

Introduction to Ab Initio Nuclear Structure Theory

Part I: Hamiltonian

Robert Roth



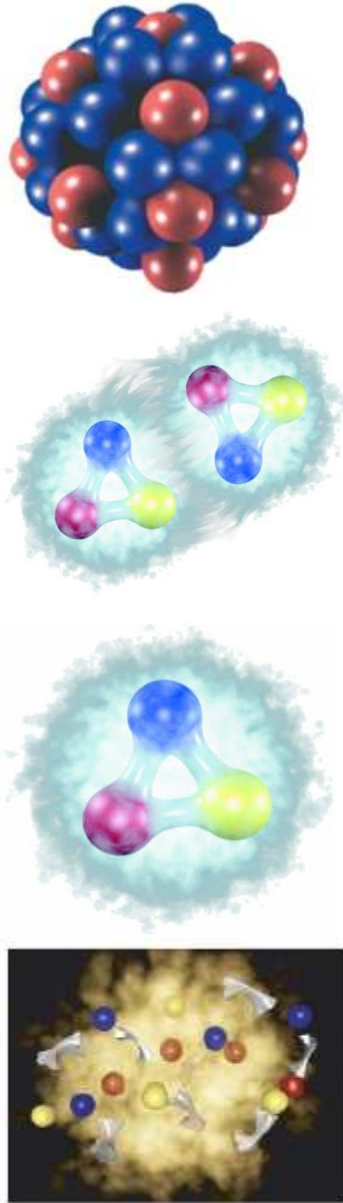
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Theoretical Context

better resolution / more fundamental

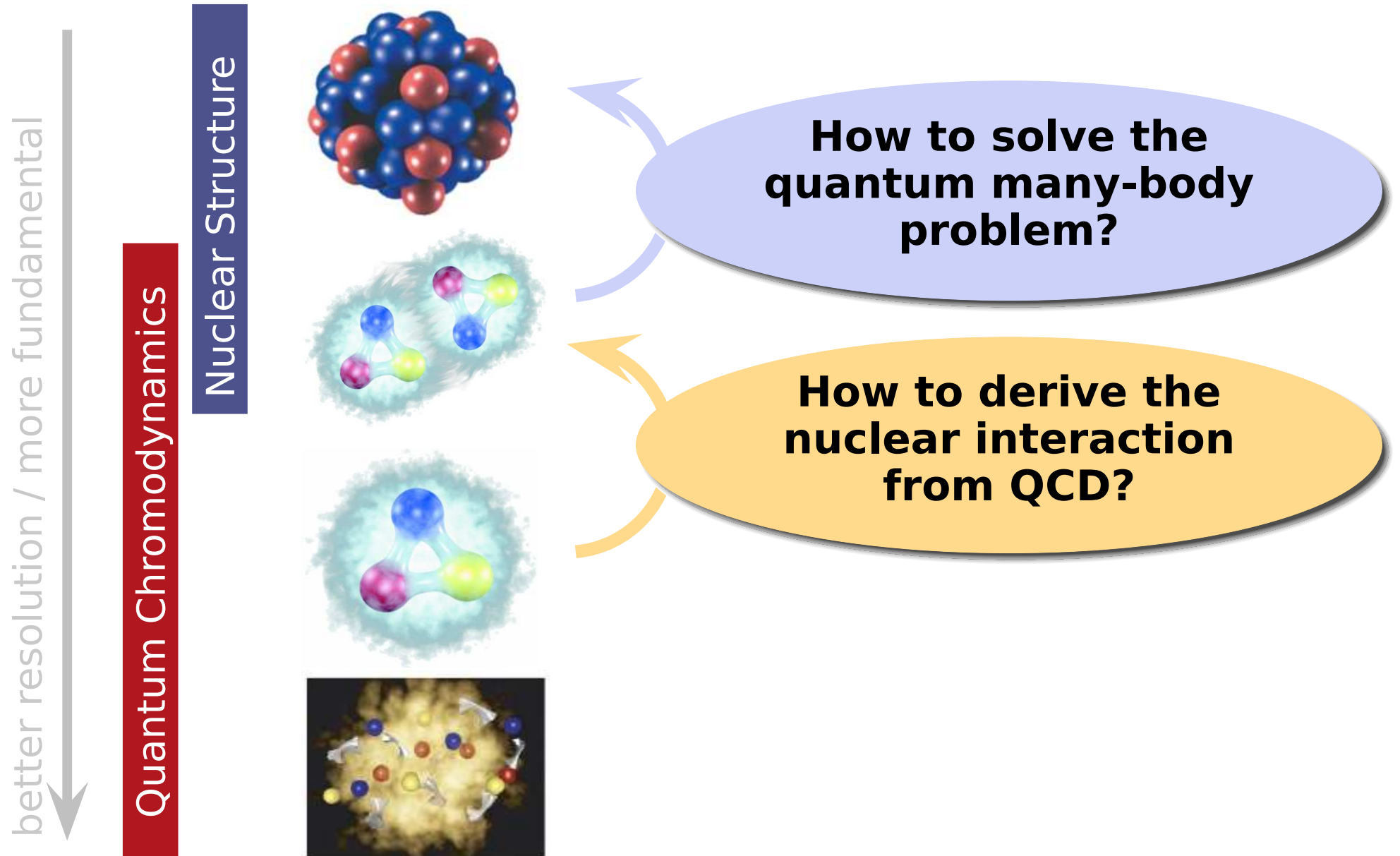
Quantum Chromodynamics

Nuclear Structure



- finite nuclei
- few-nucleon systems
- nuclear interaction
- hadron structure
- quarks & gluons
- deconfinement

Theoretical Context



Nuclear Theory — Wish List

- **nuclear structure as low-energy effective theory based on QCD**
- **robust & quantitative predictions for nuclei far-off stability**
- **systematic, controlled & improvable many-body approaches**
- **theoretical toolbox covering all masses and observables**

Building Blocks

Nuclear Structure Observables

Nuclear Lattice Sim.

chiral EFT on lattice

Exact Ab-Initio Solutions

few-body et al.

Exact Ab-Initio Solutions

few-body, no-core shell model, etc.

Approx. Many-Body Methods

controlled & improvable schemes

Similarity Transformations

physics-conserving transform. of observables

Chiral Interactions

consistent & improvable NN, 3N,... interactions

Chiral Effective Field Theory

systematic low-energy effective theory of QCD

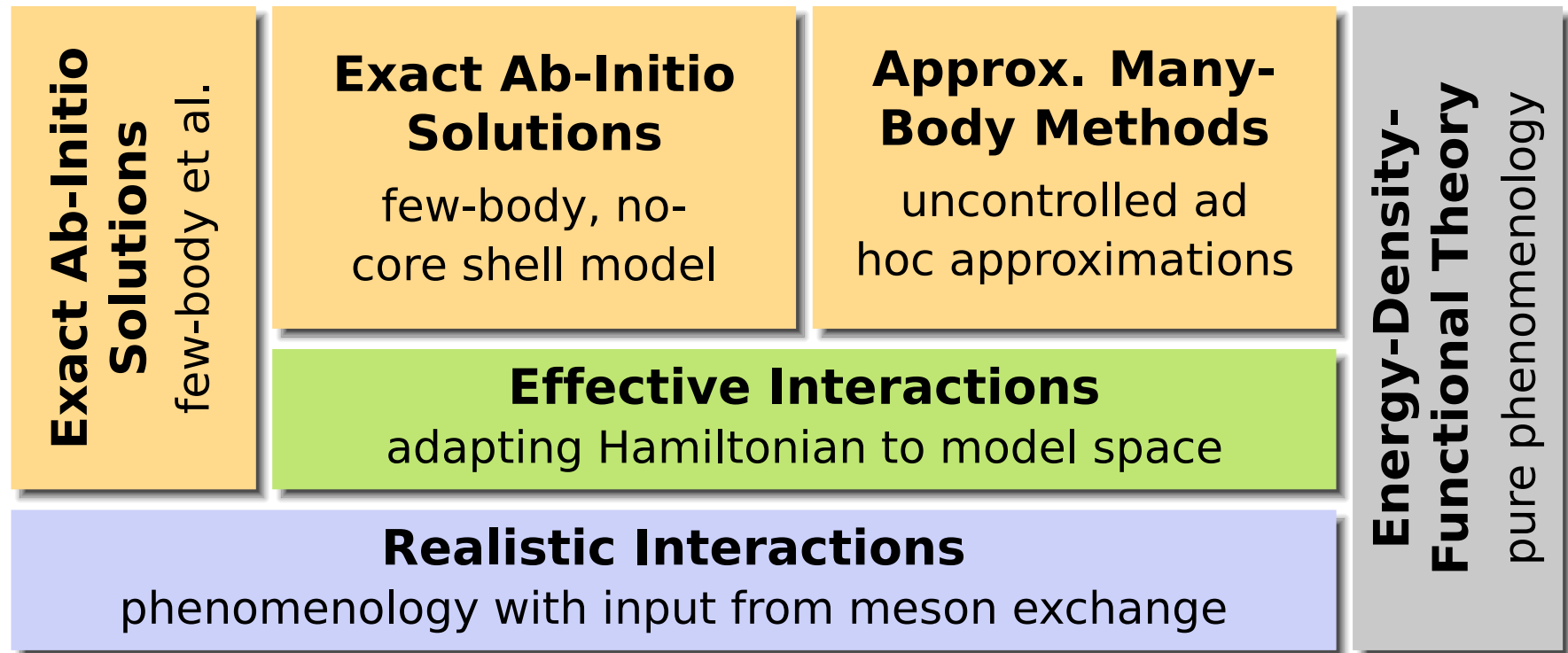
Quantum Chromodynamics

Energy-Density-Functional Theory

guided by chiral EFT

Building Blocks — Anno 2000

Nuclear Structure Observables

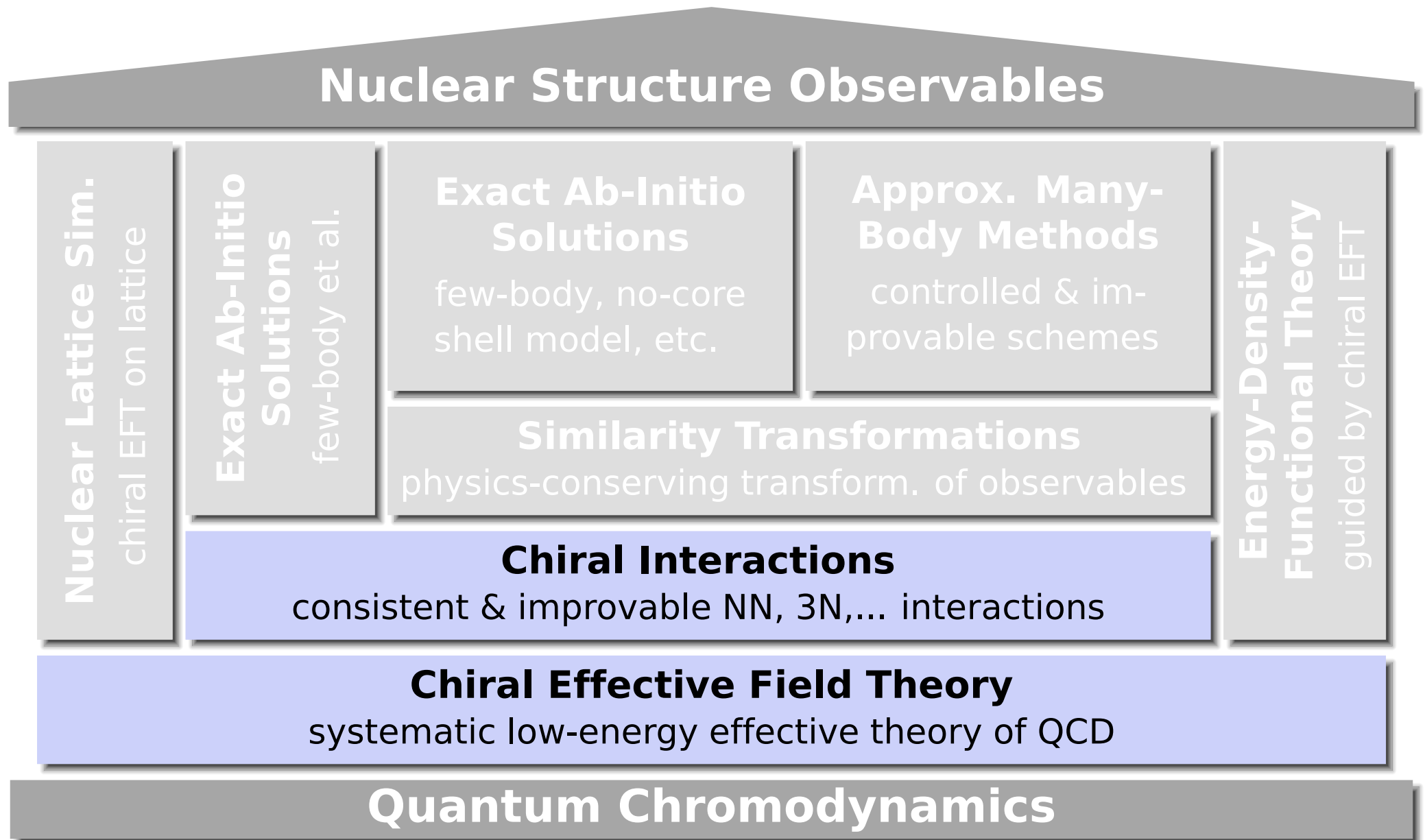


??? Connection to QCD ???

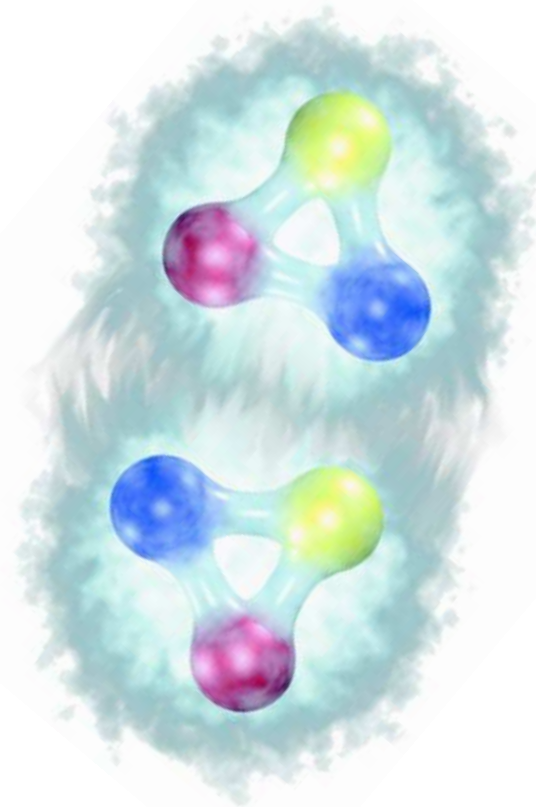
Quantum Chromodynamics

Nuclear Interactions from QCD

Building Blocks



Nature of the Nuclear Interaction

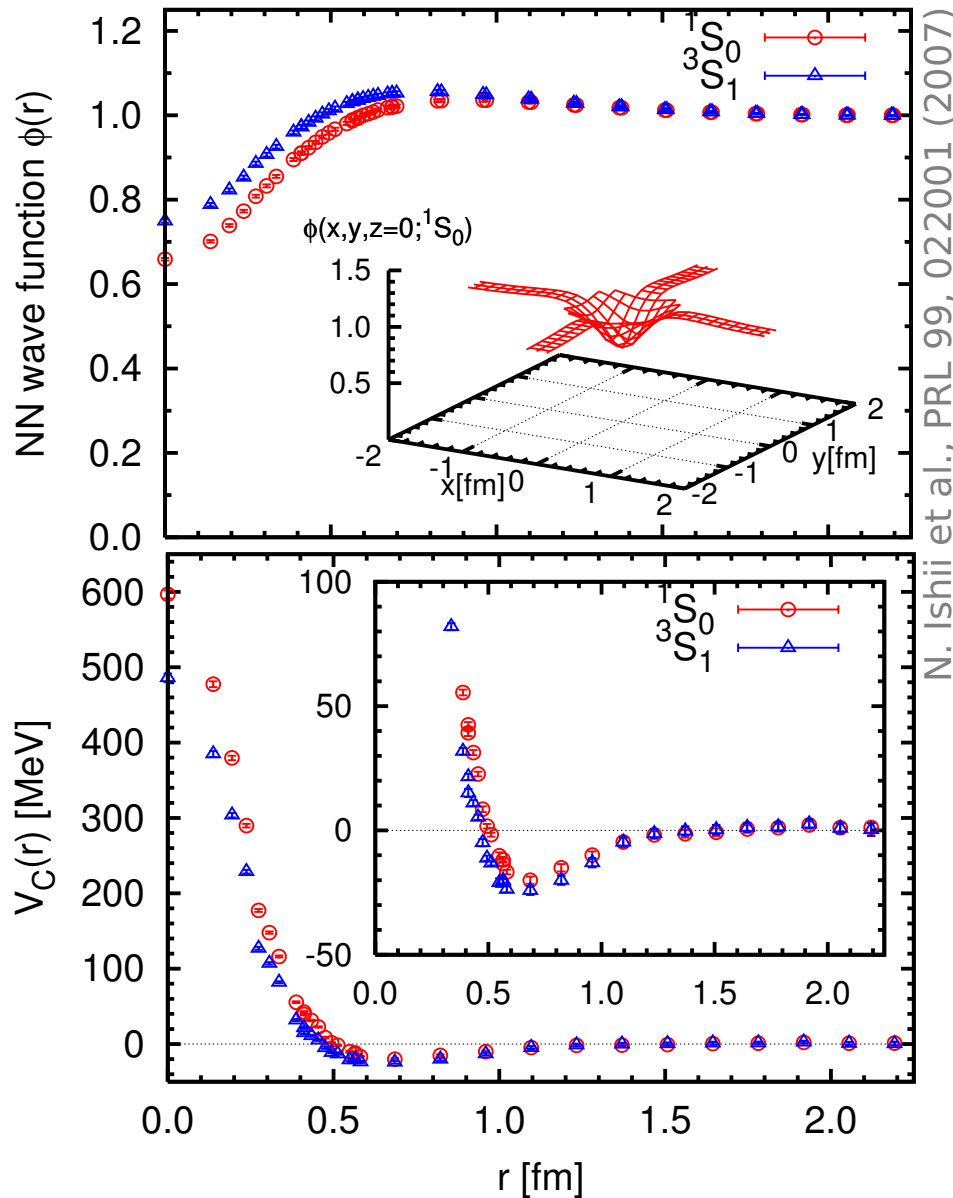


~ 1.6fm

$$\rho_0^{-1/3} = 1.8\text{fm}$$

- NN-interaction is **not fundamental**
- analogous to **van der Waals** interaction between neutral atoms
- induced via mutual **polarization** of quark & gluon distributions
- acts only if the nucleons overlap, i.e. at **short ranges**
- genuine **3N-interaction** is important

Nuclear Interaction from Lattice QCD



- first steps towards construction of a nuclear interaction through **lattice QCD simulations**
- compute relative **two-nucleon wavefunction** on the lattice
- invert Schrödinger equation to obtain **local 'effective' two-nucleon potential**
- schematic results so far (unphysical quark masses, S-wave interactions only,...)

