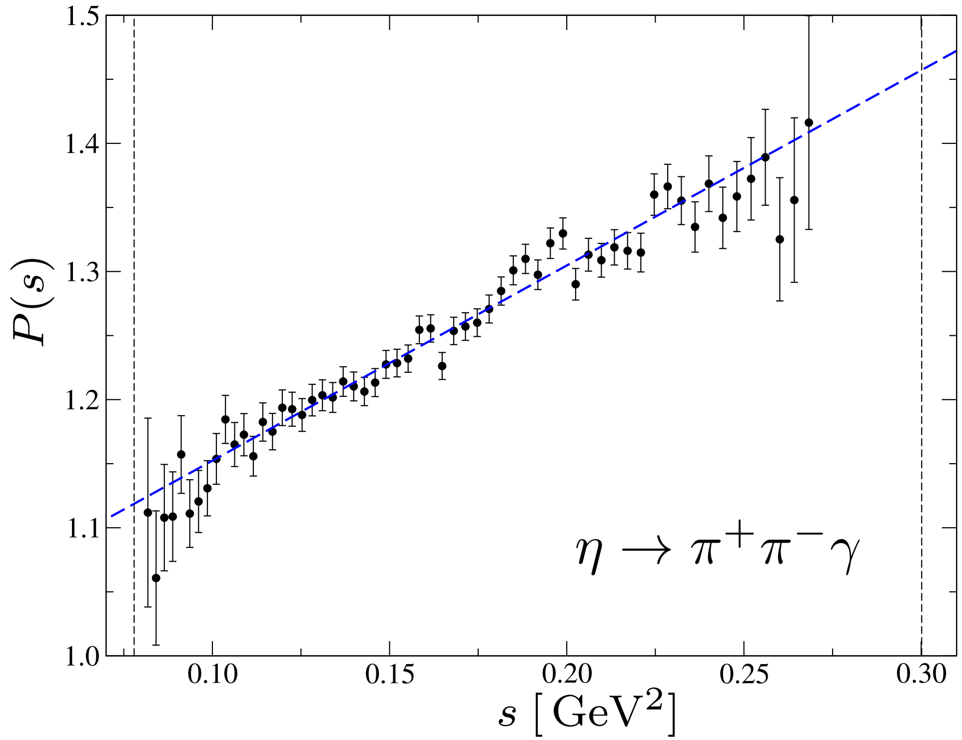


García-Martín et al. 2011

- strong constraints on data from analyticity and unitarity!

Final-state universality: $\eta,\ \eta' \to \pi^+\pi^-\gamma$

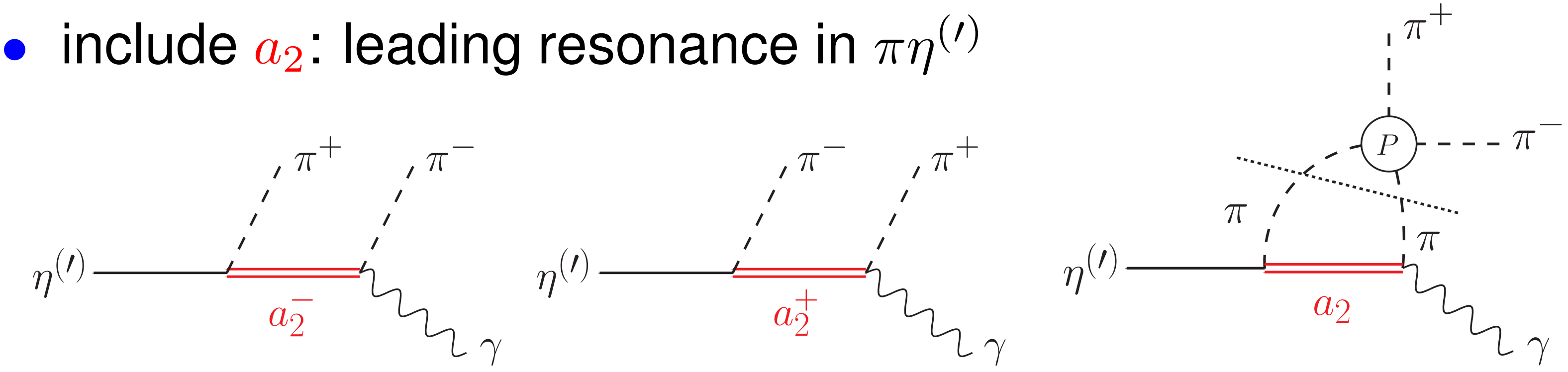
- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave
  $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{F}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(s) \times \Omega(s),\ \ P(s) = 1 + \alpha^{(\prime)} s,\ \ s = M^2_{\pi\pi}$
- divide data by pion form factor $\longrightarrow P(s)$ Stollenwerk et al. 2012

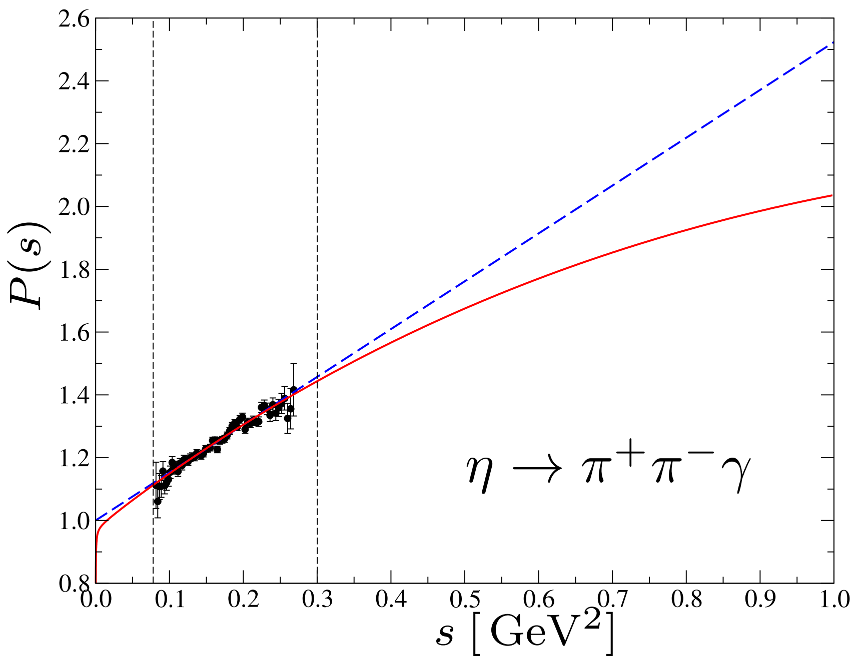


$\longrightarrow$ exp.: $\alpha_{\text{KLOE}} = (1.52 \pm 0.06)\ \text{GeV}^{-2}$ cf. KLOE 2013

