Prediction for a four-neutron resonance

A. M. Shirokov,[1,2,3,*] G. Papadimitriou,[4,†] A. I. Mazur,[3] I. A. Mazur,[3] R. Roth,[5] and J. P. Vary[2,‡]

[1] *Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow, 119991, Russia*
[2] *Department of Physics and Astronomy, Iowa State University, Ames, IA 50011-3160, USA*
[3] *Pacific National University, 136 Tikhookeanskaya st., Khabarovsk 680035, Russia*
[4] *Nuclear and Chemical Science Division, Lawrence Livermore National Laboratory, Livermore, CA 94551, USA*
[5] *Institut für Kernphysik, Technische Universität Darmstadt, 64289 Darmstadt, Germany*

We utilize various *ab initio* approaches to search for a low-lying resonance in the four-neutron ($4n$) system using the JISP16 realistic NN interaction. Our most accurate prediction is obtained using a J-matrix extension of the No-Core Shell Model and suggests a $4n$ resonant state at an energy near $E_r = 0.8$ MeV with a width of approximately $\Gamma = 1.4$ MeV.

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With interest sparked by a recent experiment [1] on the possibility of a resonant four neutron ($4n$) structure (see also [2] for a recent communication) and while awaiting for forthcoming experiments on the same system [3–5], we search for $4n$ (tetraneutron) resonances using the high precision nucleon-nucleon interaction JISP16 [6]. The experiment has found a candidate $4n$ resonant state with an energy of 0.83 ± 0.65 (stat) $\pm$ 1.25 (syst) MeV above the $4n$ disintegration threshold and with an upper limit of 2.6 MeV for the width. The $4n$ system was probed by studying the reaction between the bound ^{4}He nucleus and the weakly bound Helium isotope, ^{8}He. It has been shown [7] that the four neutrons in ^{8}He form a relatively compact geometry. Hence the experimental study of the ^{4}He+^{8}He collisions is a promising avenue for the isolation of the $4n$ subsystem.

The experimental quest for the very exotic $4n$ structure started almost fifteen years ago when the possibility of a bound $4n$ (or tetraneutron) was proposed [8] in ^{14}Be breakup reactions (^{14}Be $\to$ ^{10}Be + $4n$). This experimental result however has not been confirmed. Early calculations of the $4n$ system in a small basis [9] found it unbound by about 18.5 MeV. More recent state-of-the-art theoretical calculations have concluded that without altering fundamental characteristics of the nuclear forces [10], the tetraneutron should not be bound. More theoretical calculations were performed [11, 12], all of them agreeing that a bound tetraneutron is not supported by theory. Calculations performed in the complex energy plane to search of multi-neutron resonances within the Complex Scaling Method [13–15] give quantitatively similar results and point to the fact that the $4n$ resonance, if it exists, would have a very large width ($\sim$15 MeV), likely prohibitive for experimental detection. The tetraneutron could however exist if confined in a strong external field. In Nature, this would be the case of ^{8}He, where the nuclear mean-field is strong enough to confine the tetraneutron around the tightly bound α-core. Once the field is suddenly removed by knocking out ^{4}He, it is expected that the tetraneutron will disintegrate very fast due to its anticipated large width.

There is also a work [16] where the continuum response of the tetraneutron was studied. The outcome was that there exists a resonant-like structure at around 4–5 MeV above threshold, however this structure depends on the tetraneutron production reaction mechanism represented by the source term in this study, and the conclusion was that the $4n$ probably cannot be interpreted as a well-defined resonance but most probably as a few-body continuum response in a reaction.

Nevertheless, our current knowledge of nuclear interactions and many-body methods provide new opportunities to probe exotic states above thresholds. We are further motivated by the conclusion in Ref. [10] that even though the existence of a bound tetraneutron is ruled out, extrapolations of (artificialy) bound state results to the unbound regime, imply that there may be a $4n$ resonance at about 2 MeV above the four-neutron threshold.