Realistic Nuclear Interactions

■ QCD ingredients

- chiral effective field theory
- meson-exchange theory

■ short-range phenomenology

- contact terms or parameterization of short-range potential

■ experimental two-body data

- scattering phase-shifts & deuteron properties reproduced with high precision

■ supplementary 3N interaction

- adjusted to spectra of light nuclei

Argonne
V18

CD Bonn

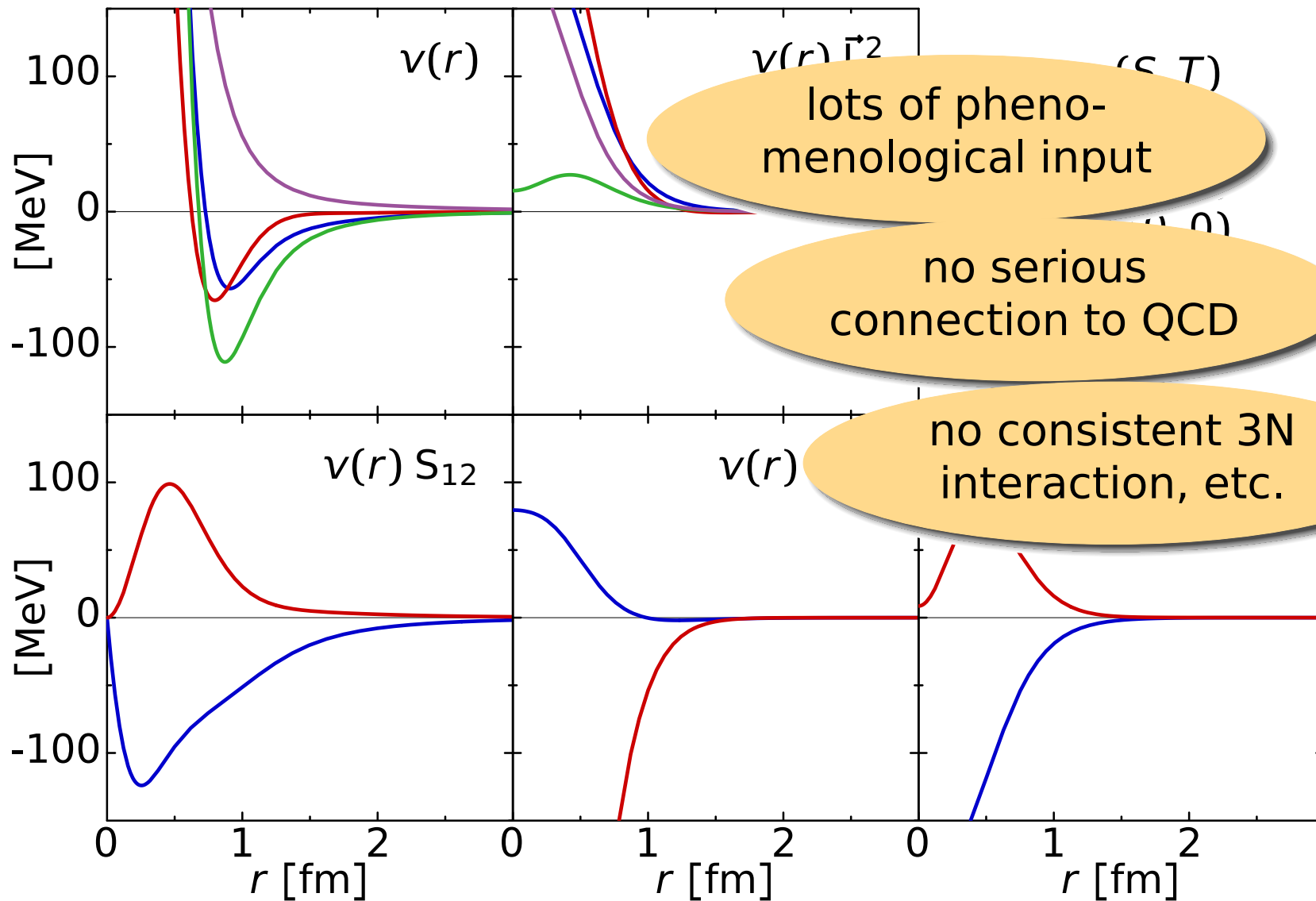
Nijmegen
I/II

Chiral
N3LO

Argonne V18
+ Illinois X

Chiral N3LO
+ N2LO

Argonne V18 Potential



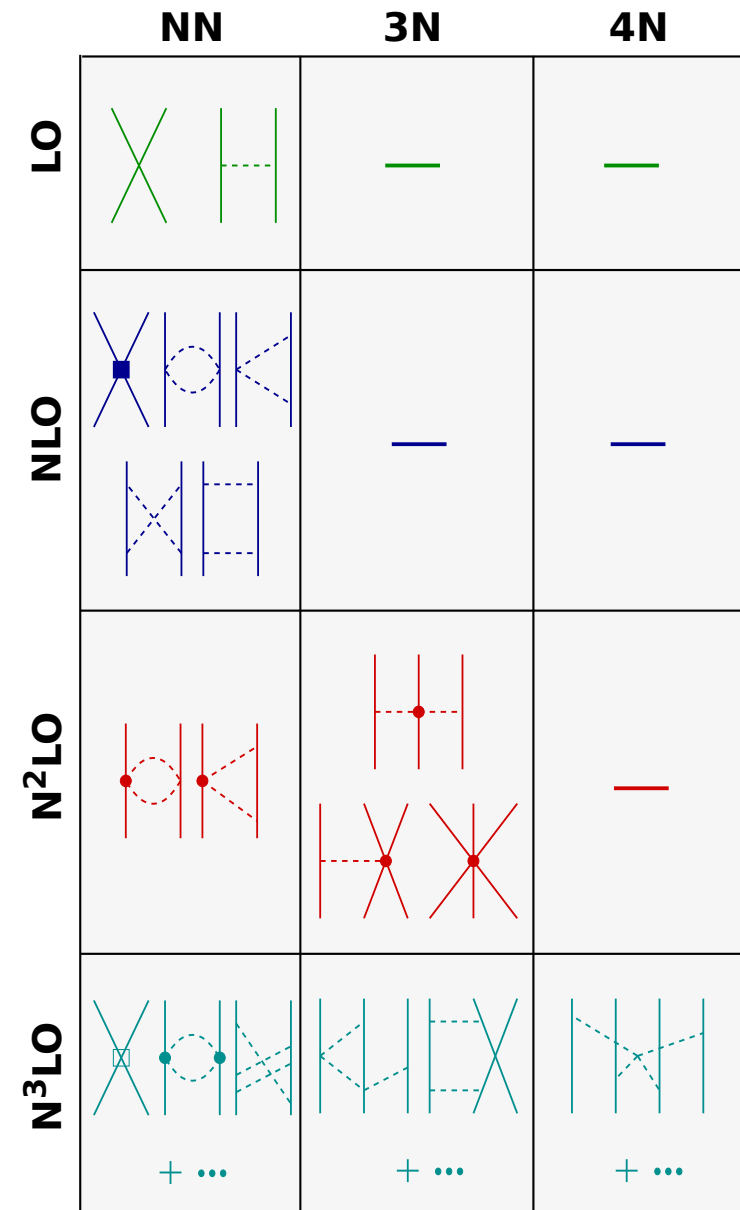
lots of phenomenological input

no serious connection to QCD

no consistent 3N interaction, etc.

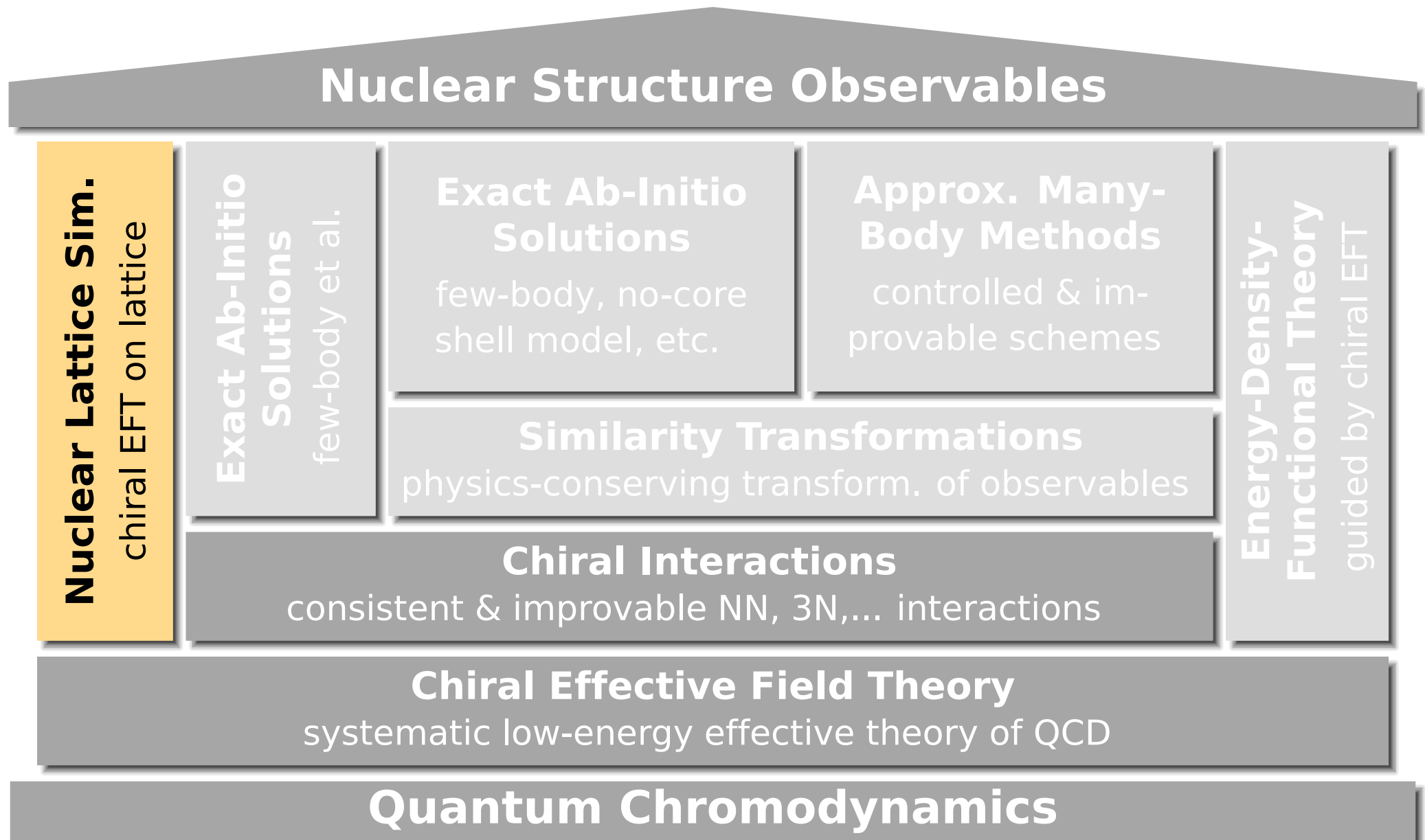
Nuclear Interactions from Chiral EFT

- low-energy **effective field theory** for relevant degrees of freedom (π, N) based on symmetries of QCD
- long-range **pion dynamics** explicitly
- short-range physics absorbed in **contact terms**, low-energy constants fitted to experiment ($NN, \pi N, \dots$)
- hierarchy of **consistent NN, 3N, ... interactions** (plus currents)
 - 3N interaction at N^3LO
 - explicit inclusion of Δ -resonance
 - formal issues: power counting, renormalization, cutoff choice, ...



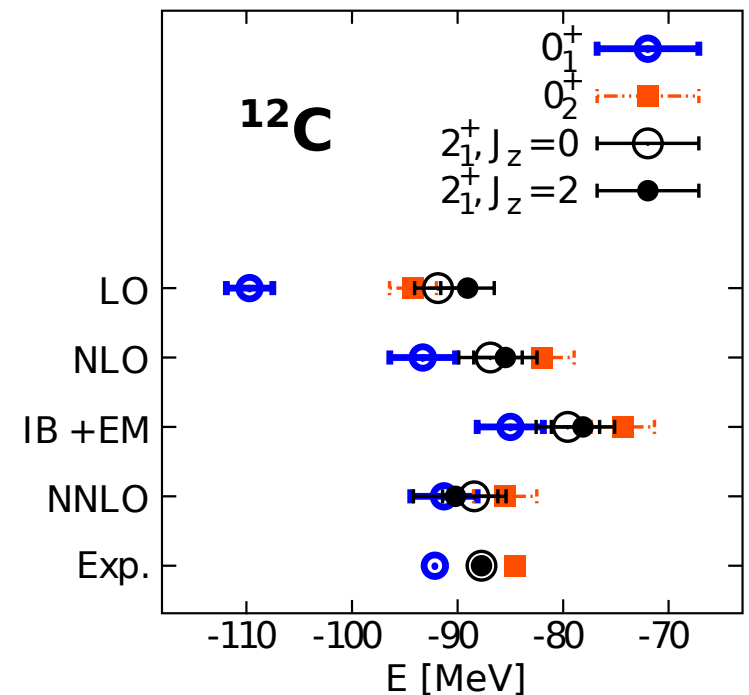
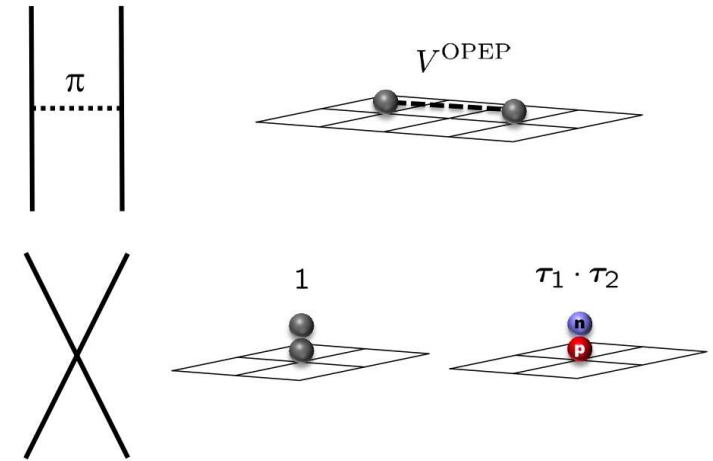
adapted from Meißner (2005)