$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with left-hand cuts

- include a_2: leading resonance in $\pi\eta^{(\prime)}$



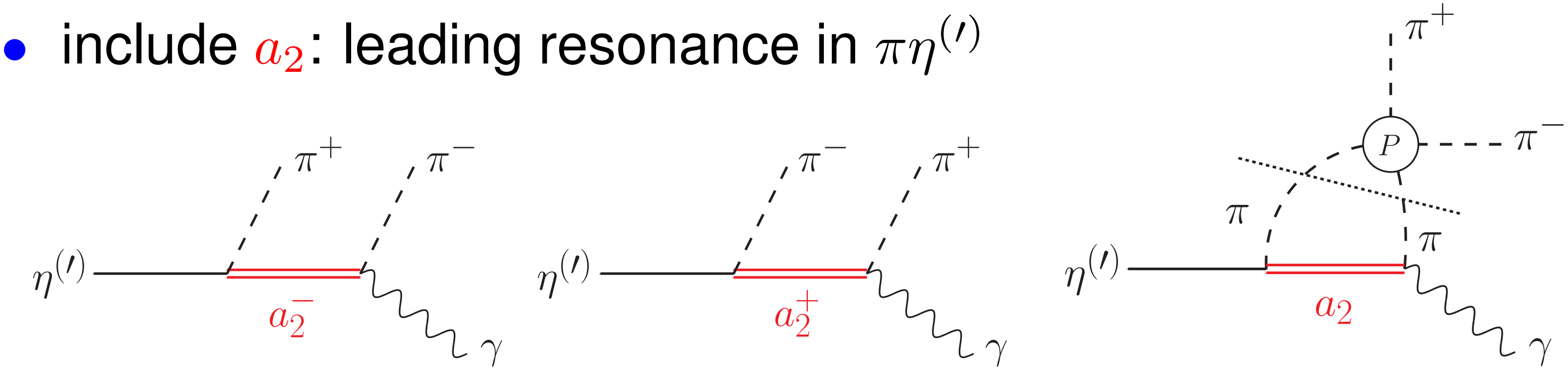


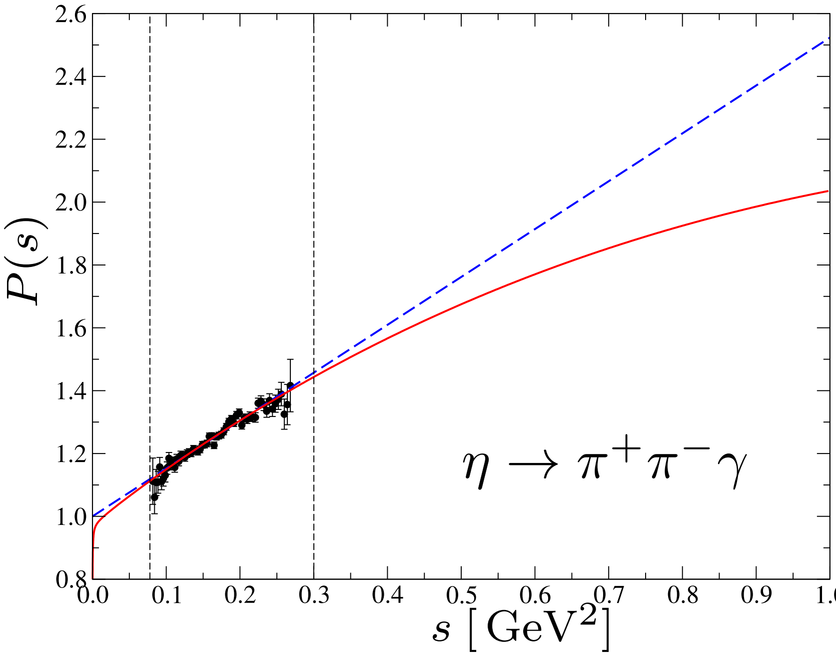
KLOE 2013; BK, Plenter 2015

- induces curvature in $P(s)$

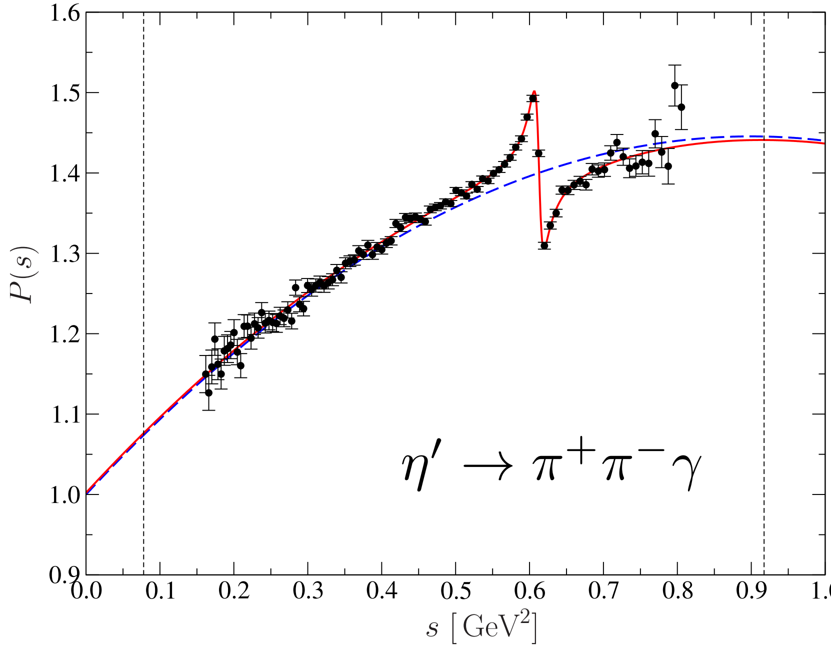
$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with left-hand cuts

- include a_2: leading resonance in $\pi\eta^{(\prime)}$





KLOE 2013; BK, Plenter 2015



BESIII 2017; Hanhart et al. 2017

- induces curvature in $P(s)$
- curvature, plus ρ–ω mixing

Part III:

π^0-pole contribution

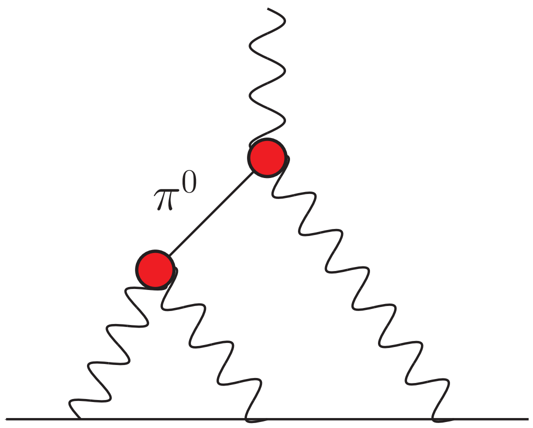
Hadronic light-by-light: the π^0 pole

- largest individual HLbL contribution:

  π^0-pole term

  singly / doubly virtual transition form factors (TFFs)

  $F_{\pi^0\gamma^*\gamma^*}(q^2, 0)$ and $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$



- normalisation fixed by Wess–Zumino–Witten (WZW) anomaly (= full leading-order ChPT prediction):

$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi}$$

$\longrightarrow$ measured at 0.75% (F_π: pion decay constant) PrimEx 2020

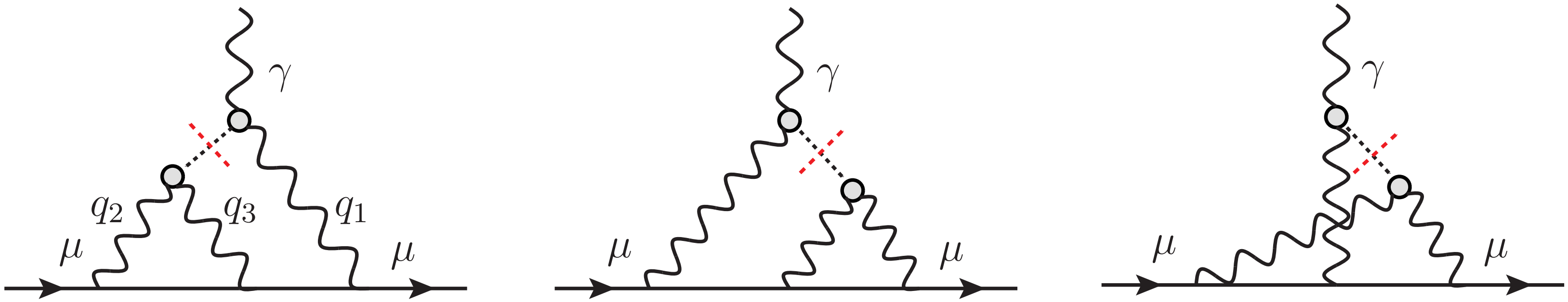
- two-loop integral with constant form factors does not converge

  $\longrightarrow$ no full prediction from e.g. chiral perturbation theory

  $\longrightarrow$ sensible high-energy behaviour required!

Pion-pole contribution to a_μ

- 3-dimensional integral representation: Jegerlehner, Nyffeler 2009

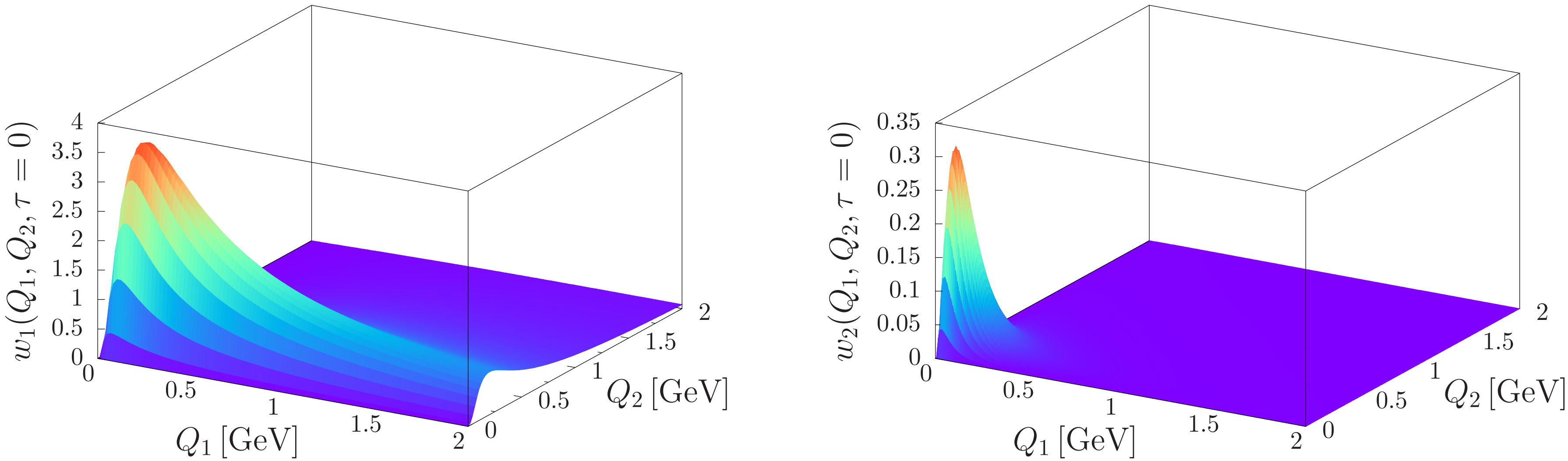


$$a_\mu^{\pi^0\text{-pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau$$
$$\times \Big[w_1(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0)$$
$$+ w_2(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_3^2,0) \Big]$$

- $w_{1/2}(Q_1,Q_2,\tau)$: kinematical weight functions, $\tau = \cos\theta$
- $F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)$: space-like on-shell π^0 TFF

Pion-pole contribution to a_μ

- weight functions $w_{1/2}(Q_1, Q_2, \tau = 0)$: Nyffeler 2016



- concentrated for $Q_i \leq 0.5$ GeV
  $\longrightarrow$ pion-pole contribution dominantly from low-energy region
  $\longrightarrow$ pion transition form factor can be determined model-independently and with high precision using dispersion relations