A complete investigation of the tetraneutron as a resonant state, would consist of performing calculations of the actual experimental reaction ^{4}He(^{8}He,^{8}Be). However, such a realistic calculation is currently out of reach, though we are witnessing the first steps for such theoretical calculations to become a reality [17, 18].

We treat the $4n$ system with a realistic non-relativistic Hamiltonian which consists of the kinetic energy and the realistic inter-neutron potential defined by the JISP16 interaction [6]. We solve for the $4n$ energies by employing basis expansion techniques for the Hamiltonian. Specifically, we employ the No-Core Shell Model (NCSM) [19] and artificially bind the $4n$ system by scaling the interaction to track its lowest state as a function of that scaling. We also employ the No-Core Gamow Shell Model (NCGSM) [20, 21] which provides resonant parameters directly in the complex energy plane. Finally, we extend NCSM using the Single-State Harmonic Oscillator Representation of Scattering Equations (SS-HORSE) for-

* shirokov@nucl-th.sinp.msu.ru
† papadimitrio1@llnl.gov
‡ jvary@iastate.edu

malism [22, 23] for calculations of the S-matrix resonant parameters.

First, to get an estimate of whether JISP16 can provide a $4n$ resonant state, we exploit the technique suggested in Ref. [10] and perform pure NCSM calculations by constructing an artificially bound $4n$ system by scaling up the NN interaction. Our extrapolations to the unbound regime are in quantitative agreement with Ref. [10] that predicts a resonance at around 2 MeV above threshold but without any indication of the width. We tried also a much more elaborate technique of Analytic Continuation in the Coupling Constant (ACCC) [24, 25]. The ACCC requires exact results for the $4n$ energy with scaled interactions while NCSM provides only variational energy upperbounds; extrapolations to the infinite basis space appear to lack the precision needed for a definite prediction of the resonance energy and width.

In order to shed further light on a possible $4n$ resonance, we solve the NCGSM with the JISP16 interaction. In the NCGSM one employs a basis set that is spanned by the Berggren states [26] which includes bound, resonant and non-resonant states; they correspond to solutions of the single particle (s. p.) Schrödinger equation obeying outgoing (bound-resonant states) and scattering (non-resonant states) boundary conditions. In this basis the Hamiltonian matrix becomes complex symmetric and its eigenvalues acquire both real and imaginary parts. The real part is identical to the position of the resonant state above the threshold and the imaginary part is related to its width $\Gamma = -2\,\mathrm{Im}(E)$.

We adopt the basis provided by a Woods–Saxon (WS) potential for a neutron in relative motion with a $3n$ system. We modify the WS parameters in a way that it will support a weakly bound $0s_{1/2}$ state and a resonant $0p_{3/2}$ state. For the $s_{1/2}$ and $p_{3/2}$ shells we include the $0s_{1/2}$ bound state, the $0p_{3/2}$ resonant state and the associated non-resonant states. We additionally include the $p_{1/2}$ real scattering continuum along the real momentum axis. We performed calculations for several WS parameterizations supporting both narrow and broad s. p. states. States with an angular momentum of $\ell > 2$ are taken as HO states. We retain states through the $3g_{9/2}$ shells. For our NCGSM calculations, the $\hbar\Omega$ parameter of the HO basis was varied from 4 MeV to 14 MeV. Due to the use of Berggren states for low angular momentum partial waves, we observe a weak dependence of the results on the $\hbar\Omega$ parameter.

For the $4n$ calculation we constructed Slater determinants allowing two neutrons to occupy continuum orbits, called the 2p-2h approximation. Taking the dependence on basis space parameters into account, the NCGSM results indicate a broad resonant state in the energy range $E_r \sim 2.5$ to 3 MeV above the $4n$ threshold and a width ranging from $\Gamma \sim 2.5$ to 6 MeV. These variations reflect the omission of additional p-h excitations. Nevertheless the real part of the resonance exhibits a robust character at the current level of p-h truncation, i. e., it is nearly independent of the WS parameterizations and independent of the frequency of the HO basis.