Building Blocks



Nuclear Lattice Simulations

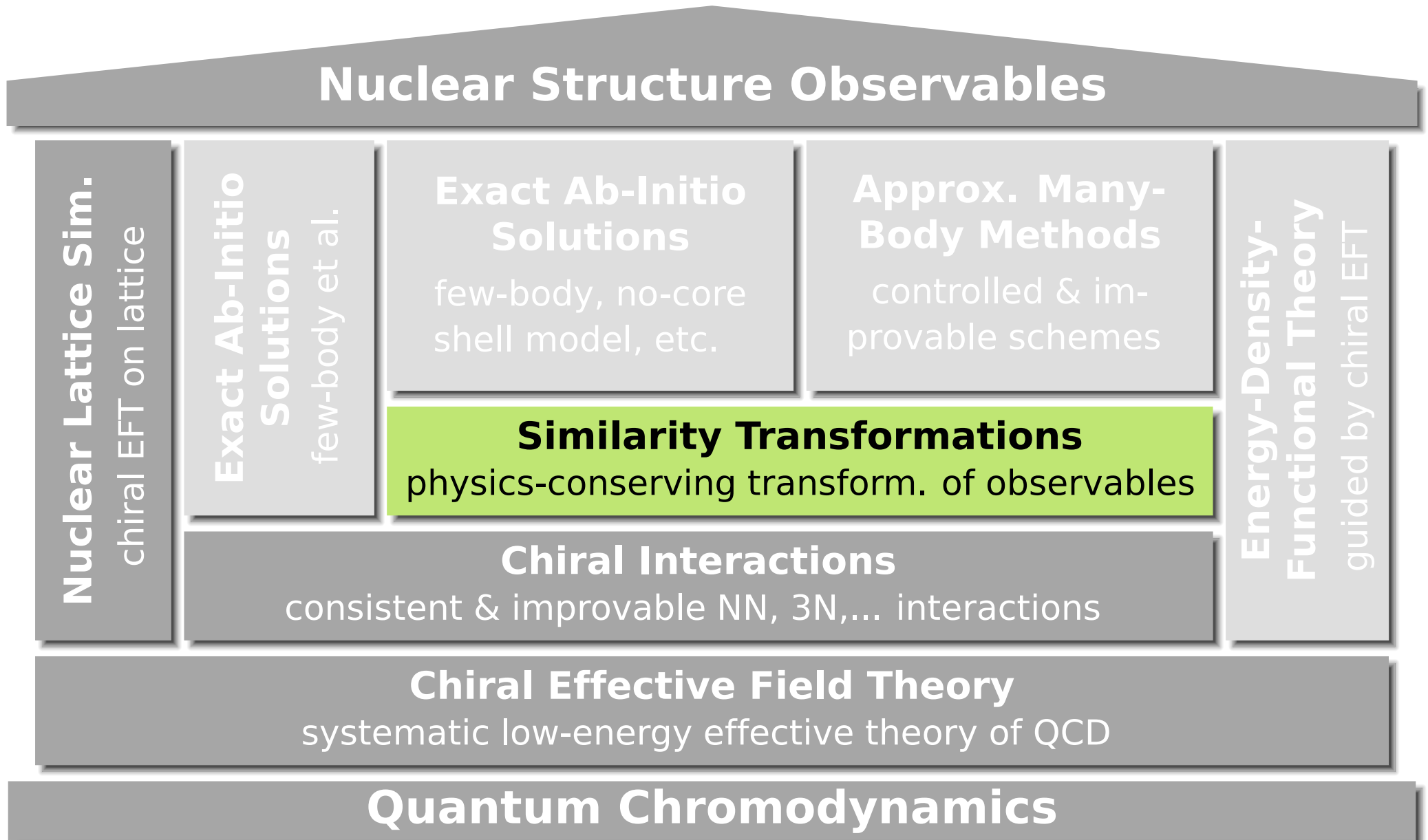
- put **chiral EFT on a space-time lattice** and use Lattice-QCD technology
- lattice defines IR and UV cutoffs
- fit LECs to scattering and ground-state observables on the lattice
- **Euclidean time projection** to extract ground and excited states
- ground states of ${}^4\text{He}$, ${}^8\text{Be}$, ${}^{12}\text{C}$,...
- highlight: **Hoyle state in ${}^{12}\text{C}$**
- **statistical and extrapolation errors** still too large for spectroscopy



Epelbaum, Krebs, Lee, Meißner (2011)

Similarity Transformations

Building Blocks



Why Similarity Transformations?

Realistic Interactions

- generate strong correlations in many-body states
- short-range central & tensor correlations most important

Many-Body Methods

- rely on truncated many-nucleon Hilbert spaces
- not capable of describing short-range correlations
- extreme: Hartree-Fock based on single Slater determinant

Similarity Transformation

- adapt realistic potential to the available model space
- conserve experimentally constrained properties (phase shifts, deuteron properties)

What are Correlations?

correlations:
everything beyond the
independent-particle picture

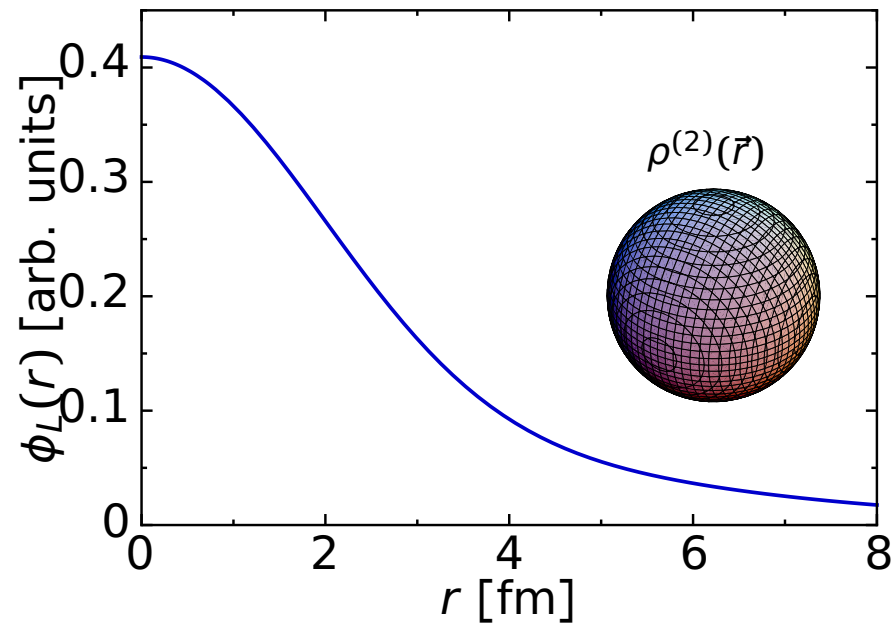
- the quantum state of A independent (non-interacting) fermions is a **Slater determinant**

$$|\Phi^{\text{SD}}\rangle \propto \sum_{\pi} \text{sgn}(\pi) P_{\pi} |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_A\rangle$$

- **any two-body interaction induces correlations** which cannot be described by a single Slater determinant

Deuteron: Manifestation of Correlations

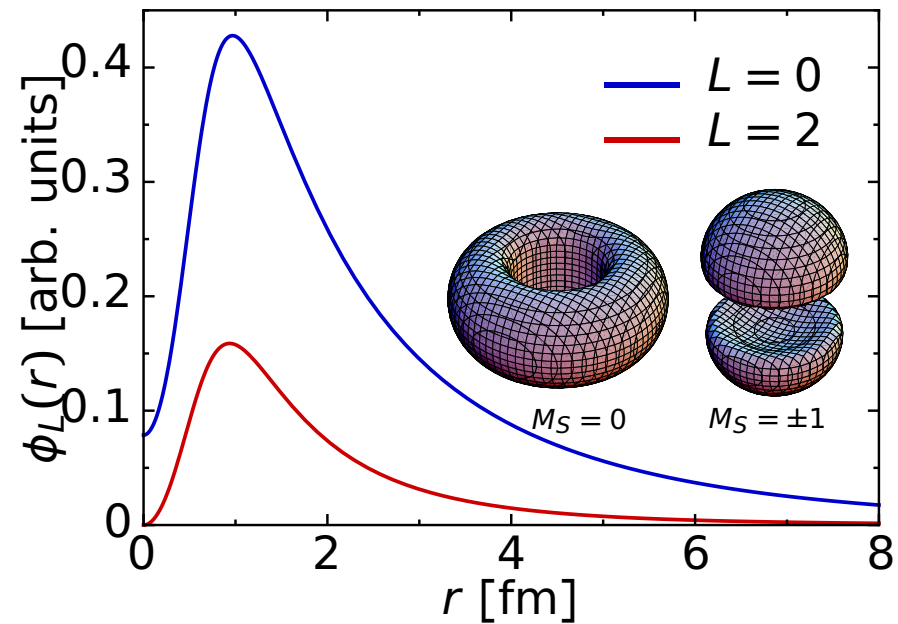
independent n+p in a trap



short-range repulsion
suppresses wavefunction
at small distances r

central correlations

realistic deuteron solution



tensor interaction
generates $L=2$ admixture
to ground state

tensor correlations

Similarity Transformations

Unitary Correlation Operator Method

Unitary Correlation Operator Method

Correlation Operator

define a unitary operator C to describe the effect of short-range correlations

$$C = \exp[-iG] = \exp\left[-i \sum_{i<j} g_{ij}\right]$$

Correlated States

imprint short-range correlations onto uncorrelated many-body states

$$|\tilde{\psi}\rangle = C |\psi\rangle$$

Correlated Operators

adapt Hamiltonian to uncorrelated states (pre-diagonalization)

$$\tilde{O} = C^\dagger O C$$

$$\langle \tilde{\psi} | O | \tilde{\psi}' \rangle = \langle \psi | C^\dagger O C | \psi' \rangle = \langle \psi | \tilde{O} | \psi' \rangle$$

Unitary Correlation Operator Method

explicit ansatz for unitary transformation operator **motivated by the physics of short-range correlations**

- **explicit unitary transformation** with correlation operator C

$$C = C_{\Omega} C_r = \exp\left(-i \sum_{i < j} g_{\Omega, ij}\right) \exp\left(-i \sum_{i < j} g_{r, ij}\right)$$

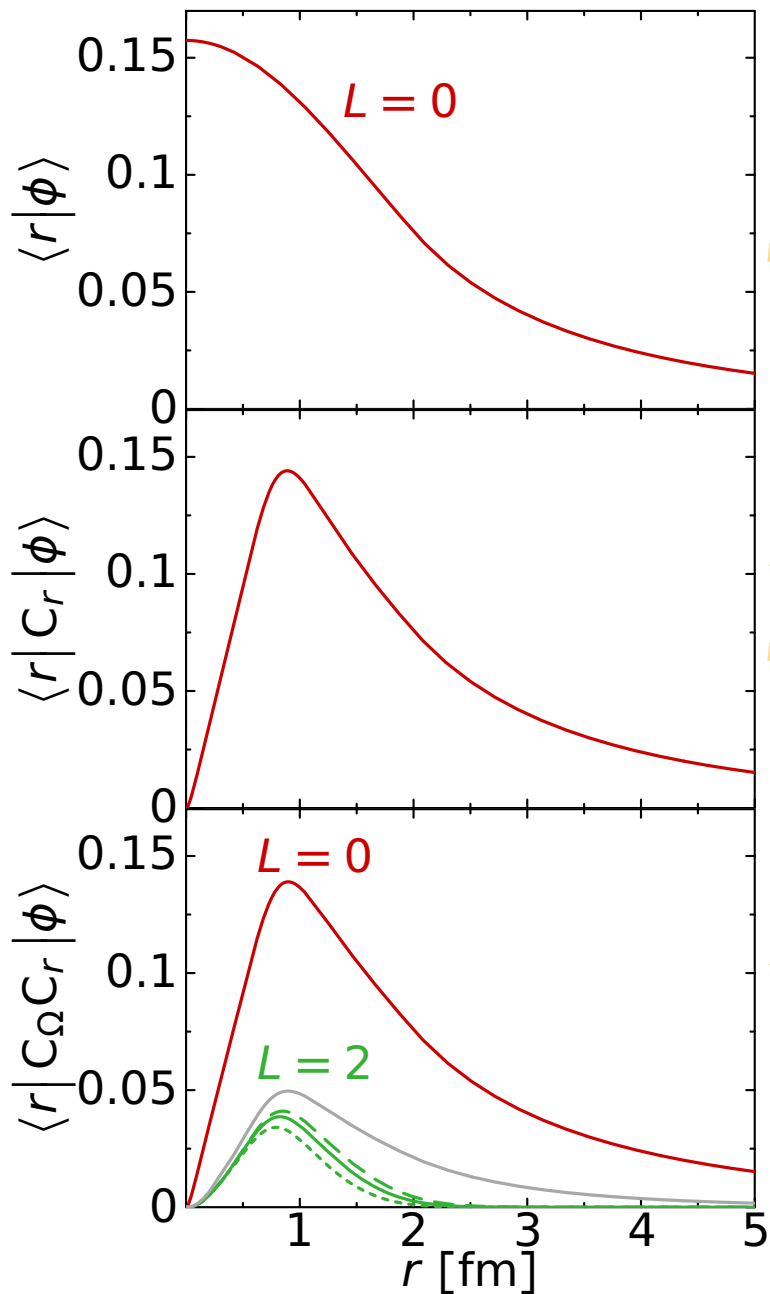
- **central correlator** C_r : radial distance-dependent shift in the relative coordinate of a nucleon pair

$$g_r = \frac{1}{2} [s(r) q_r + q_r s(r)] \qquad q_r = \frac{1}{2} \left[\frac{\vec{r}}{r} \cdot \vec{q} + \vec{q} \cdot \frac{\vec{r}}{r} \right]$$

- **tensor correlator** C_{Ω} : angular shift depending on the orientation of spin and relative coordinate of a nucleon pair

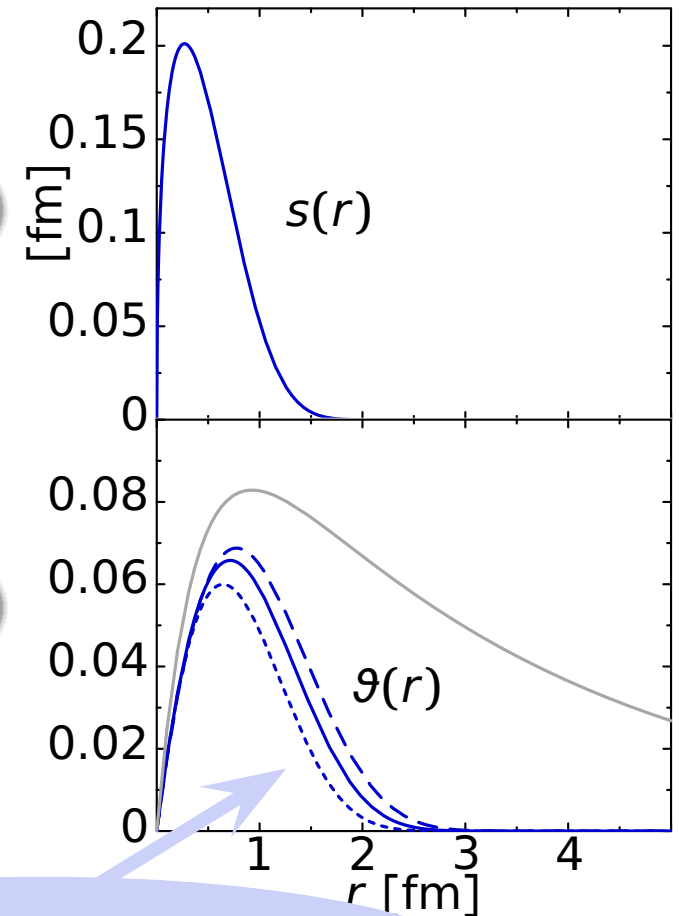
$$g_{\Omega} = \frac{3}{2} \vartheta(r) [(\vec{\sigma}_1 \cdot \vec{q}_{\Omega})(\vec{\sigma}_2 \cdot \vec{r}) + (\vec{r} \leftrightarrow \vec{q}_{\Omega})] \qquad \vec{q}_{\Omega} = \vec{q} - \frac{\vec{r}}{r} q_r$$

Correlated States: The Deuteron



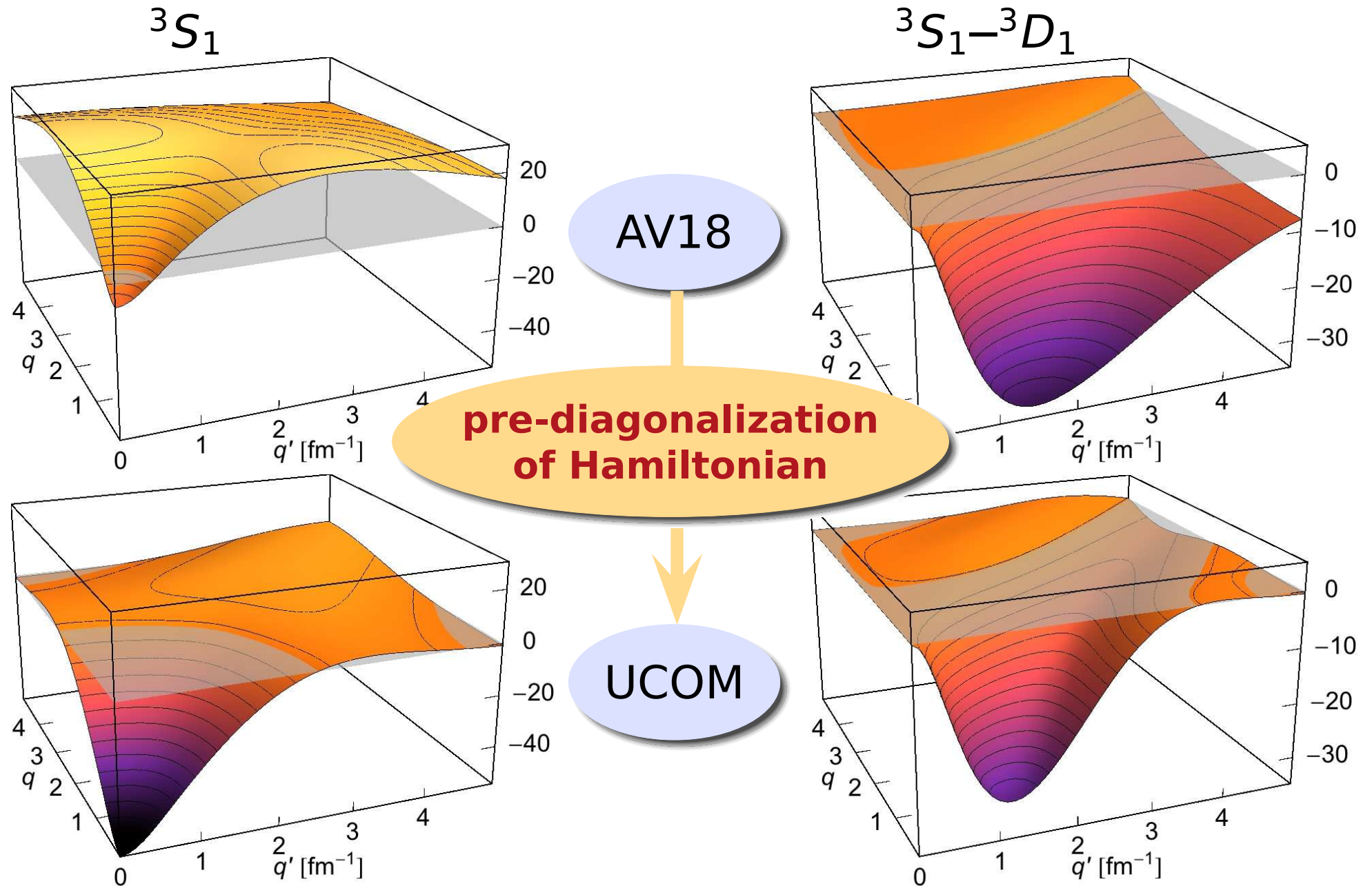
central correlations

tensor correlations



only short-range tensor correlations treated by C_{Ω}

Correlated Operators: V_{UCOM}



Similarity Transformations

Similarity Renormalization Group

Similarity Renormalization Group

continuous transformation driving
Hamiltonian to band-diagonal form
with respect to a chosen basis