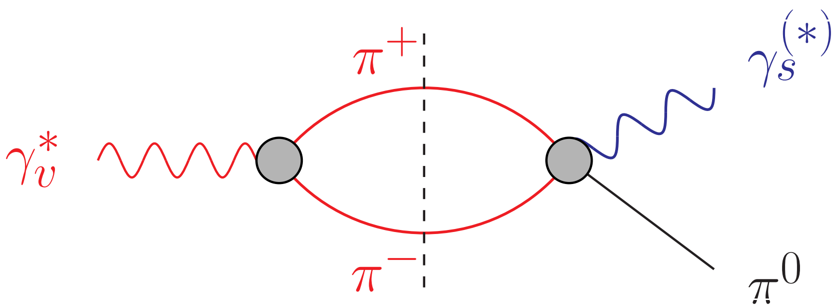
Dispersive analysis of $\pi^0 \to \gamma^*\gamma^*$

- isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$

- analyse the leading hadronic intermediate states:

Hoferichter et al. 2014



▷ isovector photon: 2 pions

$\propto$ pion vector form factor — well known from $e^+e^- \to \pi^+\pi^-$

$\times$ $\gamma^* \to 3\pi$ P-wave amplitude — Khuri–Treiman formalism

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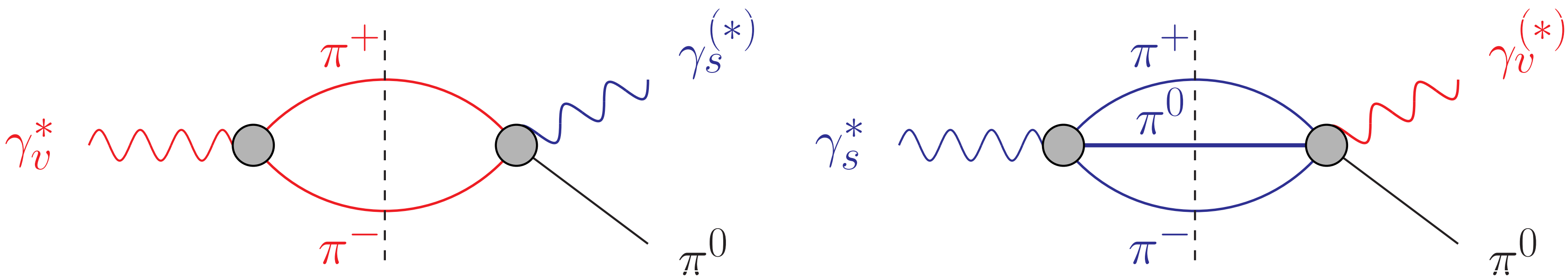

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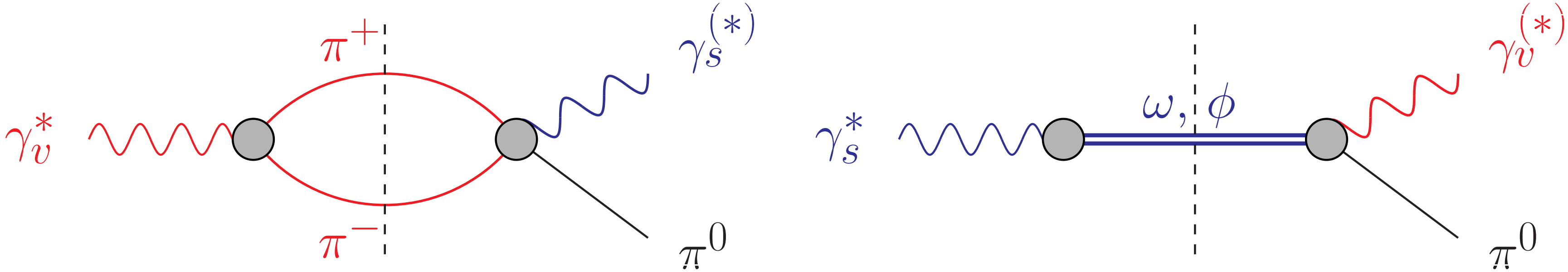

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  - $\times$ $\gamma^* \to 3\pi$ P-wave amplitude — Khuri–Treiman formalism
- ▷ isoscalar photon: 3 pions

  dominated by narrow resonances ω, ϕ

Khuri–Treiman representation $\gamma^* \to 3\pi$

- $\gamma^*(q) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ amplitude:

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma\, \mathcal{F}(s,t,u;q^2)$$

  s,t,u: pion–pion invariant masses, $s+t+u=q^2+3M_\pi^2$

- "reconstruction theorem": neglect discontinuities in F-waves...
  $\longrightarrow$ decomposition into crossing-symmetric isobars

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

- normalisation fixed from Wess–Zumino–Witten anomaly:

$$\mathcal{F}(0,0,0;0) = F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3}$$

- (s-channel) P-wave projection: $f_1(s,q^2) = \mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2)$
  $\hat{\mathcal{F}}(s,q^2)$: contribution from crossed channels $\mathcal{F}(t/u,q^2)$

Khuri–Treiman representation $\gamma^* \to 3\pi$

Unitarity relation for $\mathcal{F}(s, q^2 = \text{fixed})$:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i\{\underbrace{\mathcal{F}(s,q^2)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s,q^2)}_{\text{left-hand cut}}\} \times \theta(s-4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

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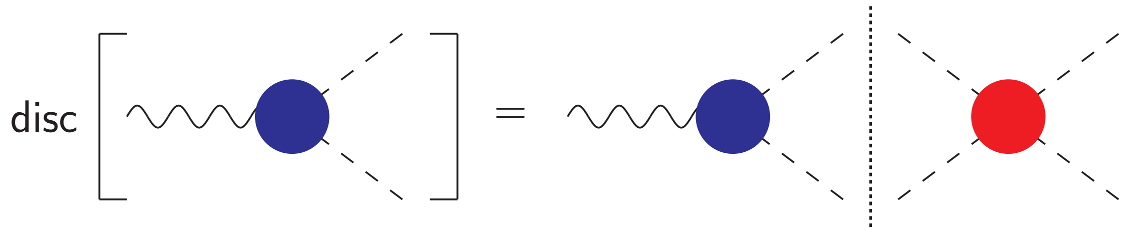
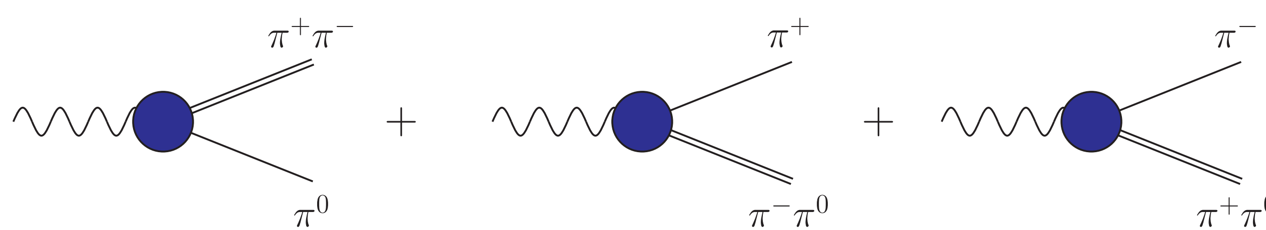


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s,q^2) = \Omega(s)\, a(q^2)\,, \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

$\mathcal{F}(s,t,u;q^2) =$



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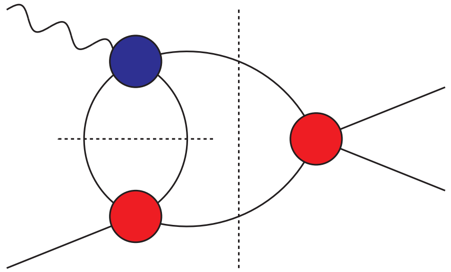
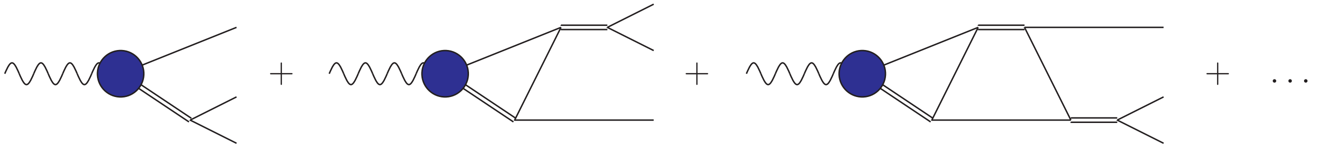


- inhomogeneities $\hat{\mathcal{F}}(s,q^2)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s,q^2) = \Omega(s)\left\{a(q^2) + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s,q^2) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z),q^2\big)$$

$$\mathcal{F}(s,q^2) =$$



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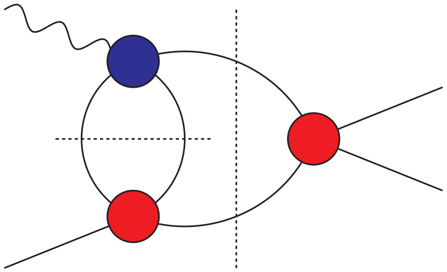

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- crossed-channel scatt. between s-, t-, u-channel (left-hand cuts)

Dispersive representation $\gamma^* \to 3\pi$

- parameterisation of subtraction function $a(q^2)$

  $\longrightarrow$ to be fitted to $e^+e^- \to 3\pi$ cross section data:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi}\int_{\rm thr}^{\infty} ds' \frac{\operatorname{Im}\mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

- $\mathcal{A}(q^2)$ includes resonance poles:

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\Gamma_V(q^2)} \qquad V = \omega, \phi, \omega', \omega''$$

  c_V real

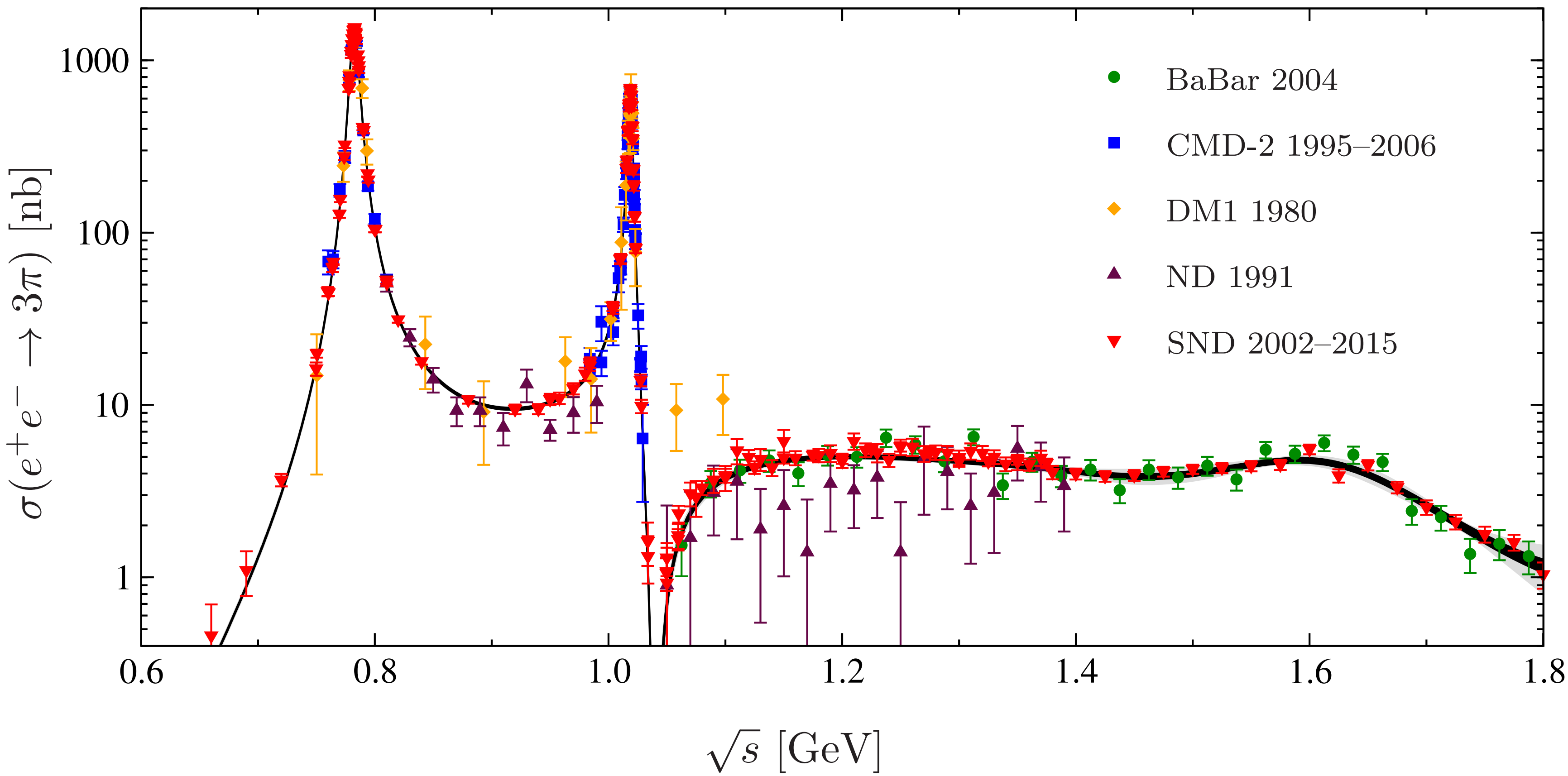
- conformal polynomial (inelasticities)

$$C_n(q^2) = \sum_{i=1}^{n} c_i \Big(z(q^2)^i - z(0)^i \Big), \qquad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}}$$

- exact implementation of $\gamma^* \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\operatorname{Im} a(s')}{s'}$$

Fit results $e^+e^- \to 3\pi$ data up to 1.8 GeV

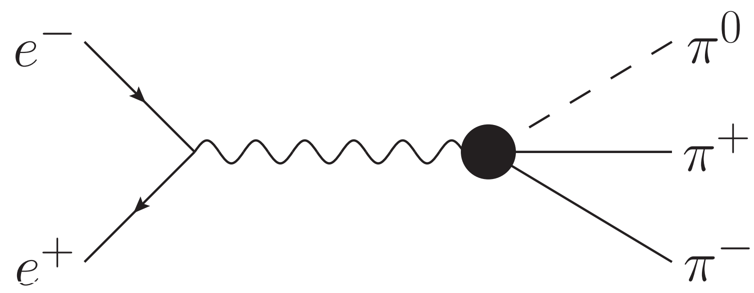


Hoferichter, Hoid, BK 2019

$\longrightarrow$ updated in Hoferichter, Hoid, BK, Schuh 2023 to include isospin breaking

- black / gray bands represent fit and total uncertainties
- vacuum polarisation removed from the cross section

From $e^+e^- \to 3\pi$ to $e^+e^- \to \pi^0\gamma^*$

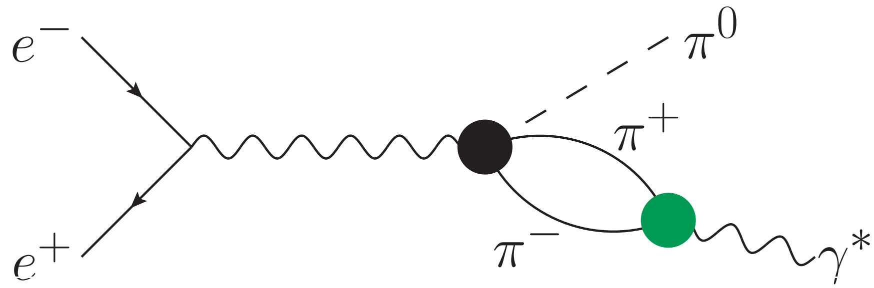


- amplitude for $e^+e^- \to 3\pi \quad \propto \; \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$

$$\mathcal{F}(s,q^2) = \Omega(s)\left\{ a(q^2) + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)} \right\}$$

subtraction function $a(q^2)$ adjusted to reproduce $e^+e^- \to 3\pi$

From $e^+e^- \to 3\pi$ to $e^+e^- \to \pi^0\gamma^*$



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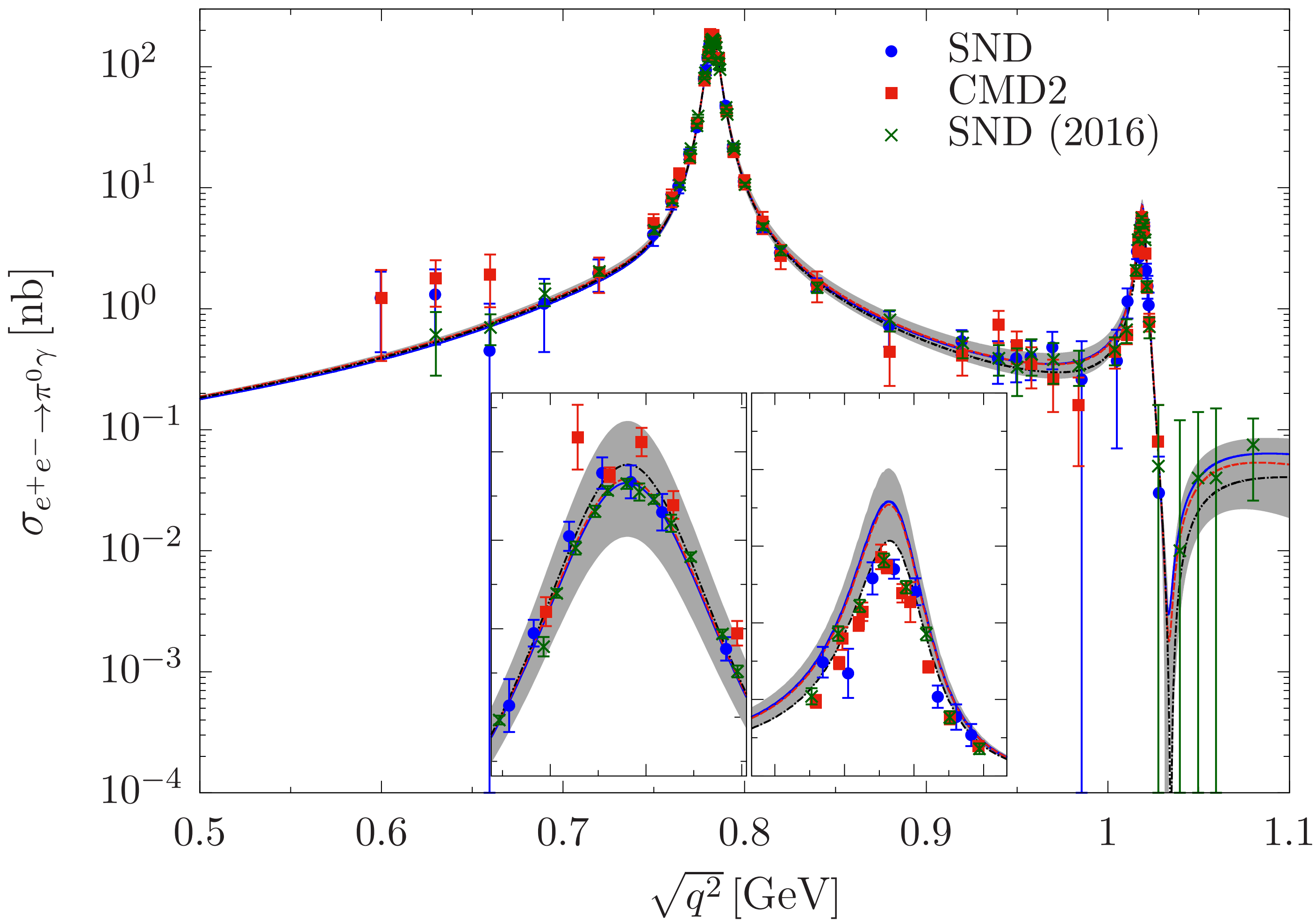
  subtraction function $a(q^2)$ adjusted to reproduce $e^+e^- \to 3\pi$