At the same time, we observe that the resonance energy decreases together with the width as the NCGSM basis increases. Getting the converged resonance pole position in this approach requires the NCGSM basis spaces beyond our current reach.

Finally, following the J-matrix formalism in scattering theory [27] as represented in the HORSE method [28], we extend the finite NCSM Hamiltonian matrix in the harmonic oscillator (HO) basis into the continuum by appending to it the infinite kinetic energy matrix.

For the kinetic energy extension of the NCSM Hamiltonian, we use the *democratic decay* approximation (also known as *true four-body scattering* or *4 → 4 scattering* suggested [29, 30]) and first applied to the tetraneutron problem [31–33] by Jibuti et al. Later it was exploited in other tetraneutron studies (see, e. g., Refs. [13, 16, 34, 35]). Democratic decay implies a description of the continuum using a complete hyperspherical harmonics (HH) basis. In practical applications, a limited set of HH is selected which is adequate for the systems like the $4n$ which has no bound subsystems.

The general theory of the democratic decay within the HORSE formalism was proposed in Ref. [36]. We use here the minimal approximation for the four-neutron decay mode, i. e., only HH with hyperspherical momentum $K = K_{\min} = 2$ are retained in the kinetic energy extension to the NCSM. This approximation relies on the fact that the decay in the hyperspherical states with $K > K_{\min}$ is strongly suppressed by a large hyperspherical centrifugal barrier $\frac{\mathscr{L}(\mathscr{L}+1)}{\rho^2}$ where the effective momentum $\mathscr{L} = K+3$ and the hyperradius $\rho^2 = \sum_{i=1}^{4}(\mathbf{r}_i - \mathbf{R})^2$, $\mathbf{R}$ is the tetraneutron center-of-mass coordinate and $\mathbf{r}_i$ are the coordinates of individual neutrons. Note, all possible HH are retained in the NCSM basis. The accuracy of this approximation was confirmed in studies of democratic decays in cluster models [37–40].

Realistic NN interactions require large NCSM basis spaces and extensive computational resources. For computational economy, we also adopt the SS HORSE approach [22, 23] where we calculate the $4 \to 4$ S-matrix $S(E)$ at one of the positive eigenenergies of the NCSM Hamiltonian, $E = E_\lambda$. In this case, the general HORSE formula for the S-matrix simplifies: expressing $S(E)$ through the $4 \to 4$ phase shifts $\delta(E)$,

$$S(E) = e^{2i\delta(E)}, \tag{1}$$

we obtain for the phase shifts [22, 23]

$$\delta(E_\lambda) = -\tan^{-1}\frac{S_{N^{tot}_{\max}+2,\mathscr{L}}(E_\lambda)}{C_{N^{tot}_{\max}+2,\mathscr{L}}(E_\lambda)}. \tag{2}$$

Here the maximal total quanta in the NCSM basis $N^{tot}_{\max} = N_{\min} + N_{\max}$, $N_{\min} = 2$ is the quanta of the