- **unitary transformation** of Hamiltonian

$$\tilde{H}_\alpha = U_\alpha^\dagger H U_\alpha$$

simplicity and flexibility
are great advantages of
the SRG approach

- **evolution equations** for \tilde{H}_α and U_α

$$\frac{d}{d\alpha} \tilde{H}_\alpha = [\eta_\alpha, \tilde{H}_\alpha]$$

$$\frac{d}{d\alpha} U_\alpha = \dots$$

other transformation
approaches (UCOM, V_{lowk})
follow as special cases

- **dynamic generator**: commutator with the operator in whose eigenbasis H shall be diagonalized

$$\eta_\alpha = (2\mu)^2 [T_{int}, \tilde{H}_\alpha]$$

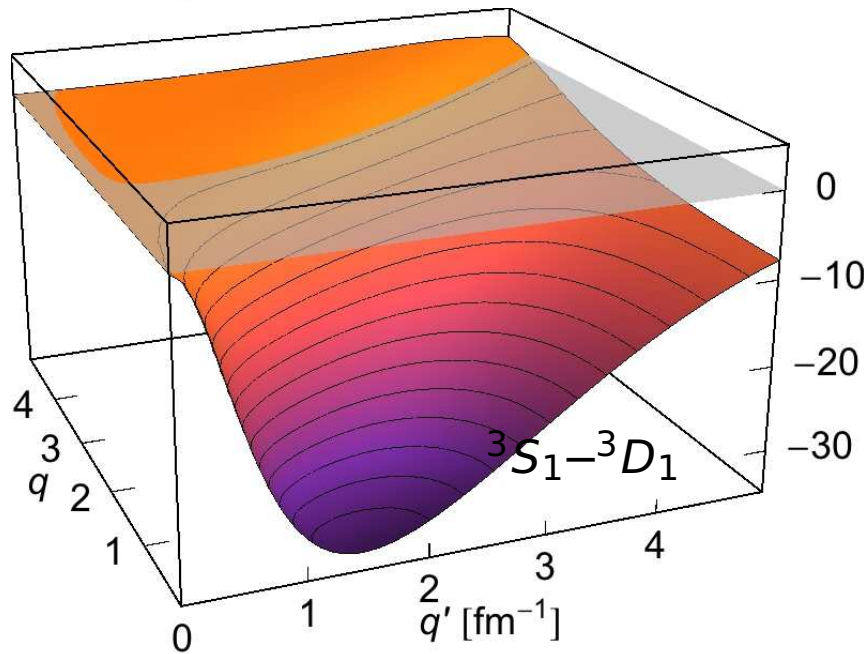
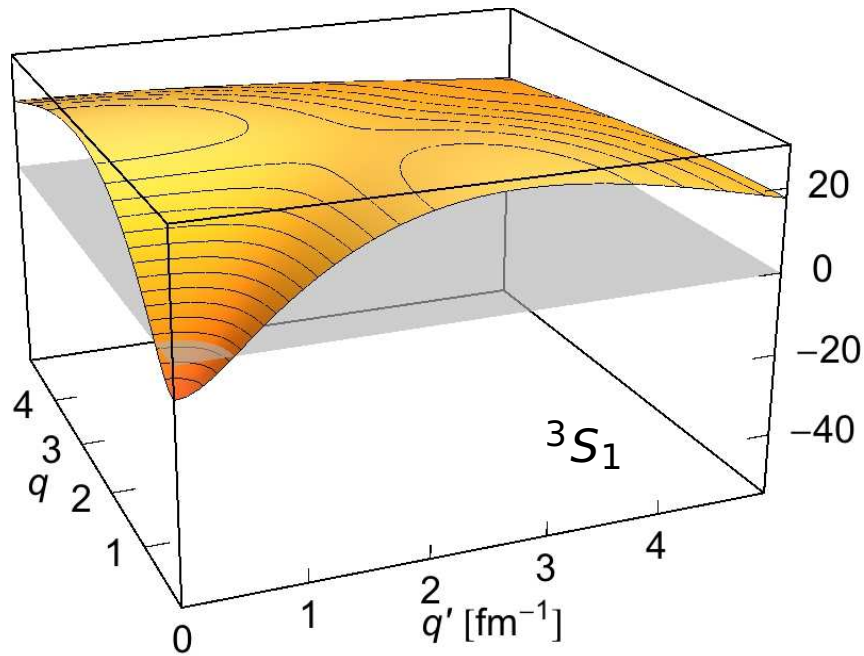
SRG Evolution of Matrix Elements

- convert Fock-space operator equations into **coupled evolution equations for matrix elements** in n -body Hilbert space
- $n = 2$: use **antisym. relative LS -coupled two-body states**
 - momentum space: $|q(LS)JT\rangle$
 - harmonic oscillator: $|n(LS)JT\rangle$
- system of **coupled evolution equations** for each $J^\pi ST$ -block

$$\frac{d}{d\alpha} \langle n(LS)JT | \tilde{H}_\alpha | n'(L'S)JT \rangle = (2\mu)^2 \sum_{n''L''} \sum_{n'''L'''} \left[\begin{aligned} & \langle nL\dots | T_{\text{int}} | n''L''\dots \rangle \langle n''L''\dots | \tilde{H}_\alpha | n'''L'''\dots \rangle \langle n'''L'''\dots | \tilde{H}_\alpha | n'L'\dots \rangle \\ & - 2 \langle nL\dots | \tilde{H}_\alpha | n''L''\dots \rangle \langle n''L''\dots | T_{\text{int}} | n'''L'''\dots \rangle \langle n'''L'''\dots | \tilde{H}_\alpha | n'L'\dots \rangle \\ & + \langle nL\dots | \tilde{H}_\alpha | n''L''\dots \rangle \langle n''L''\dots | \tilde{H}_\alpha | n'''L'''\dots \rangle \langle n'''L'''\dots | T_{\text{int}} | n'L'\dots \rangle \end{aligned} \right]$$

SRG Evolution in Two-Body Space

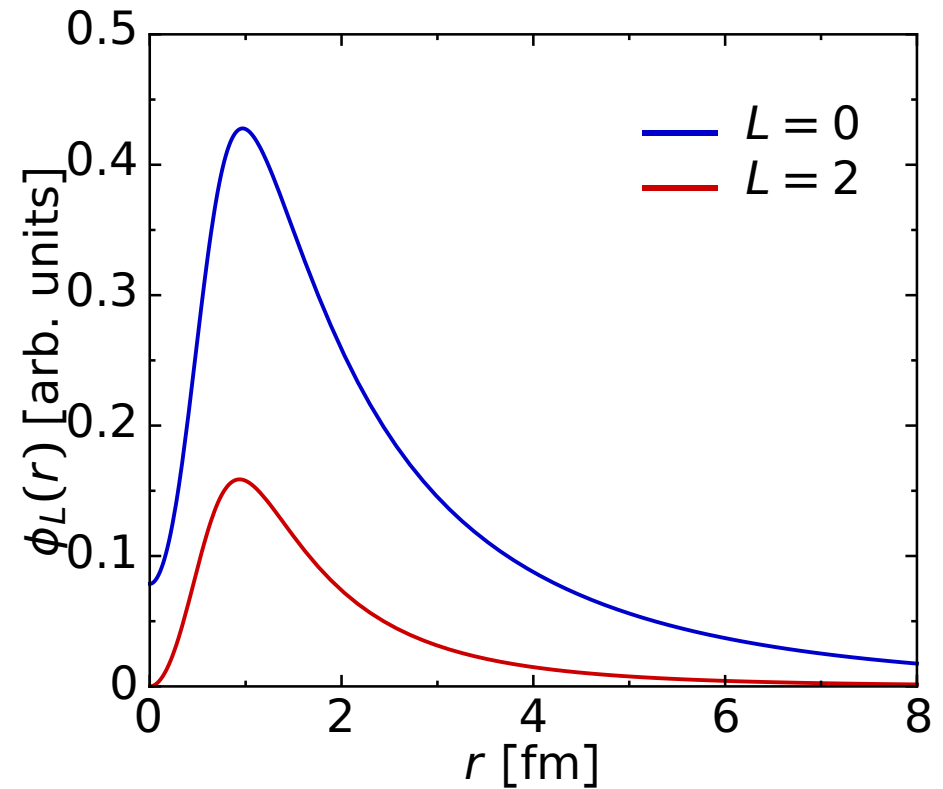
momentum-space matrix elements



Argonne V18

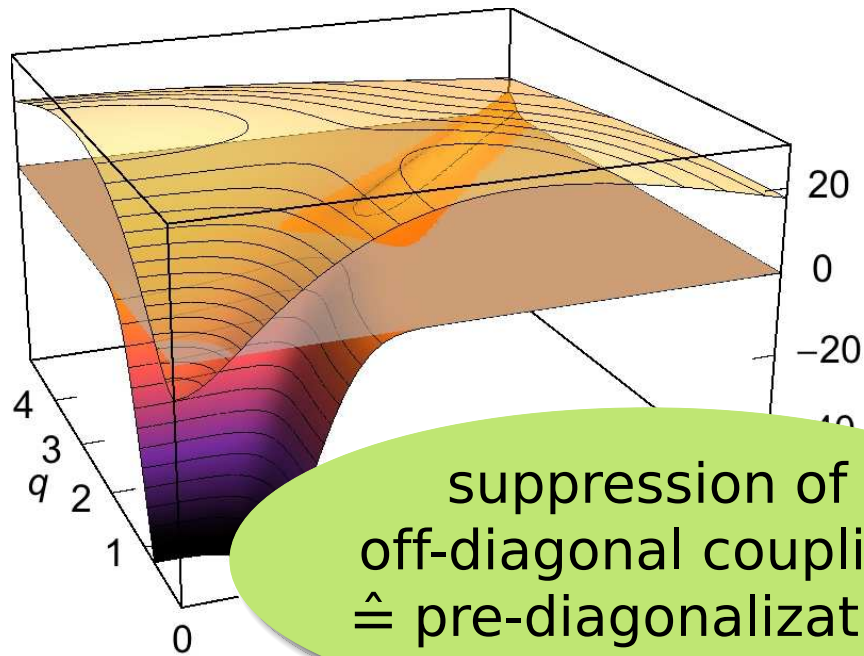
$$J^\pi = 1^+, T = 0$$

deuteron wave-function

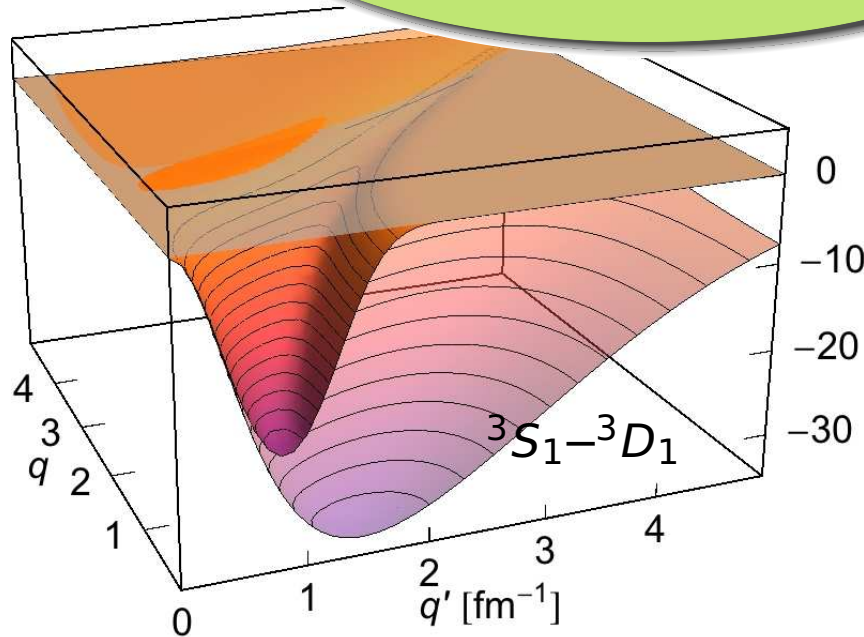


SRG Evolution in Two-Body Space

momentum-space matrix elements



suppression of off-diagonal coupling $\hat{=}$ pre-diagonalization

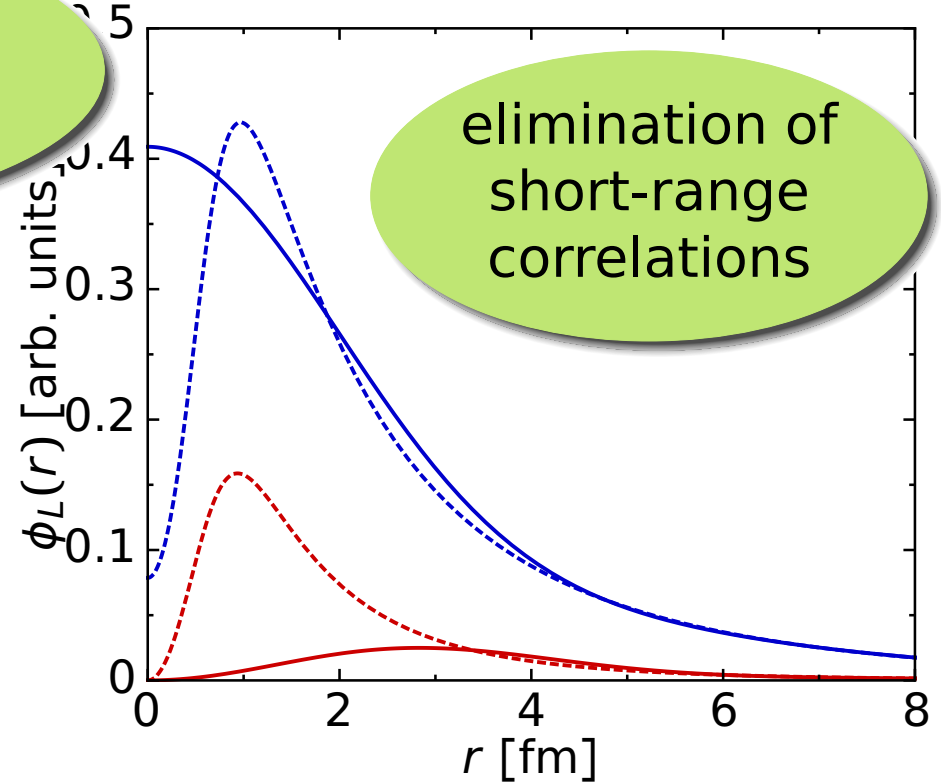


$$\alpha = 0.320 \text{ fm}^4$$

$$\Lambda = 1.33 \text{ fm}^{-1}$$

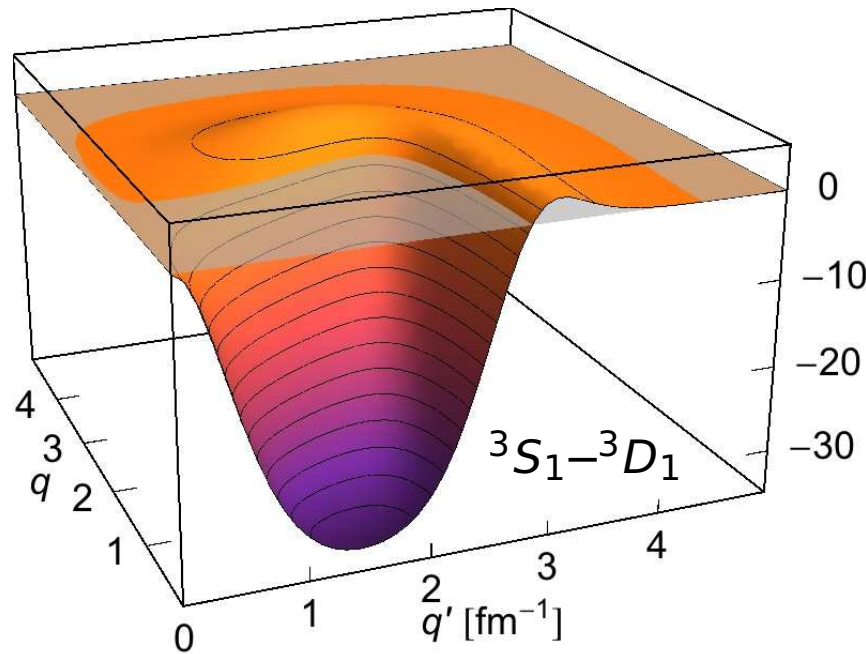
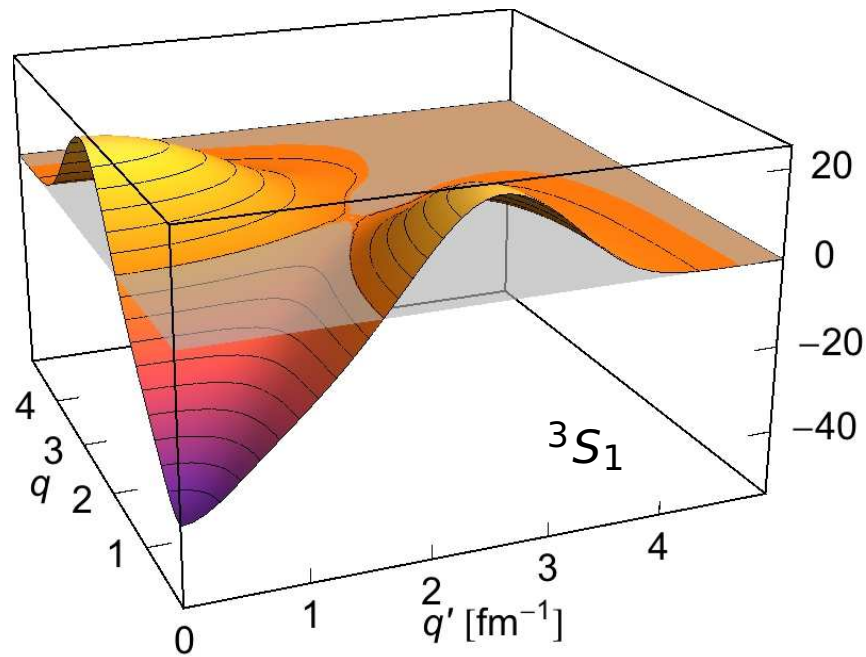
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deuteron wave-function