- fit to $e^+e^- \to 3\pi$ data

  combine with $e^+e^- \to \pi^+\pi^-$ form factor

  $\longrightarrow$ prediction for $e^+e^- \to \pi^0\gamma^{(*)}$

Comparison to $e^+e^- \to \pi^0\gamma$ data



Hoferichter, Hoid, BK, Leupold, Schneider 2018

- "prediction"—no further parameters adjusted
- timelike π^0 transition form factor data very well reproduced

Asymptotics and pQCD constraints (1)

- so far: dispersion relation based on (dominant) 2π, 3π
  $\longrightarrow$ high precision at low energies
- double-spectral-function representation:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dx \int_{s_{\rm thr}}^{\infty} dy \frac{\rho^{\rm disp}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

$$\rho^{\rm disp}(x,y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}} {\rm Im}\left[F_\pi^{V*}(x) f_1(x,y)\right] + [x \leftrightarrow y]$$

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- asymptotically: pion wave function $\phi_\pi(x) = 6x(1-x)+\ldots$

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = -\frac{2F_\pi}{3}\int_0^1 dx \frac{\phi_\pi(x)}{xq_1^2+(1-x)q_2^2} + \mathcal{O}(Q^{-4})$$

implies asymptotically — Brodsky, Lepage 1979–1981

$$F_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2) \sim \frac{2F_\pi}{3Q^2}\,, \qquad F_{\pi^0\gamma^*\gamma^*}(-Q^2,0) \sim \frac{2F_\pi}{Q^2}$$

$\longrightarrow$ rewrite this as double-spectral representation $\rho^{\rm pQCD}(x,y)$

Khodjamirian 1999; Hoferichter et al. 2018

Asymptotics and pQCD constraints (2)

- dispersion-theoretical $\rho^{\text{disp}}(x,y)$ at low energies $x,y \leq s_m$
- doubly-asymptotic $\rho^{\text{pQCD}}(x,y)$ for $x,y > s_m$
  $\longrightarrow$ does not contribute to singly-virtual TFF

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_\text{m}} dx \int_0^{s_\text{m}} dy \frac{\rho^{\text{disp}}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_\text{m}}^{\infty} dx \int_{s_\text{m}}^{\infty} dy \frac{\rho^{\text{pQCD}}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

- pQCD piece alone: $\quad F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,-Q^2\big) = \dfrac{2F_\pi}{3Q^2} + \mathcal{O}\big(Q^{-4}\big)$

  dispersive part: $\quad \dfrac{1}{\pi^2}\displaystyle\int_0^{s_\text{m}} dx \int_0^{s_\text{m}} dy \frac{\rho^{\text{disp}}(x,y)}{(x+Q^2)(y+Q^2)} = \mathcal{O}\big(Q^{-4}\big)$

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$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_{\rm m}} dx \int_0^{s_{\rm m}} dy \frac{\rho^{\rm disp}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_{\rm m}}^{\infty} dx \int_{s_{\rm m}}^{\infty} dy \frac{\rho^{\rm pQCD}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

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  dispersive part: $\dfrac{1}{\pi^2}\displaystyle\int_0^{s_{\rm m}} dx \int_0^{s_{\rm m}} dy \frac{\rho^{\rm disp}(x,y)}{(x+Q^2)(y+Q^2)} = \mathcal{O}\big(Q^{-4}\big)$

- anomaly and Brodsky–Lepage: $\rho^{\rm disp}(x,y)$ fulfils two sum rules

  $\longrightarrow$ add effective pole: $\rho^{\rm eff} = \dfrac{g_{\rm eff}}{4\pi^2 F_\pi}\pi^2 M_{\rm eff}^4 \delta\big(x-M_{\rm eff}^2\big)\delta\big(y-M_{\rm eff}^2\big)$

  find $g_{\rm eff} \sim 10\%$ (small), $M_{\rm eff} \sim 1.5\ldots2.0\,{\rm GeV}$ (reasonable)

Uncertainties in the π^0-pole contribution

Normalisation

- uncertainty on $\pi^0 \to \gamma\gamma$ $\pm 1.5\%$ PrimEx 2020

Dispersive input

- different $\pi\pi$ phase shift inputs:
  - ▷ Bern vs. Madrid Colangelo et al. 2011, García-Martín et al. 2011
  - ▷ effective form factor phase (incl. ρ', ρ'') Schneider et al. 2012
- cutoff in Khuri–Treiman integrals $1.8 \ldots 2.5\,\mathrm{GeV}$