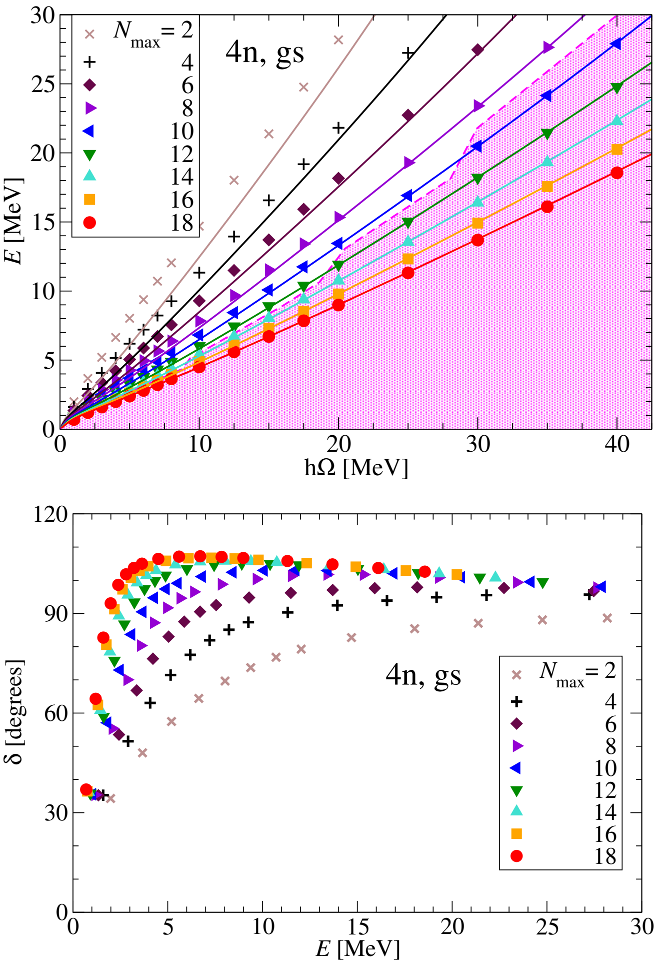

FIG. 1. NCSM results for the tetraneutron ground state energy obtained with various $N_{\max}$ (symbols) plotted as functions of $\hbar\Omega$ (upper panel). The shaded area shows the NCSM result selection for the S-matrix parametrization; the solid curves are obtained from the phase shifts parametrized with a single resonance pole by solving Eq. (2) for the eigenenergies at given $N_{\max}$ and $\hbar\Omega$ values. The $4 \to 4$ phase shifts obtained directly from the NCSM results using Eq. (2), are shown in the lower panel.

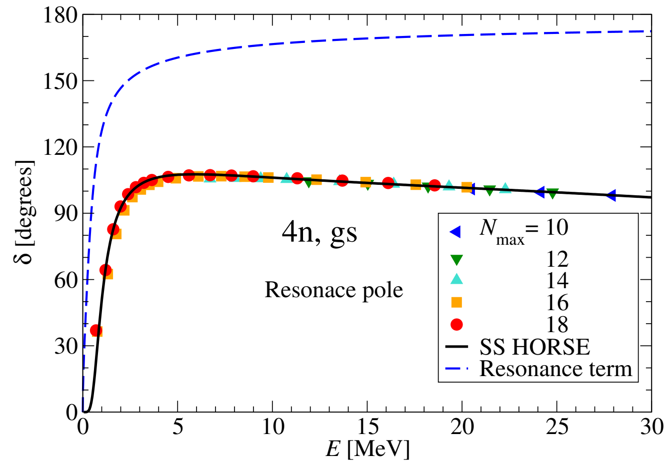

FIG. 2. The $4 \to 4$ scattering phase shifts: parametrization with a single resonance pole (solid line) and obtained directly from the selected NCSM results using Eq. (2) (symbols). The dashed line shows the contribution of the resonance term.

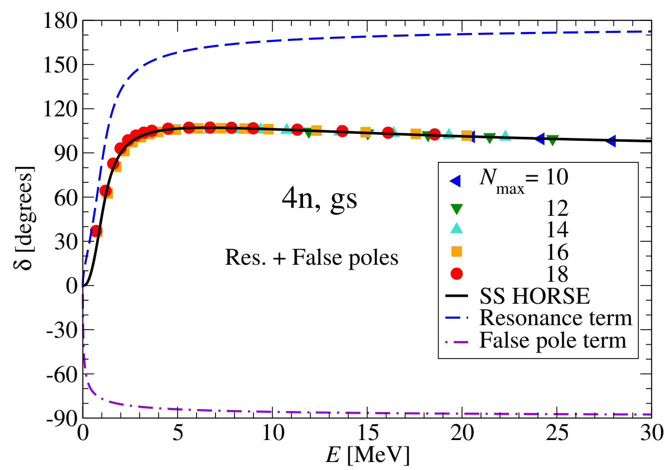

FIG. 3. Same as Fig. 2 but for the parametrization with resonance and false state poles. The dashed-dotted line shows the contribution of the false state pole term.