SRG Evolution in Two-Body Space

momentum-space matrix elements

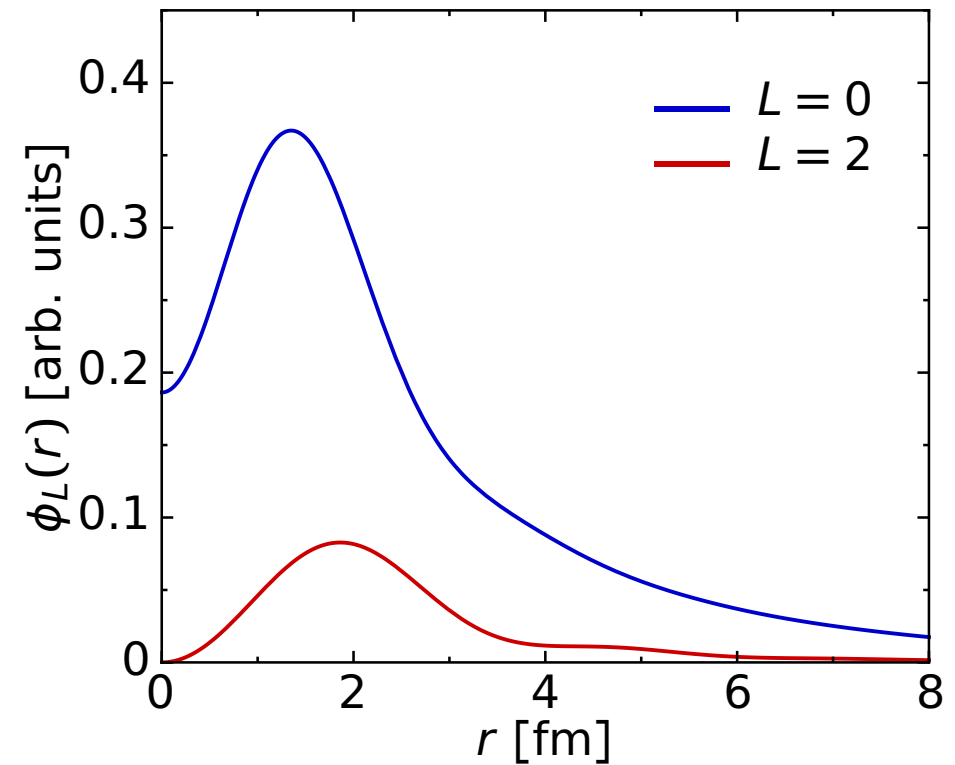


chiral NN

Entem & Machleidt. N³LO, 500 MeV

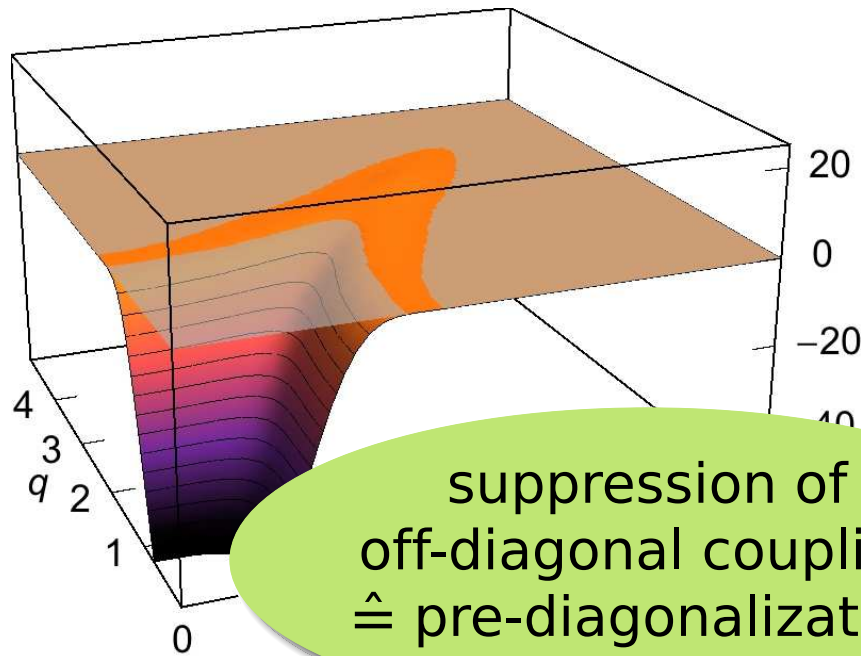
$J^\pi = 1^+, T = 0$

deuteron wave-function

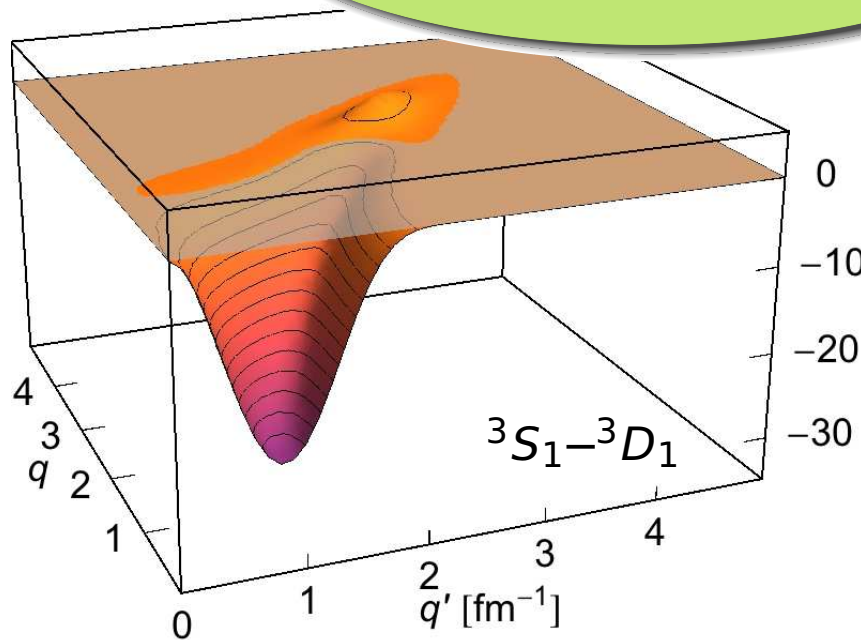


SRG Evolution in Two-Body Space

momentum-space matrix elements



suppression of
off-diagonal coupling
 $\hat{=}$ pre-diagonalization

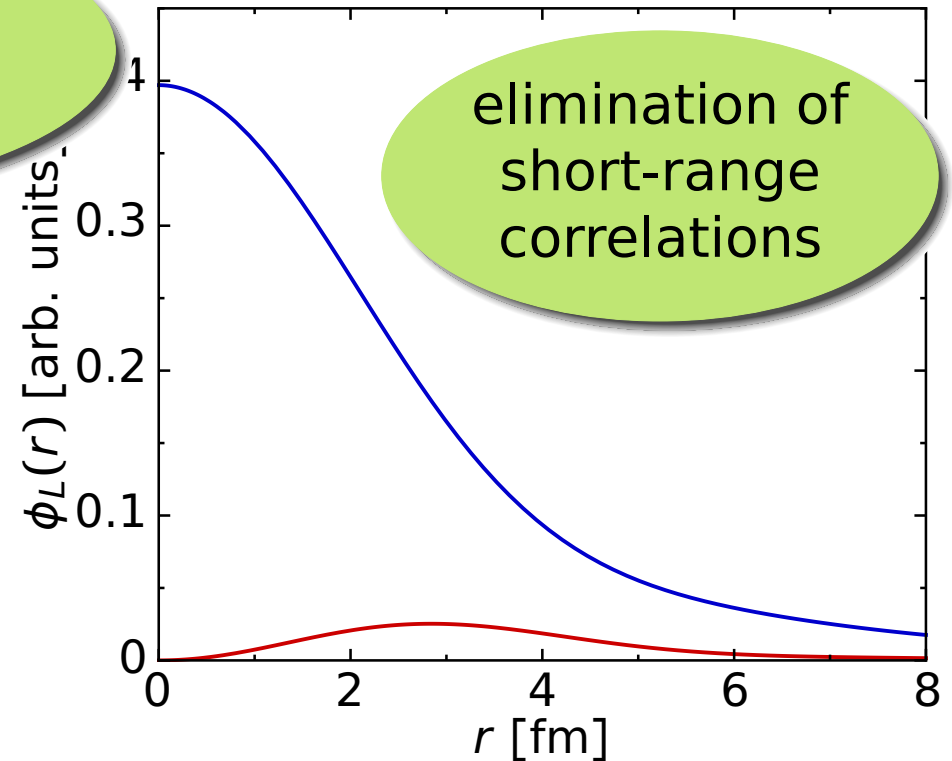


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$$J^\pi = 1^+, T = 0$$

deuteron wave-function



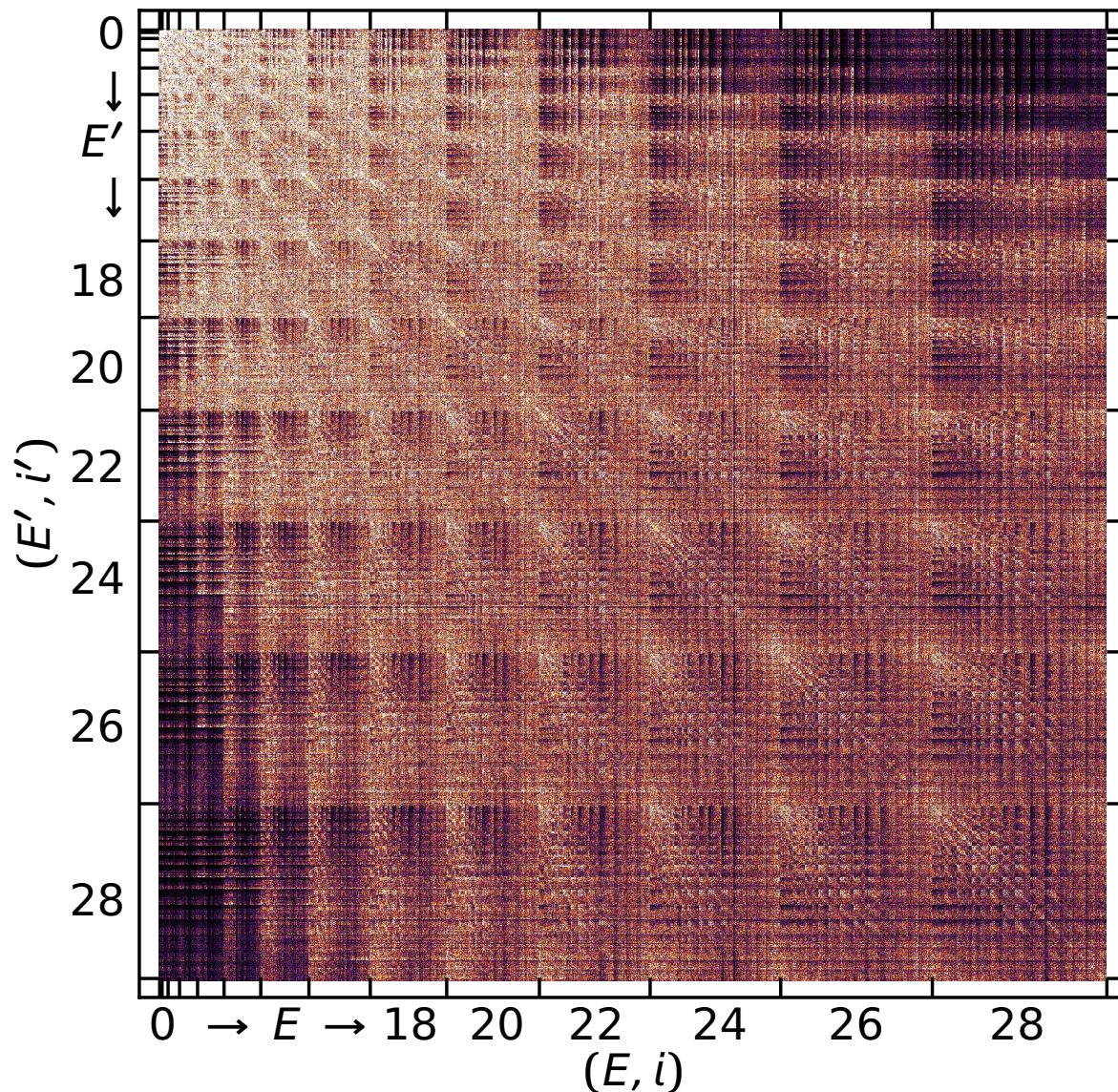
SRG Evolution of Matrix Elements

- convert Fock-space operator equations into **coupled evolution equations for matrix elements** in n -body Hilbert space
- $n = 3$: use **antisym. Jacobi-coordinate three-body states**
 - harmonic oscillator: $|Eij^\pi T\rangle$
- system of **coupled evolution equations** for each $(J^\pi T)$ -block

$$\frac{d}{d\alpha} \langle Eij^\pi T | \tilde{H}_\alpha | E' i' J^\pi T \rangle = (2\mu)^2 \sum_{E'' i''} \sum_{E''' i'''} \left[\begin{aligned} & \langle Ei\dots | T_{\text{int}} | E'' i'' \dots \rangle \langle E'' i'' \dots | \tilde{H}_\alpha | E''' i''' \dots \rangle \langle E''' i''' \dots | \tilde{H}_\alpha | E' i' \dots \rangle \\ & - 2 \langle Ei\dots | \tilde{H}_\alpha | E'' i'' \dots \rangle \langle E'' i'' \dots | T_{\text{int}} | E''' i''' \dots \rangle \langle E''' i''' \dots | \tilde{H}_\alpha | E' i' \dots \rangle \\ & + \langle Ei\dots | \tilde{H}_\alpha | E'' i'' \dots \rangle \langle E'' i'' \dots | \tilde{H}_\alpha | E''' i''' \dots \rangle \langle E''' i''' \dots | T_{\text{int}} | E' i' \dots \rangle \end{aligned} \right]$$