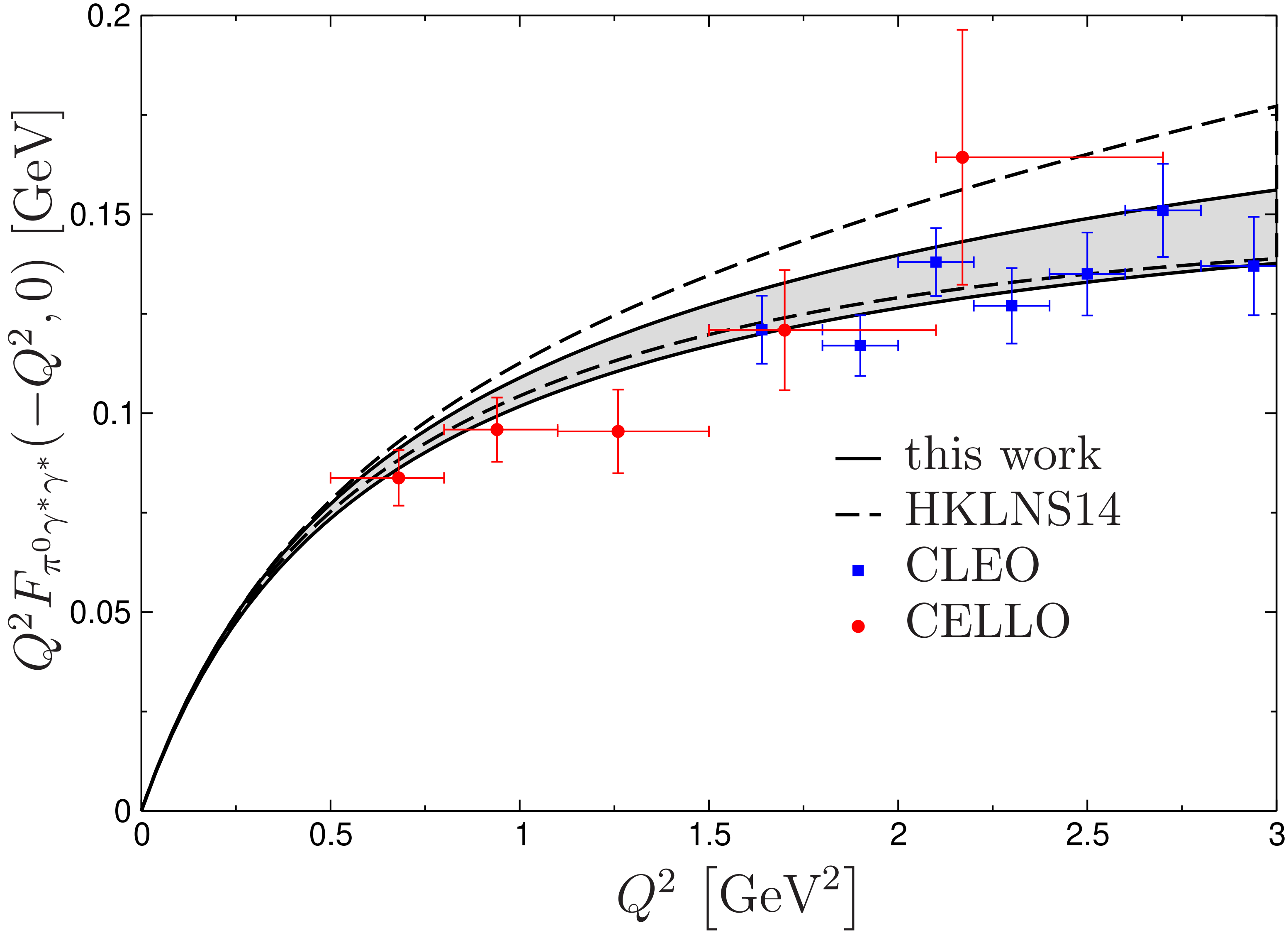
Brodsky–Lepage limit uncertainty

- allow for $^{+20\%}_{-10\%}$, 3σ band around data BaBar 2009, Belle 2012

Onset of pQCD asymptotics

- vary $s_m = 1.7(3)$GeV^2

Results: singly-virtual



Hoferichter, Hoid, BK, Leupold, Schneider 2018

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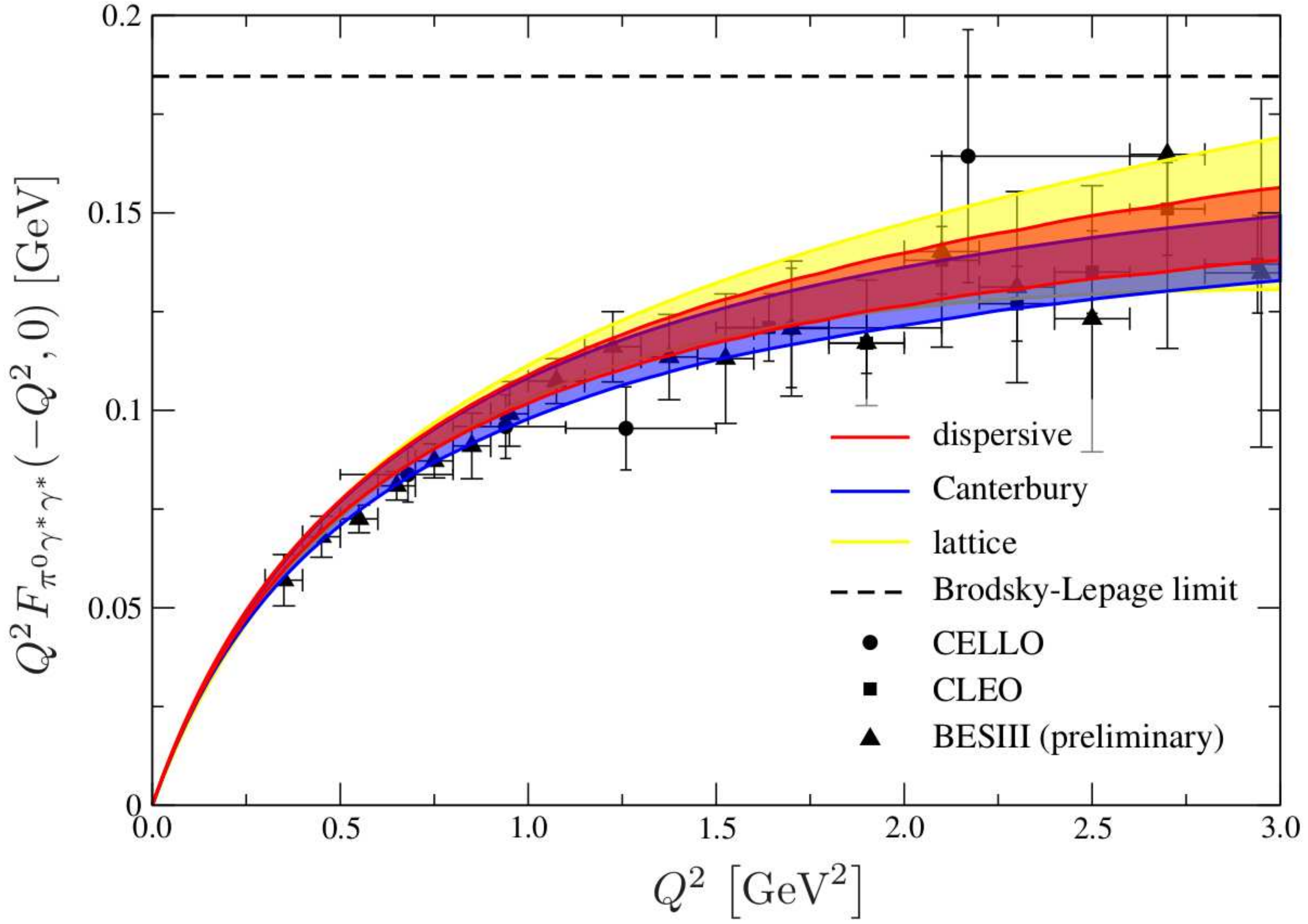
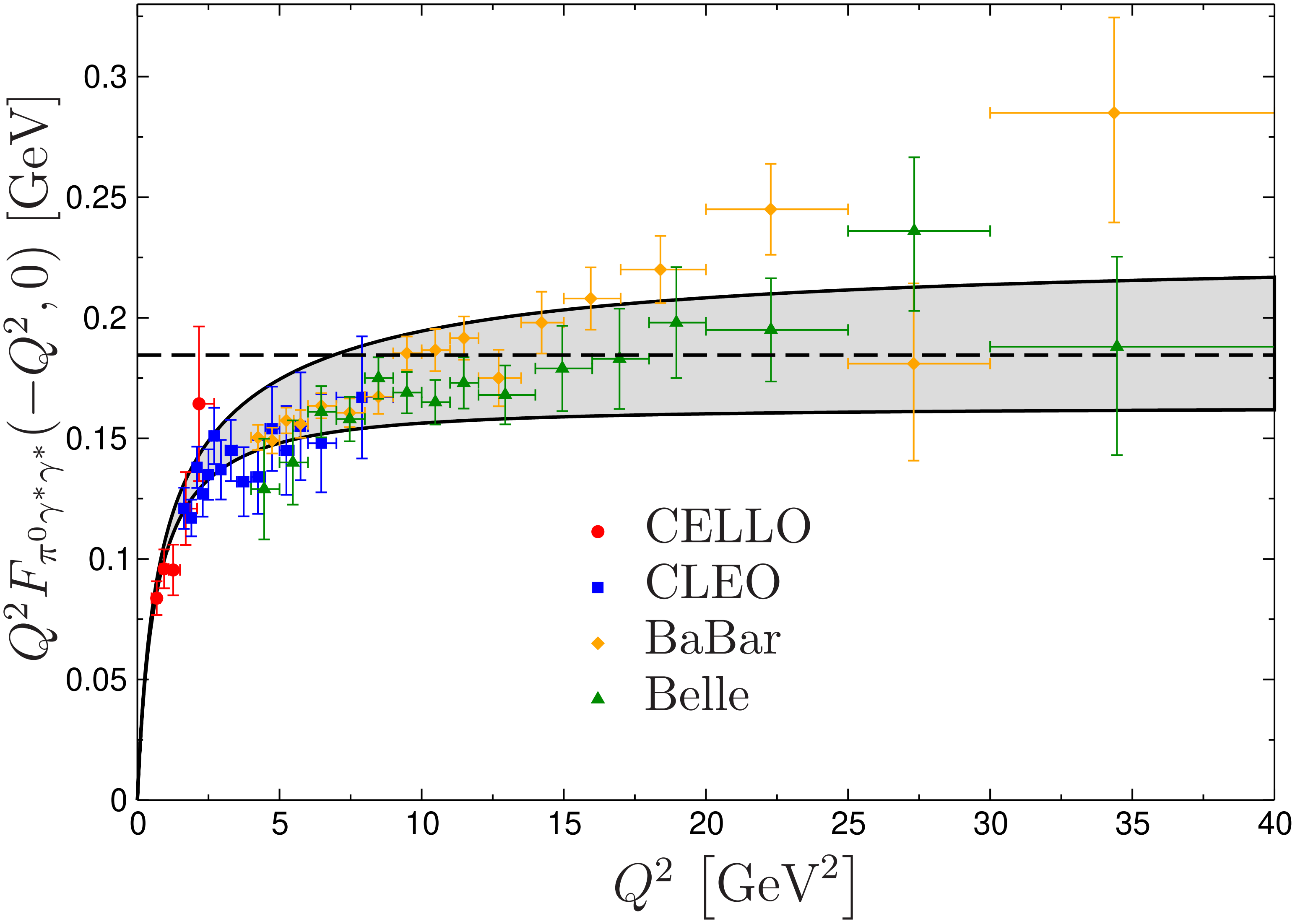


Aoyama et al. 2020

Results: singly-virtual



Hoferichter, Hoid, BK, Leupold, Schneider 2018

Result: $(g-2)_\mu$ from π^0 pole

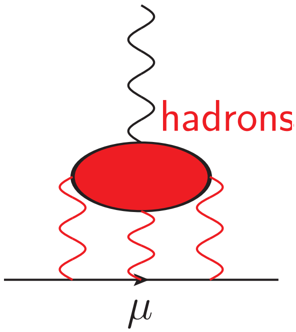
Final result for the π^0-pole contribution $[10^{-11}]$

63.0 ± 0.9	chiral anomaly / $\pi^0 \to \gamma\gamma$
± 1.1	dispersive input
${}^{+2.2}_{-1.4}$	Brodsky–Lepage
± 0.6	onset of pQCD contribution s_m
$= 63.0\, {}^{+2.7}_{-2.1}$	Hoferichter, Hoid, BK, Leupold, Schneider 2018

- model-independent, data-driven determination with all physical low- and high-energy constraints implemented
- perfectly consistent with
  - ▷ Padé approxim. $63.6(2.7) \times 10^{-11}$ Masjuan, Sánchez-Puertas 2017
  - ▷ lattice $62.3(2.3) \times 10^{-11}$ Gérardin et al. 2019
  - ▷ Dyson–Schw. $62.6(1.3) \times 10^{-11}$ Eichmann et al. 2019