lowest possible oscillator state of the $4n$ system, $N_{\max}$ is the maximal excitation quanta in the NCSM basis; analytical expressions for the regular $S_{N\mathscr{L}}(E)$ and irregular $C_{N\mathscr{L}}(E)$ solutions of the free many-body Hamiltonian in the oscillator representation can be found elsewhere [36]. Varying $N_{\max}$ and $\hbar\Omega$ in the NCSM calculations, we obtain the phase shifts and S-matrix over an energy interval. Parametrizing the S-matrix in this energy interval, we obtain information about its nearby poles and hence resonances in the system.

The NCSM calculations were performed with $N_{\max} = 2, 4, \ldots, 18$ using the code MFDn [41, 42] and with $\hbar\Omega$ values, 1 MeV $\leq \hbar\Omega \leq$ 40 MeV. The results for the 0^+ tetraneutron ground state are shown in the upper panel of Fig. 1.

The convergence patterns of the NCSM SS-HORSE approach to the $4 \to 4$ phase shifts using Eq. (2) are shown in the lower panel of Fig. 1. We observe that the phase shifts tend to the same curve when $N_{\max}$ is increased. The convergence is first achieved at the higher energies while larger $N_{\max}$ yield converged phase shifts at smaller energies. We obtain nearly completely converged phase shifts at all energies with $N_{\max} = 16$ and 18.

We need only phase shifts close to convergence for the phase shift parametrization. Our selected NCSM eigenenergies are enclosed by the shaded area on the top panel of Fig. 1 since their resulting phase shifts form a single smooth curve (see Figs. 2 and 3).

We will describe now how we utilize the NCSM solutions within the SS-HORSE method in order to obtain resonance positions. Due to the S-matrix symmetry property, $S(k)=1/S(-k)$, and Eq. (1), the $4\to4$ phase shift $\delta(E)$ is an odd function of momentum k and its expansion in Taylor series of $\sqrt{E}\sim k$ includes only odd powers of $\sqrt{E}$:

$$\delta(E)=v_1\sqrt{E}+v_3\big(\sqrt{E}\big)^3+...+v_{11}\big(\sqrt{E}\big)^{11}+... \quad (3)$$

Furthermore, the $4\to4$ phase shifts at low energies, i. e., in the limit $k\to0$, should behave as $\delta\sim k^{2\mathscr{L}+1}$. Note, in our case, $\mathscr{L}=K_{\min}+3=5$, hence $v_1=v_3=...=v_9=0$ and expansion (3) starts at the 11^{th} power.

Supposing the existence of a low-energy resonance in the $4n$ system, we express the S-matrix as $S(E)=\Theta(E)\,S_r(E)$, where $\Theta(E)$ is a smooth function of energy E and $S_r(E)$ is a resonant pole term. The respective phase shift is

$$\delta(E)=\phi(E)+\delta_r(E), \quad (4)$$

where the pole contribution $\delta_r(E)$ takes the form

$$\delta_r(E)=-\tan^{-1}\big(a\sqrt{E}/(E-b^2)\big). \quad (5)$$

The resonance energy E_r and width Γ are expressed through parameters a and b entering Eq. (5) as

$$E_r=b^2-a^2/2, \qquad \Gamma=2a\sqrt{b^2-a^2/4}. \quad (6)$$

We use the following expression for the background phase:

$$\phi(E)=\frac{w_1\sqrt{E}+w_3\big(\sqrt{E}\big)^3+c\big(\sqrt{E}\big)^5}{1+w_2E+w_4E^2+w_6E^3+dE^4}. \quad (7)$$

The parameters w_i, $i=1,2,3,4,6$ are uniquely defined through the parameters a and b and guarantee the cancellation of the terms of powers up to 9 in the expansion (3).