SRG Evolution in Three-Body Space

3B-Jacobi HO matrix elements

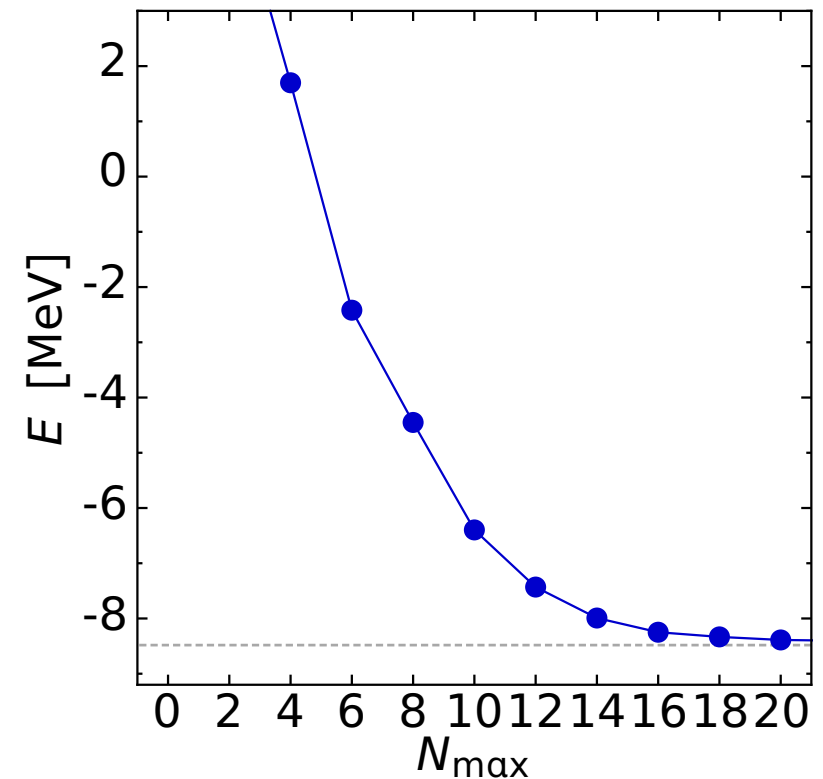


chiral NN+3N

$N^3\text{LO} + N^2\text{LO}$, triton-fit, 500 MeV

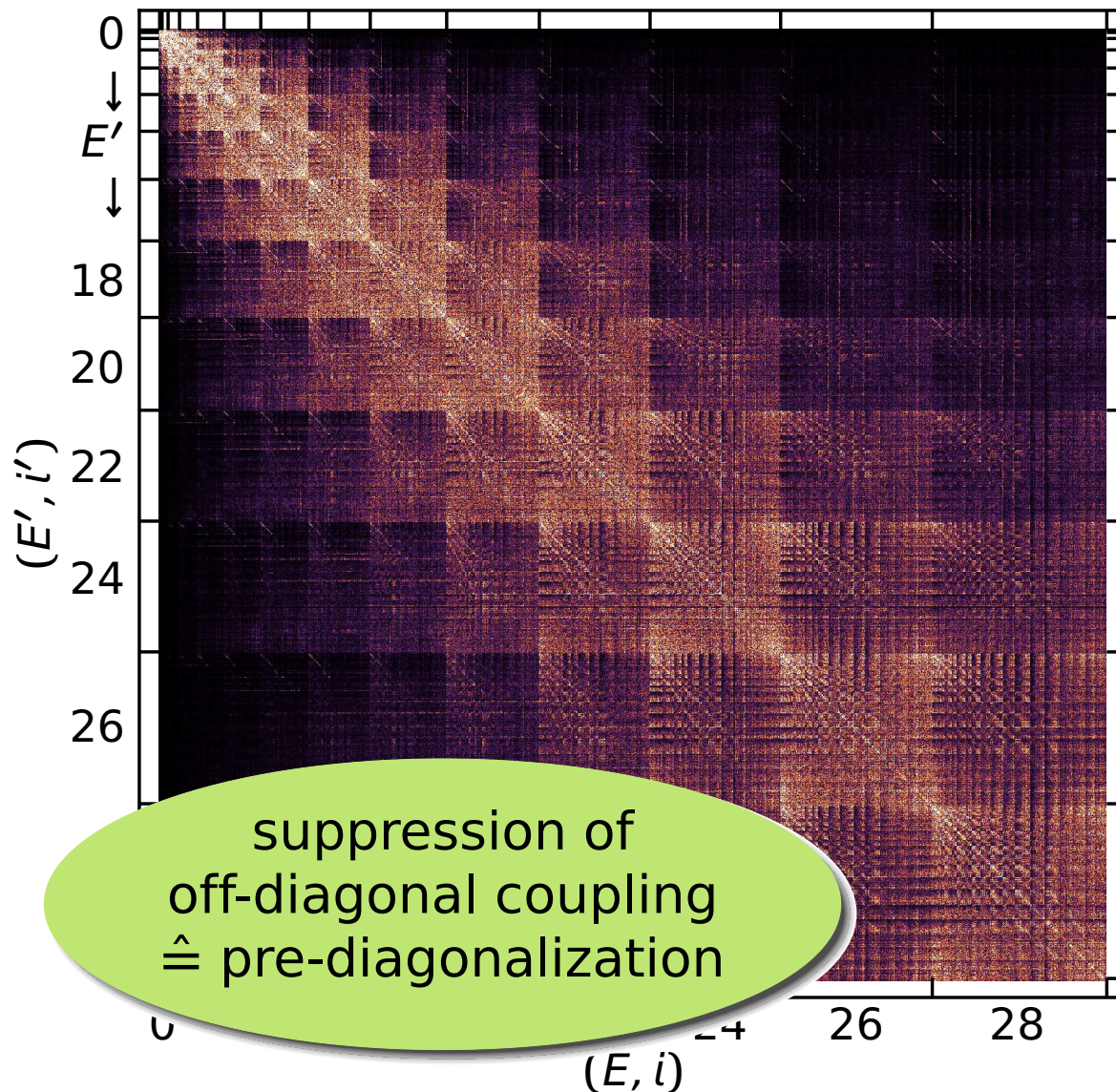
$$J^\pi = \frac{1}{2}^+, T = \frac{1}{2}, \hbar\Omega = 28 \text{ MeV}$$

NCSM ground state ${}^3\text{H}$



SRG Evolution in Three-Body Space

3B-Jacobi HO matrix elements

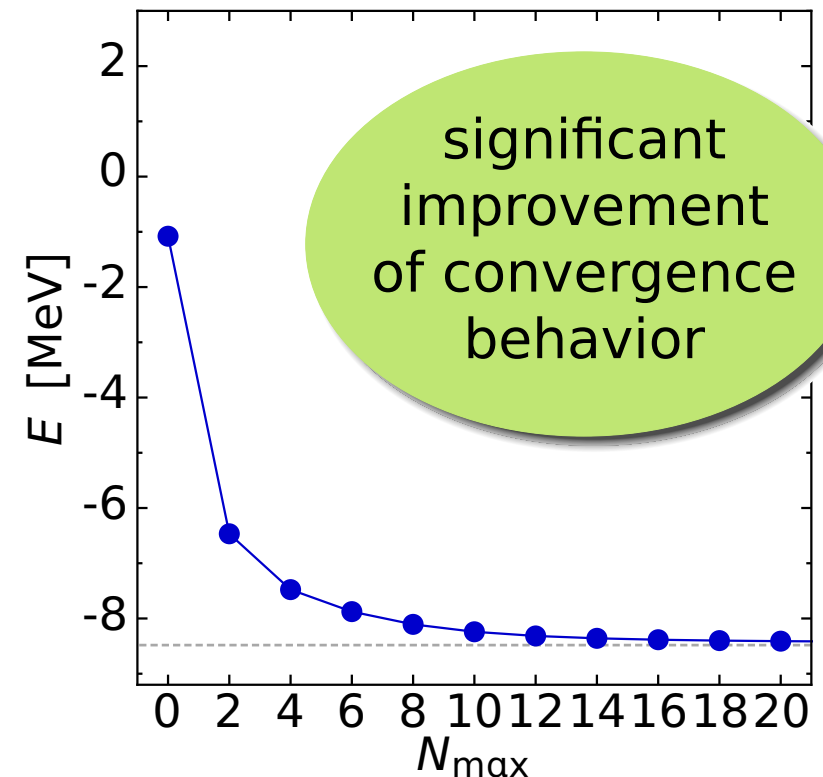


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NCSM ground state ${}^3\text{H}$



SRG Evolution in A-Body Space

- assume **initial Hamiltonian** and intrinsic kinetic energy are two-body operators written in second quantization

$$\tilde{H}_0 = \sum \dots a^\dagger a^\dagger a a, \quad T_{\text{int}} = T - T_{\text{cm}} = \sum \dots a^\dagger a^\dagger a a$$

- perform **single evolution step** $\Delta\alpha$ in Fock-space representation

$$\begin{aligned} \tilde{H}_{\Delta\alpha} &= \tilde{H}_0 + \Delta\alpha \left[[T_{\text{int}}, \tilde{H}_0], \tilde{H}_0 \right] \\ &= \sum \dots a^\dagger a^\dagger a a \\ &+ \Delta\alpha \left(\sum \dots a^\dagger a^\dagger a a \right) \left(\sum \dots a^\dagger a^\dagger a a \right) \left(\sum \dots a^\dagger a^\dagger a a \right) + \dots \end{aligned}$$

- unitary transformation **induces many-body contributions** in the Hamiltonian

Hamiltonian in A-Body Space

- **cluster decomposition**: decompose evolved Hamiltonian into irreducible n -body contributions $\tilde{H}_\alpha^{[n]}$

$$\tilde{H}_\alpha = \tilde{H}_\alpha^{[1]} + \tilde{H}_\alpha^{[2]} + \tilde{H}_\alpha^{[3]} + \dots + \tilde{H}_\alpha^{[n]} + \dots$$

- **A-body unitarity**: transformation is unitary only if all terms up to $n = A$ are kept, then all eigenvalues are independent of α
- **cluster truncation**: can construct contributions up to $n = 3$ from evolution in 2B and 3B space, but have to discard $n > 3$
- α -dependence of eigenvalues
measures impact of discarded

α -variation provides a **diagnostic tool** to assess the contributions of omitted many-body interactions

Introduction to Ab Initio Nuclear Structure Theory

Part II: Many-Body Problem

Robert Roth

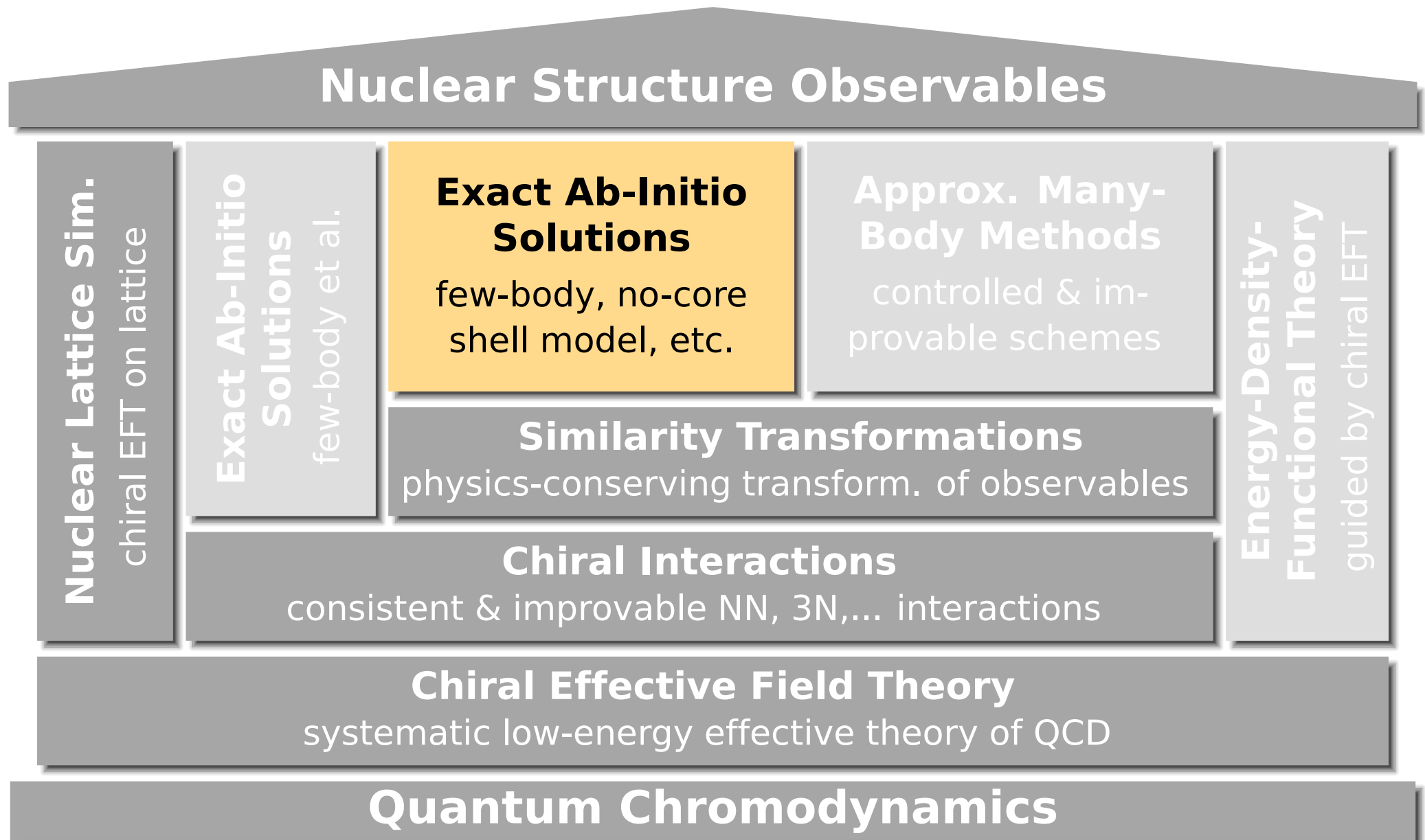


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Ab Initio Many-Body Methods

No-Core Shell Model

Building Blocks



No-Core Shell Model — Basics

- **many-body basis**: Slater determinants $|\Phi_{\nu}^{\text{SD}}\rangle$ composed of harmonic oscillator single-particle states (m-scheme)

$$|\Psi\rangle = \sum_{\nu} C_{\nu} |\Phi_{\nu}^{\text{SD}}\rangle$$

- **model space**: spanned by basis states $|\Phi_{\nu}^{\text{SD}}\rangle$ with unperturbed excitation energies of up to $N_{\text{max}}\hbar\Omega$

with increasing model space size more and more **correlations can be described** by the model space

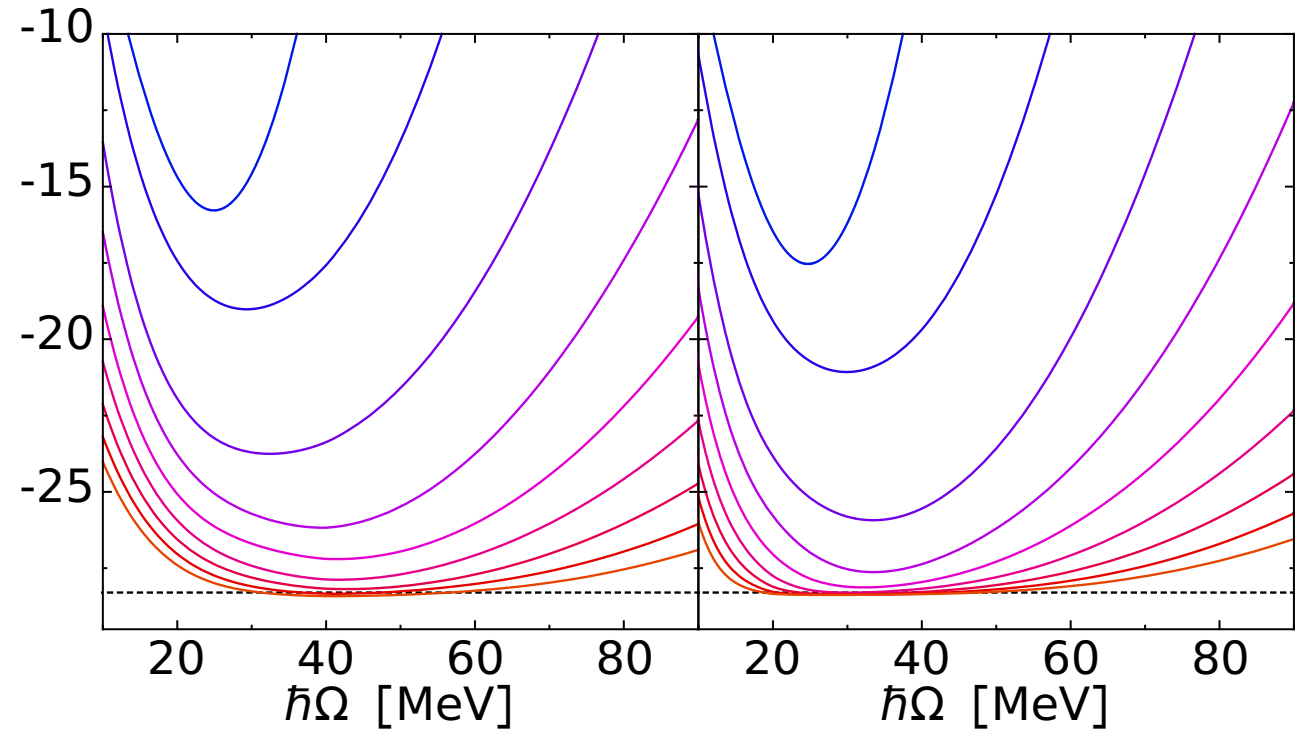
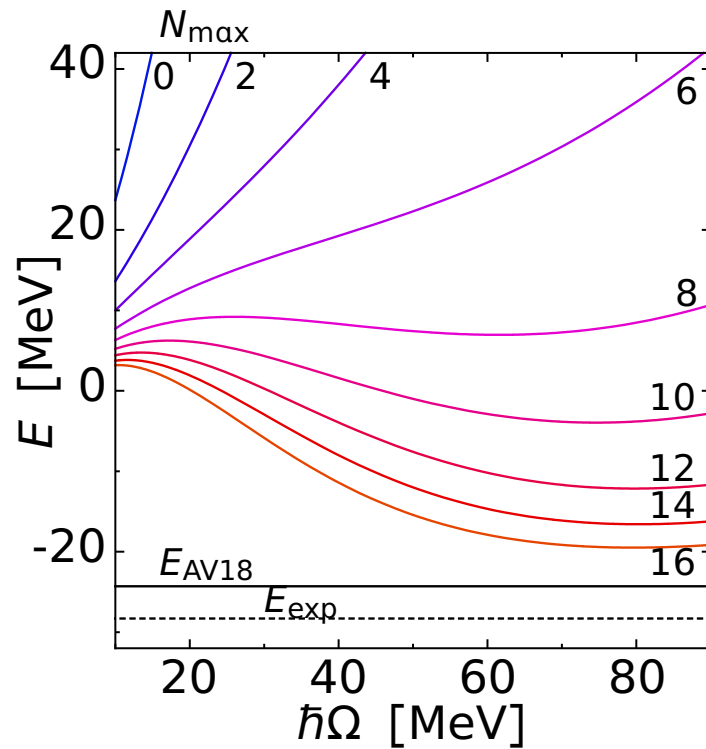
convergence of observables with N_{max} has to be investigated and is crucial