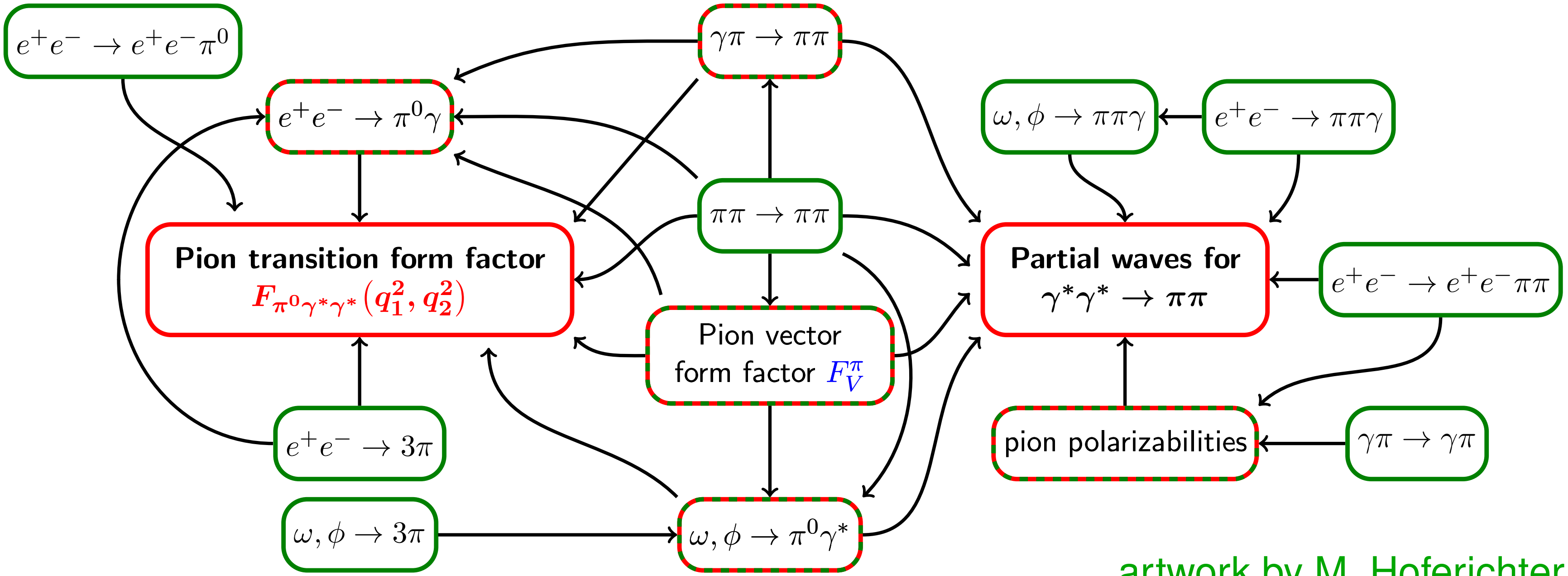
"White Paper" summary HLbL

hadronic state	$a_\mu^{\text{HLbL}}\,[10^{-11}]$	
pseudoscalar poles	$93.8^{+4.0}_{-3.6}$	η, η': Masjuan, Sánchez-Puertas 2017
pion box	$-15.9(2)$	Colangelo et al. 2017
S-wave $\pi\pi$ rescatt.	$-8(1)$	Colangelo et al. 2017
kaon box	$-0.5(1)$	
scalars+tensors $\gtrsim 1\,\text{GeV}$	$\sim -1(3)$	
axial vectors	$\sim 6(6)$	
short distance	$\sim 15(10)$	
heavy quarks	$\sim 3(1)$	
total	$92(19)$	Aoyama et al. 2020



$\longrightarrow$ further need for improvement to reach 10% accuracy for a_μ^{HLbL}

Summary: dispersion relations for HLbL



artwork by M. Hoferichter

Dispersive analyses of π^0, $\eta^{(\prime)}$ transition form factors:

- QCD constraints + high-precision data on

  $e^+e^- \to \pi^+\pi^-(\pi^0)$ var. / $\eta \to \pi^+\pi^-\gamma$ KLOE / $\eta' \to \pi^+\pi^-\gamma$ BESIII

  allow for high-precision dispersive predictions of π^0, $\eta^{(\prime)} \to \gamma^*\gamma^{(*)}$

Main challenges for HLbL at 10% accuracy:

- matching low and high energies various

Spares

Form factors constrained by analyticity and unitarity

For illustration, let's briefly derive the Omnès solution!

- use $F_\pi^V(s) = P(s)\Omega(s)$: $\Omega(s)$ free of zeros, $\Omega(0) = 1$
- begin with the following simple manipulations:

$$\begin{aligned}
\text{disc}\,\Omega(s) &= 2i\,\Omega(s+i\epsilon) \times \sin\delta(s)\,e^{-i\delta(s)} \\
\Omega(s+i\epsilon) - \Omega(s-i\epsilon) &= \Omega(s+i\epsilon) \times (1 - e^{-2i\delta(s)}) \\
\Omega(s+i\epsilon) &= \Omega(s-i\epsilon) \times e^{2i\delta(s)} \\
\text{disc}\,\log\Omega(s) &= 2i\,\delta(s)
\end{aligned}$$

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\text{disc}\,\log\Omega(s) &= 2i\,\delta(s)
\end{aligned}$$

- this allows to write a dispersion relation for disc $\log\Omega(s)$:

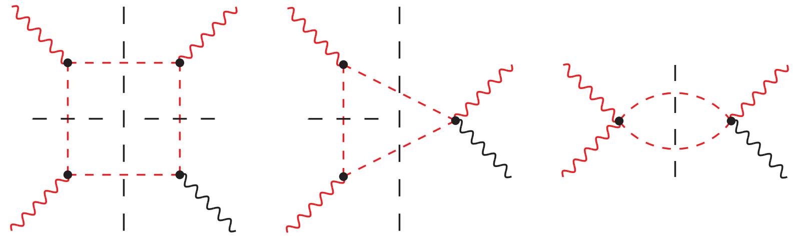
$$\begin{aligned}
\log\Omega(s) &= \frac{s}{2\pi i}\int_{4M_\pi^2}^\infty ds' \frac{\text{disc}\,\log\Omega(s')}{s'(s'-s)} = \frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta(s')}{s'(s'-s)} \\
\Omega(s) &= \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta(s')}{s'(s'-s)}\right\}
\end{aligned}$$

Pion loop contributions / $\pi\pi$ intermediate states

Colangelo, Hoferichter, Procura, Stoffer 2017 [figs. courtesy of M. Hoferichter]

Decompose light-by-light scattering tensor $\Pi_{\mu\nu\lambda\sigma}$ into

- form factor scalar QED part $\longrightarrow$ preserves gauge invariance

$$\Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2)\times \Big[$$  $$\Big]$$

$$\longrightarrow \boxed{a_\mu^{\pi\text{-box}} = -15.9(2)\times 10^{-11}}$$

Pion loop contributions / $\pi\pi$ intermediate states

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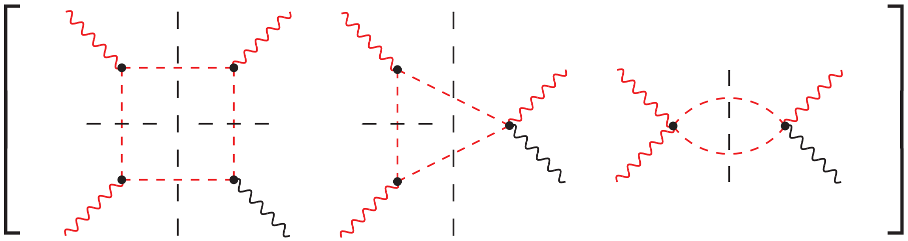
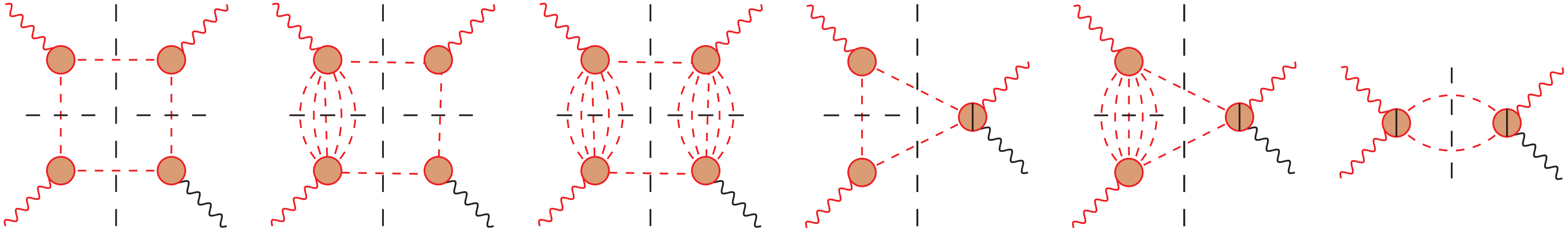


$$\longrightarrow \boxed{a_\mu^{\pi\text{-box}} = -15.9(2)\times 10^{-11}}$$

- + remainder $\bar{\Pi}_{\mu\nu\lambda\sigma}$ expanded in $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves



organised according to left-hand-cut structure

S-wave rescattering $\longrightarrow \boxed{a_\mu^{\pi\pi,S\text{-wave}} = -8(1)\times 10^{-11}}$

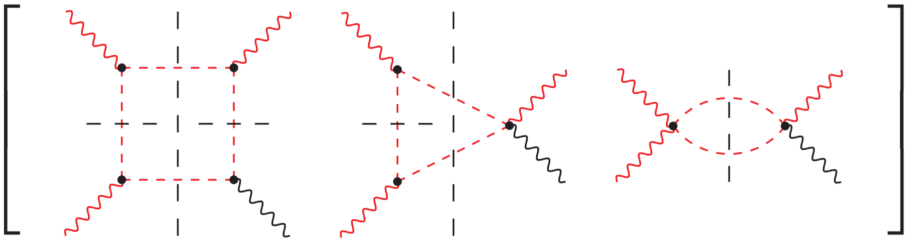
Pion loop contributions / $\pi\pi$ intermediate states

Colangelo, Hoferichter, Procura, Stoffer 2017 [figs. courtesy of M. Hoferichter]

Decompose light-by-light scattering tensor $\Pi_{\mu\nu\lambda\sigma}$ into

- form factor scalar QED part $\longrightarrow$ preserves gauge invariance

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2)\times \left[\quad\right]$$