Our phase shift parametrization is given by Eqs. (4), (5) and (7) with fitting parameters a, b, c and d. For each parameter set, we solve Eq. (2) to find the values of the energies $E_\lambda^{a,b,c,d}$, and search for the parameter set (a,b,c,d) minimizing the rms deviation of $E_\lambda^{a,b,c,d}$ from the selected set of NCSM eigenenergies E_λ. Following this route, we obtain an excellent description of the selected E_λ with an rms deviation of 5.8 keV with $a=0.724$ MeV$^{-\frac{1}{2}}$, $b^2=0.448$ MeV, $c=0.941$ MeV$^{-\frac{5}{2}}$, and $d=-9.1\cdot10^{-4}$ MeV^{-4}. The resulting predictions for the NCSM eigenenergies are shown by solid lines in the upper panel of Fig. 1 where we also describe well NCSM energies with large enough $N_{\max}$ and/or $\hbar\Omega$ not included in the minimization fit. We obtain also an excellent description of NCSM-SS-HORSE predicted phase shifts as is shown by solid line in Fig. 2.

However the resonance parameters describing the location of the S-matrix pole obtained by this fit, are surprisingly small: the resonance energy $E_r=0.186$ MeV and the width $\Gamma=0.815$ MeV. Note, looking at the phase shift in Fig. 2, we would expect the resonance at the energy of approximately 0.8 MeV corresponding to the maximum of the phase shift derivative and with the width of about 1.5 MeV. The contribution of the pole term (5) to the phase shifts is shown by the dashed line in Fig. 2. This contribution is seen to differ considerably from the resulting phase shift due to substantial contributions from the background phase (7) which is dominated by the terms needed to fulfill the low-energy theorem $\delta\sim k^{2\mathscr{L}+1}$ and to cancel low-power terms in the expansion of the resonant phase $\delta_r(E)$. Such a sizable contribution from the background in the low-energy region, impels us to search for additional poles or other singularities giving rise to a strong energy dependence which would be separate from the background phase.

After we failed to find a reasonable description of the NCSM-SS-HORSE phase shifts with a low-energy virtual state, we found the resolution of the strong background phase problem by assuming that the S-matrix has an additional low-energy false pole at a positive imaginary momentum [43]. We add the false term contribution

$$\delta_f(E)=-\tan^{-1}\sqrt{E/|E_f|} \quad (8)$$

to the phase shift to obtain the equation

$$\delta(E)=\phi(E)+\delta_r(E)+\delta_f(E) \quad (9)$$

replacing Eq. (4). This parametrization involves an additional fitting parameter E_f. We obtain nearly the same quality description of the selected $4n$ ground state energies with the rms deviation of 6.2 keV with the parameters $a=0.701$ MeV$^{-\frac{1}{2}}$, $b^2=1.089$ MeV, $c=-27.0$ MeV$^{-\frac{5}{2}}$, $d=0.281$ MeV^{-4}, and a low-lying false pole at energy $E_f=-54.9$ keV. The respective $4n$ resonance at $E_r=0.844$ MeV and width $\Gamma=1.378$ MeV appears consistent with what is expected from directly inspecting the $4n$ phase shifts. The parametrized phase shifts are shown by solid line in Fig. 3 together with separate contributions from the resonant and false pole terms. We note that corrections introduced by this new parametrization to the solid lines in Figs. 1 and 2 are nearly unseen in the scales of these figures.

Conclusions. Our results with the realistic JISP16 interaction and the SS-HORSE technique show there is a resonant structure near 0.8 MeV above threshold with a width Γ of about 1.4 MeV. This is the first theoretical calculation that predicts such a low energy $4n$ resonance, without altering any of the properties of the realistic NN interaction. Our result is compatible with the recent experiment which found a resonant structure at an energy of 0.86 ± 0.65 (stat) $\pm\,1.25$ (syst) MeV and that set an upper limit for the width at $\Gamma=2.6$ MeV. Our complex energy calculations also suggest a broad low-lying $4n$ resonance that agrees marginally with experiment due to the large error bars for both the current application of the NCGSM and the experiment.

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