${}^4\text{He}$: NCSM Convergence

V_{AV18}

V_{UCOM}
MIN, $I_9 = 0.09 \text{ fm}^3$

V_{SRG}
 $\alpha = 0.03 \text{ fm}^4$



- I_9 or $\bar{\alpha}$ adjusted such that ${}^4\text{He}$ binding energy is reproduced

High-End Applications

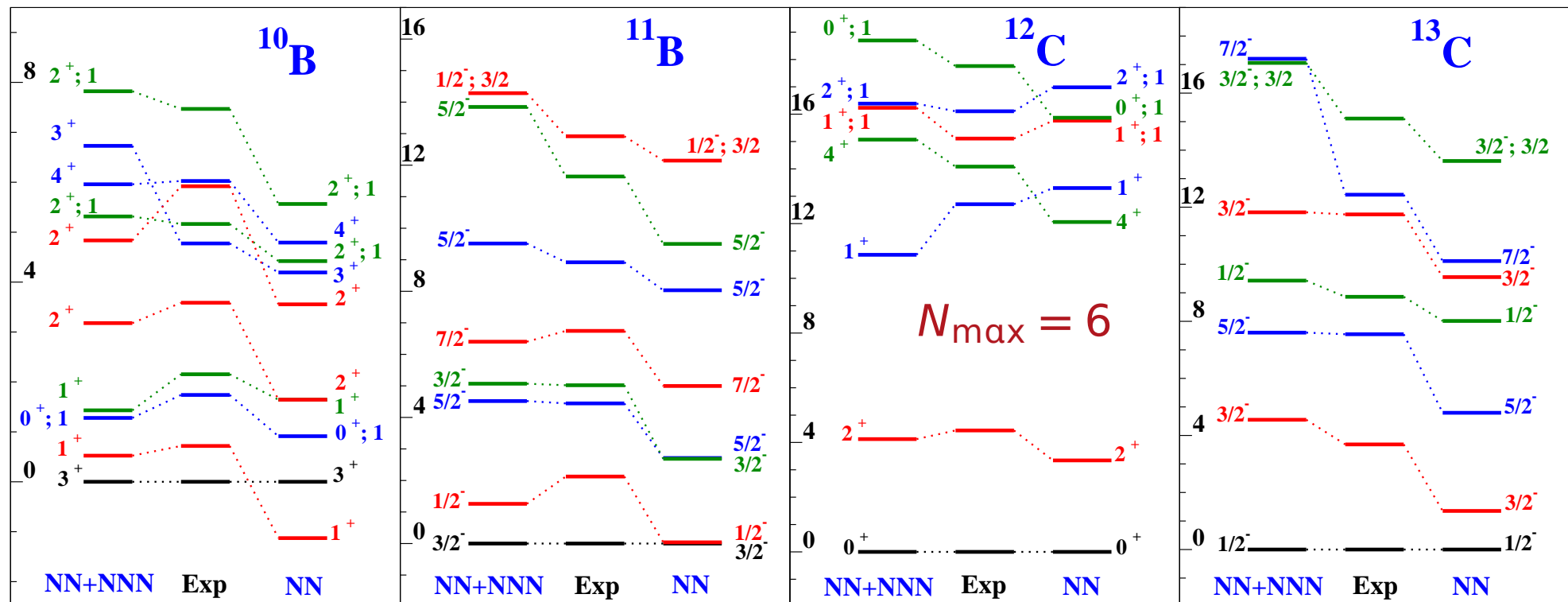
PRL 99, 042501 (2007)

PHYSICAL REVIEW LETTERS

week ending
27 JULY 2007

Structure of $A = 10-13$ Nuclei with Two- Plus Three-Nucleon Interactions from Chiral Effective Field Theory

P. Navrátil,¹ V. G. Gueorguiev,^{1,*} J. P. Vary,^{1,2} W. E. Ormand,¹ and A. Nogga³

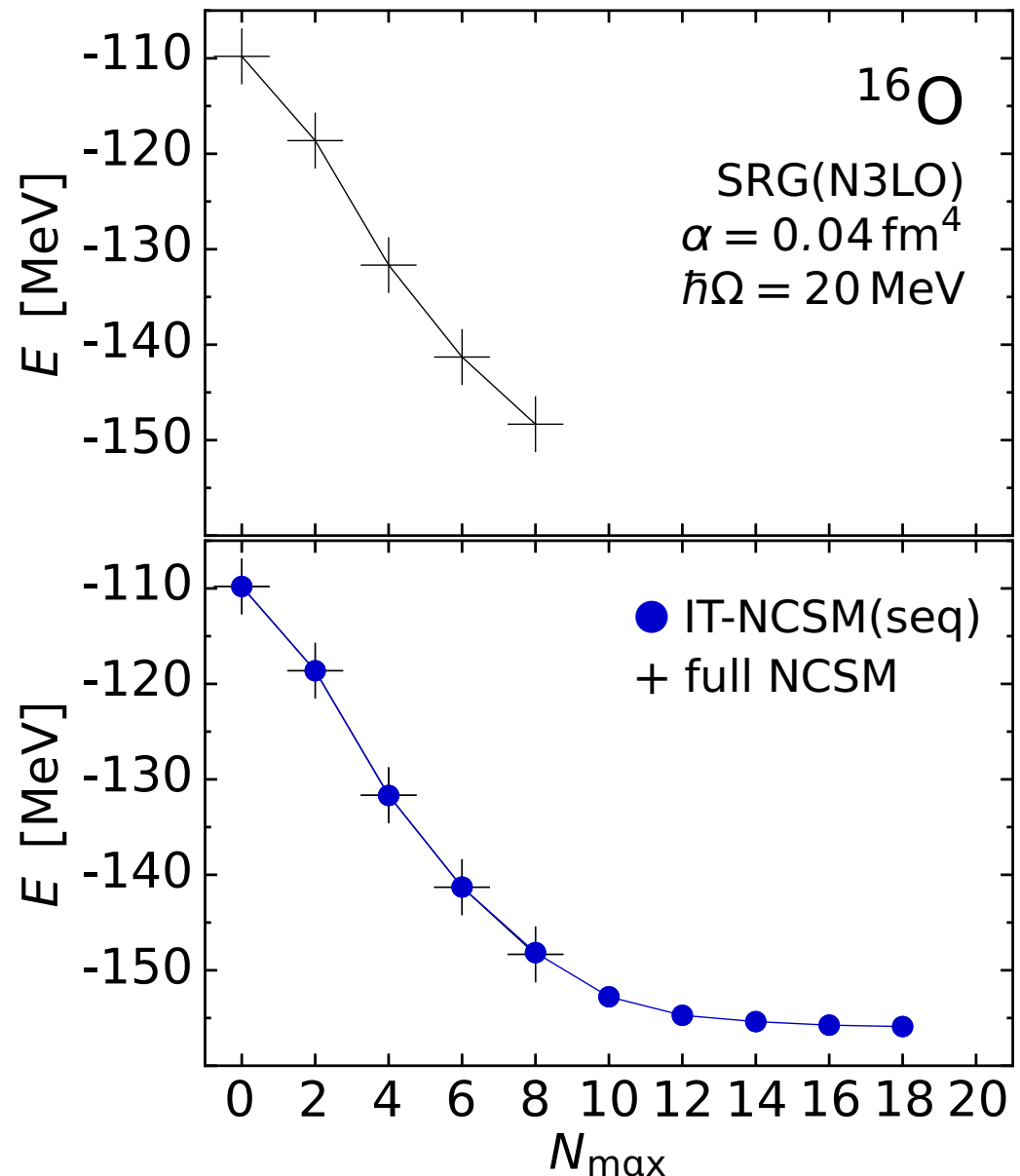


Importance Truncated NCSM

- converged NCSM calculations essentially restricted to lower/mid p-shell
- full 10 or 12 $\hbar\Omega$ calculation for ^{16}O not really feasible (basis dimension $> 10^{10}$)

Importance Truncation

reduce model space to the relevant basis states using an **a priori importance measure** derived from MBPT



A Tale of Three Hamiltonians

Initial Hamiltonian

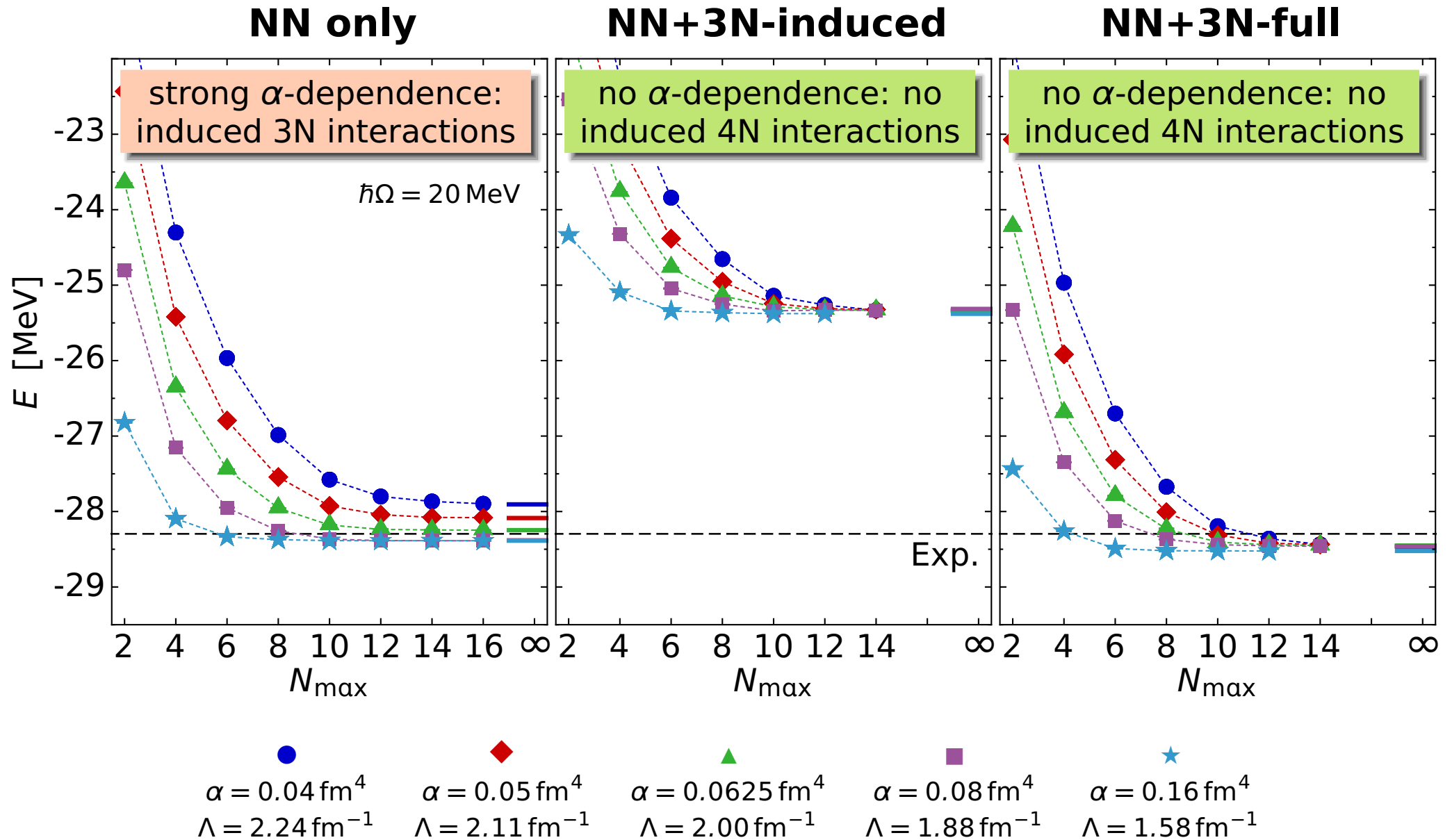
- NN: chiral interaction at $N^3\text{LO}$ (Entem & Machleidt, 500 MeV)
- 3N: chiral interaction at $N^2\text{LO}$ (c_D, c_E from ${}^3\text{H}$ binding & half-life)

SRG-Evolved Hamiltonians

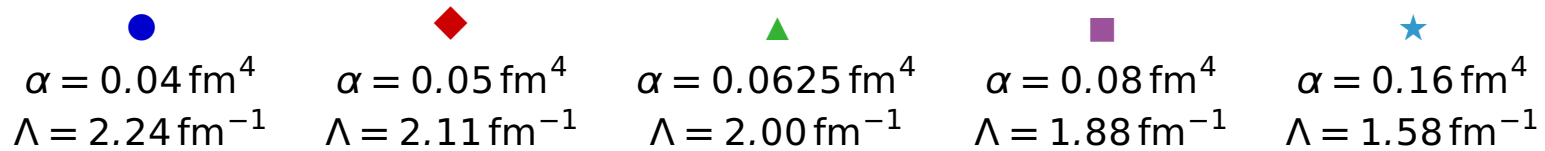
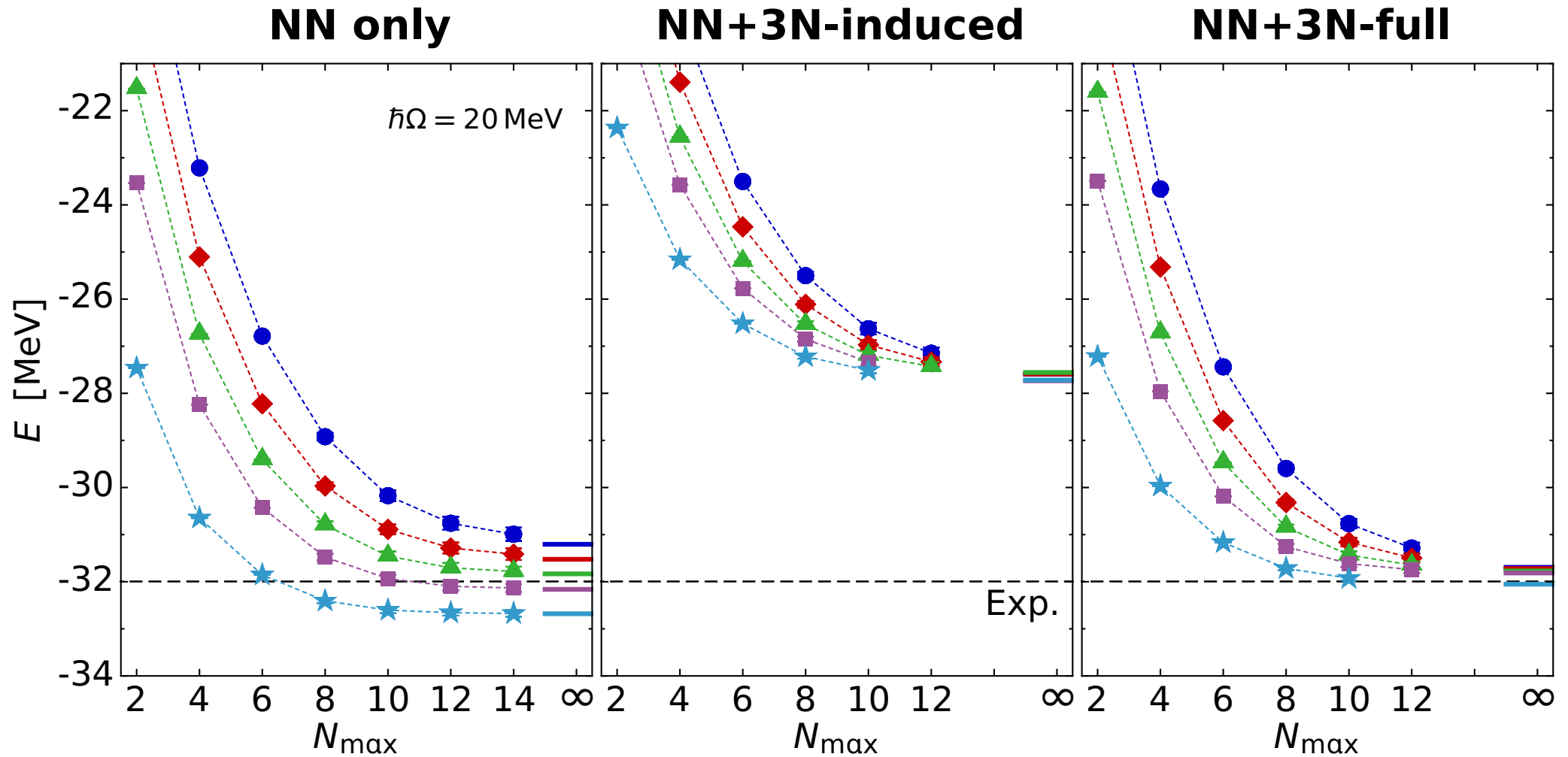
- **NN only**: start with NN initial Hamiltonian and keep two-body terms only
- **NN+3N-induced**: start with NN initial Hamiltonian and keep two- and three-body terms
- **NN+3N-full**: start with NN+3N initial Hamiltonian and keep two- and three-body terms