$\longrightarrow$ $a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$

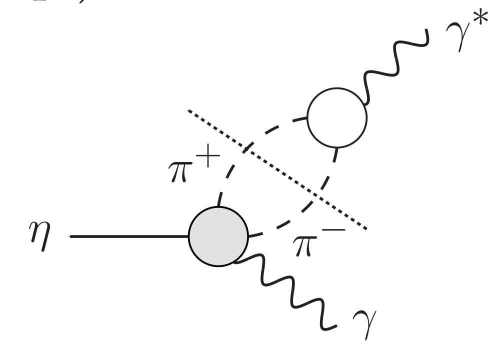
- + remainder $\bar{\Pi}_{\mu\nu\lambda\sigma}$ expanded in $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves
- contains automatically
  - ▷ polarisability effects — Engel, Patel, Ramsey-Musolf 2012
  - ▷ $\pi\pi$ resonances: $f_0(500)$ [$f_2(1270)$]
  - ▷ can be extended to $K\bar{K}$ ($\longrightarrow f_0(980)$)

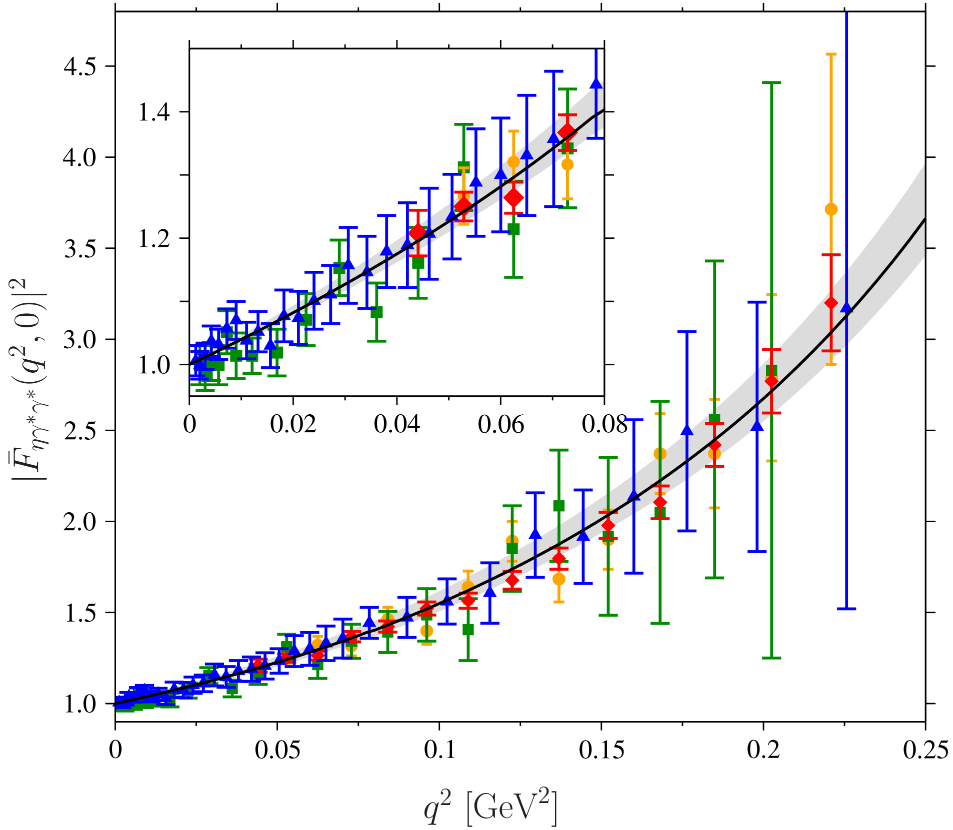
    Danilkin, Deineka, Vanderhaeghen 2019; Danilkin, Hoferichter, Stoffer 2021

Transition form factor $\eta \to \gamma^*\gamma$

Hanhart et al. 2013, BK, Plenter 2015

$$F_{\eta\gamma^*\gamma}(q^2,0) = F_{\eta\gamma\gamma} + \frac{q^2}{12\pi^2}\int_{4M_\pi^2}^{\infty} dt\, \frac{q_\pi^3(t)[F_\pi^V(t)]^* F_{\eta\pi\pi\gamma}(t)}{t^{3/2}(t-q^2)} + \Delta F^{I=0}_{\eta\gamma^*\gamma}(q^2,0)\ [\longrightarrow \text{VMD}]$$





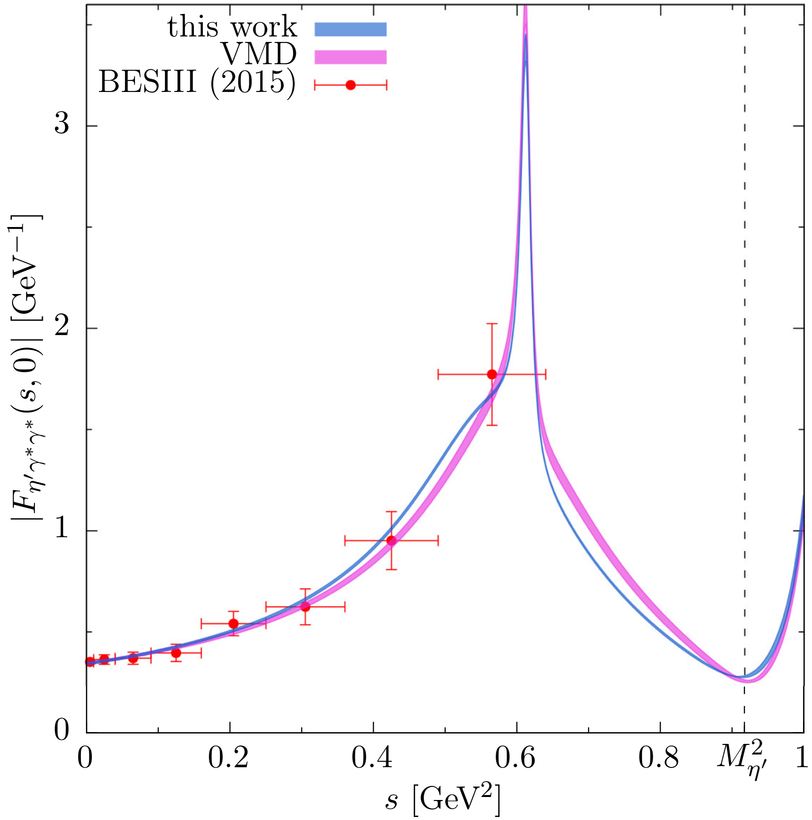
$\longrightarrow$ statistical advantage of hadronic $\eta \to \pi^+\pi^-\gamma$ over direct $\eta \to \ell^+\ell^-\gamma$ (rate suppressed $\propto \alpha_{\text{QED}}^2$)

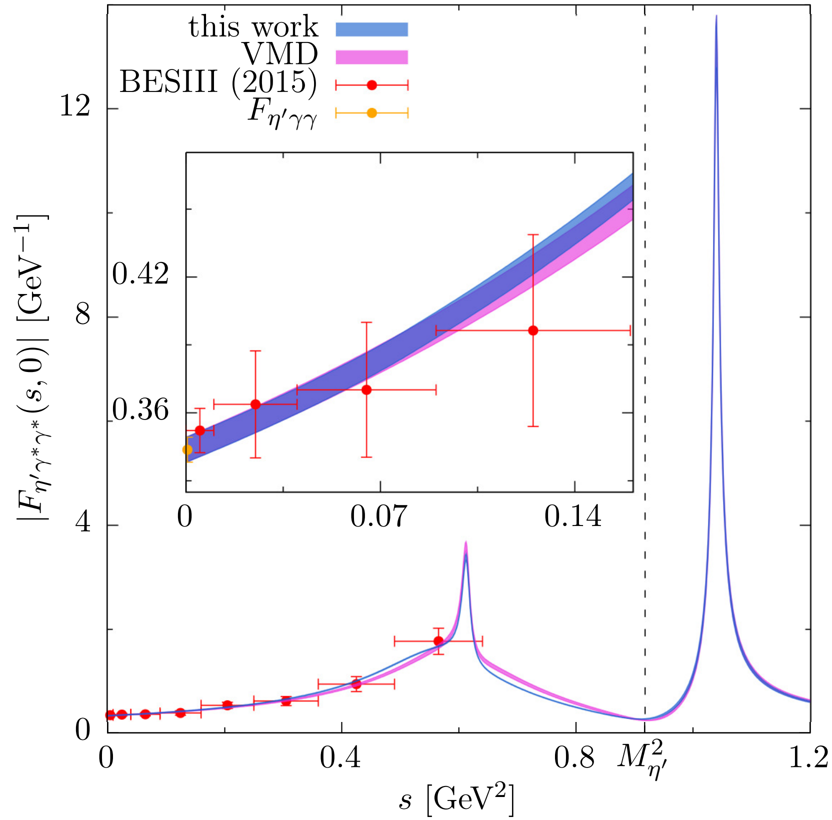
data: NA60 2009, 2016
A2 2014, 2017

Transition form factor $\eta' \to \gamma^*\gamma$

$$F_{\eta'\gamma^*\gamma}(q^2,0) = F_{\eta'\gamma\gamma} + \frac{q^2}{12\pi^2}\int_{4M_\pi^2}^{\infty} dt\, \frac{q_\pi^3(t)[F_\pi^V(t)]^* F_{\eta'\pi\pi\gamma}(t)}{t^{3/2}(t-q^2)}$$

$$+\,\Delta F_{\eta\gamma^*\gamma}^{I=0}(q^2,0)\ [\text{VMD}]\ +\ [\text{consistent isospin breaking}]$$





Holz, Hanhart, Hoferichter, BK 2022; data: BESIII 2015