α -variation provides a **diagnostic tool** to assess the contributions of omitted many-body interactions

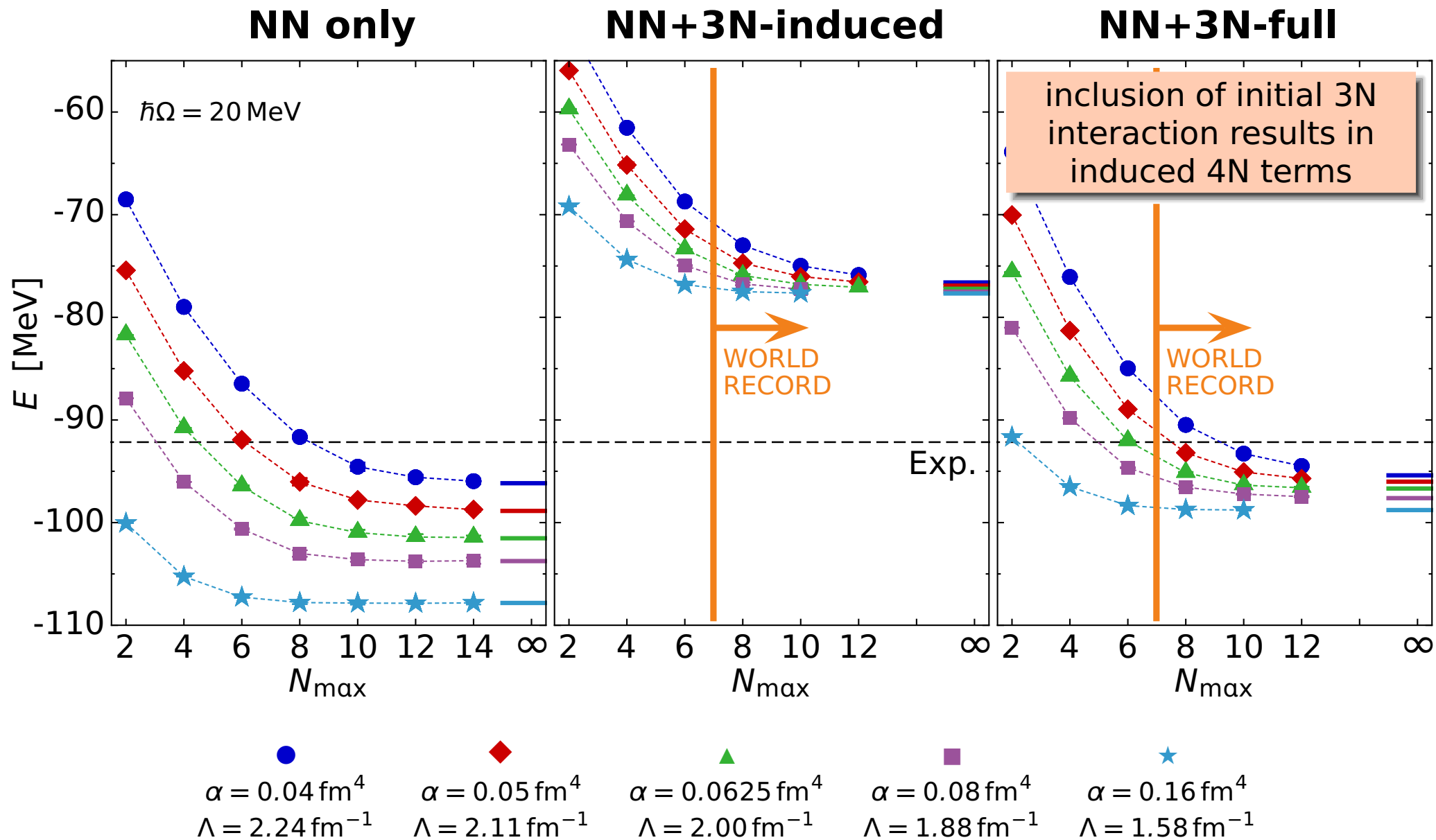
^4He : Ground-State Energies



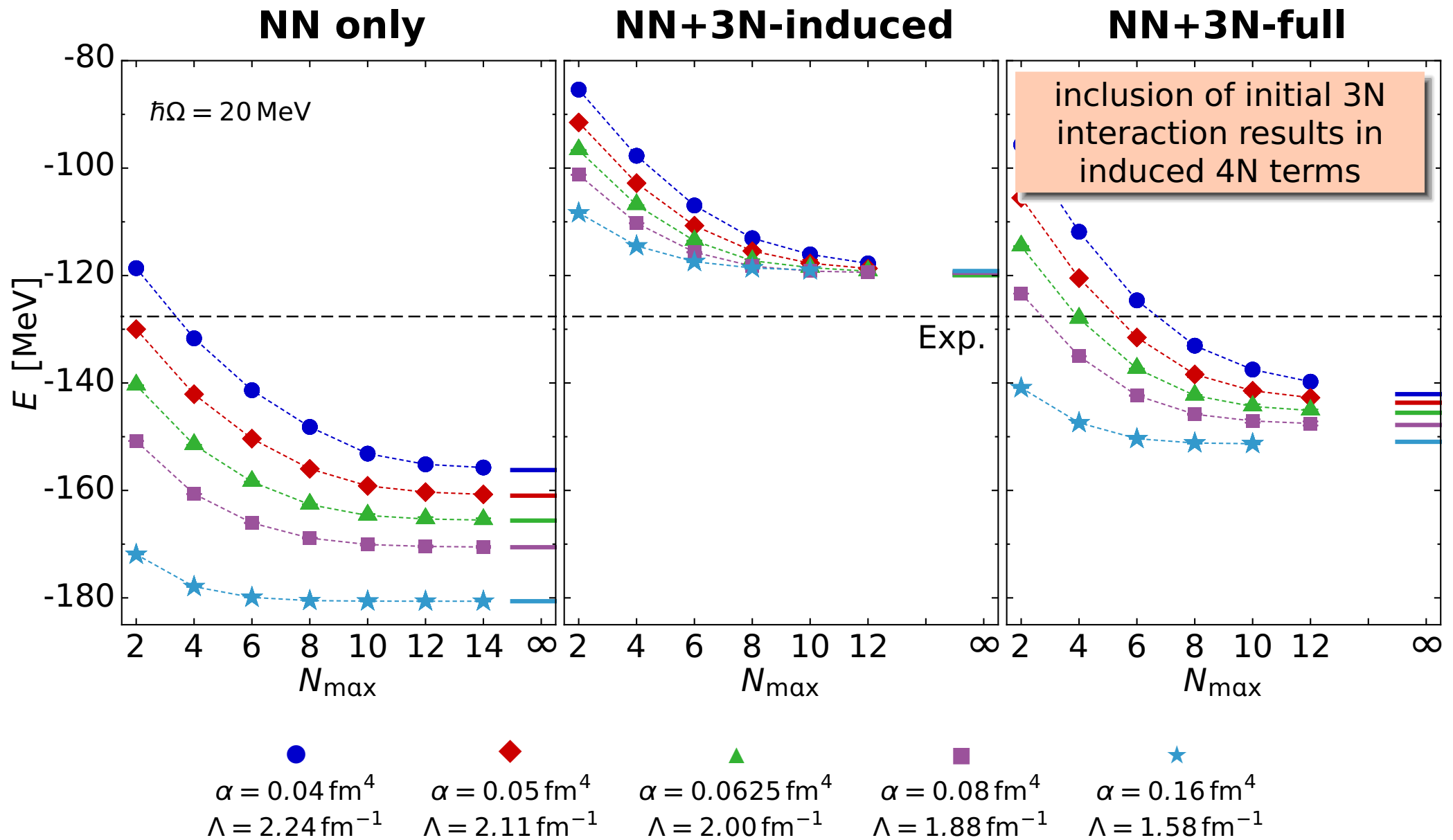
${}^6\text{Li}$: Ground-State Energies



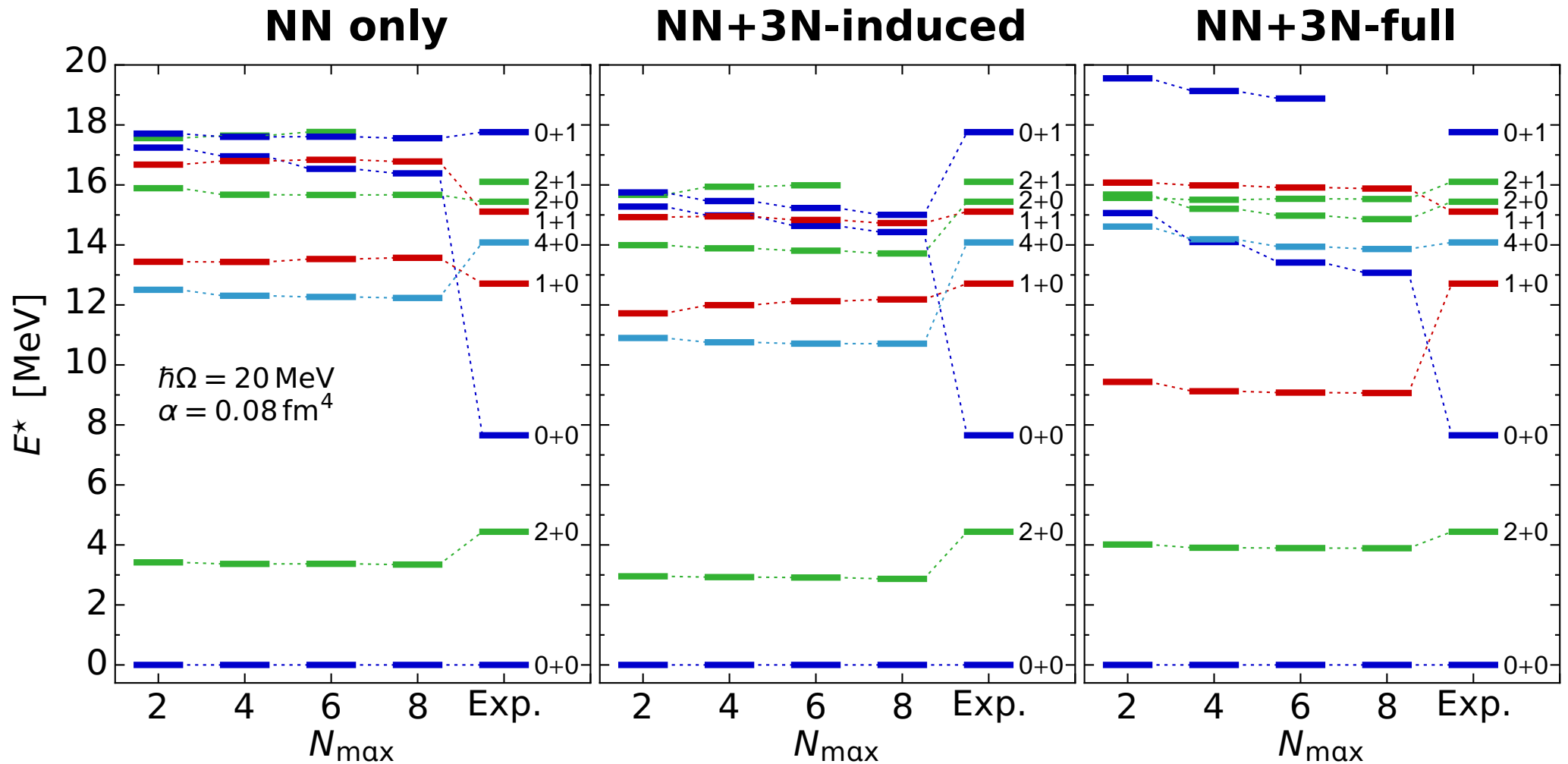
^{12}C : Ground-State Energies



^{16}O : Ground-State Energies



Spectroscopy of ^{12}C

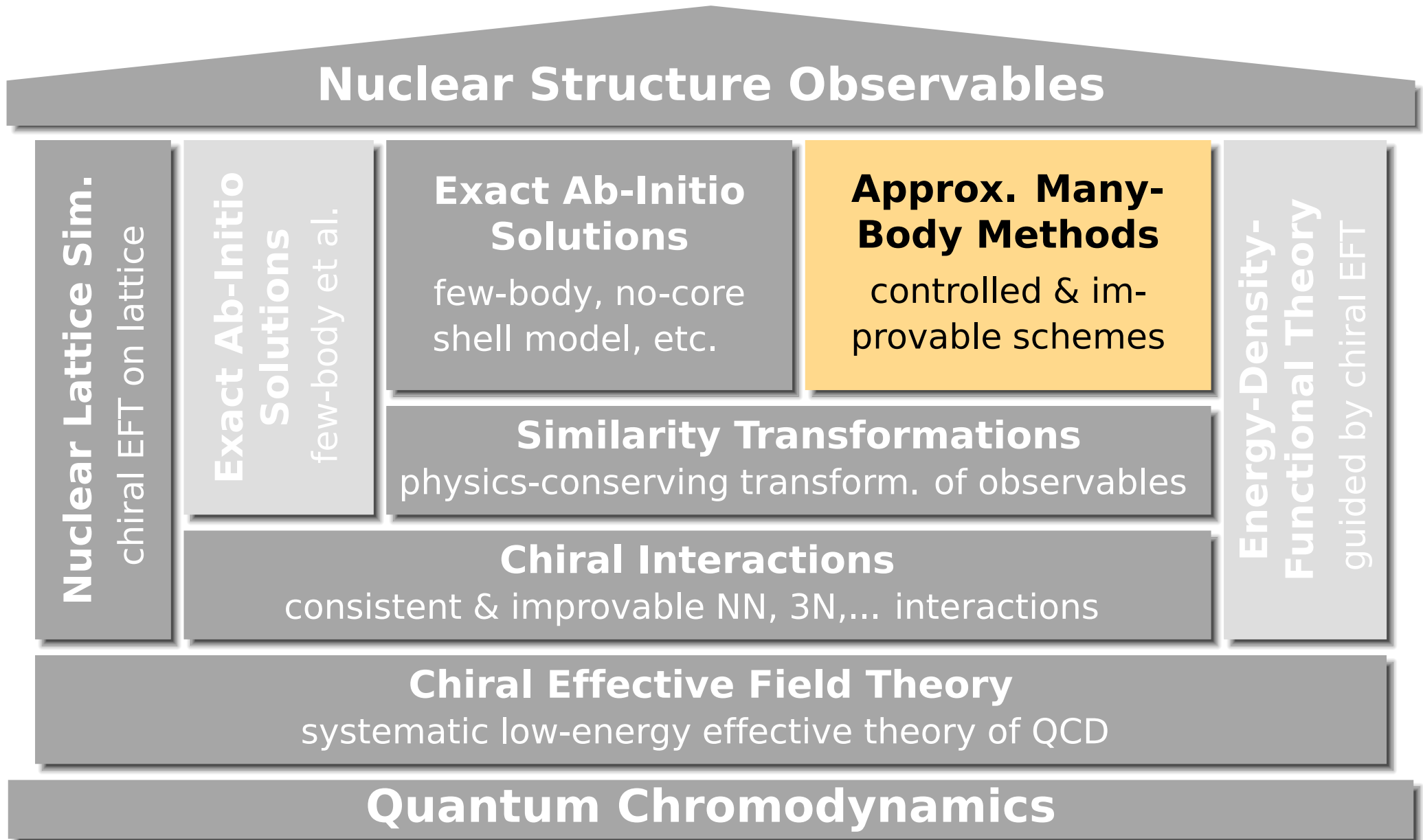


- IT-NCSM gives access to **complete spectroscopy of p- and sd-shell nuclei** starting from chiral NN+3N interactions

Approximate Many-Body Methods

Hartree-Fock & Beyond

Building Blocks



Hartree-Fock Approximation — Basics

- ground state $|\Psi\rangle$ approximated by a **single Slater determinant**

$$|\text{HF}\rangle = |\phi_1, \phi_2, \dots, \phi_A\rangle_a = \sum_{\pi} \text{sgn}(\pi) P_{\pi} |\phi_1\rangle \otimes |\phi_2\rangle \otimes \dots \otimes |\phi_A\rangle$$

- **variational calculation**: single-particle states $|\phi_i\rangle$ determined by minimizing the energy expectation value

$$E_{\text{HF}} = \langle \text{HF} | H_{\text{int}} | \text{HF} \rangle = \frac{1}{2} \sum_{i,j=1}^A \langle \phi_i \phi_j | (T_{\text{int}} + V_{\text{NN}}) | \phi_i \phi_j \rangle_a$$

single Slater determinant by definition
cannot describe any correlations

Hartree-Fock solution is **starting point for improved calculations**

Perturbation Theory — Basics

- start from HF state as zeroth-order ‘unperturbed’ state and construct **perturbative corrections** to the state and the energy

$$|\Psi\rangle = |\text{HF}\rangle + \lambda |\Delta\Psi^{(1)}\rangle + \lambda^2 |\Delta\Psi^{(2)}\rangle + \lambda^3 |\Delta\Psi^{(3)}\rangle + \dots$$

$$E = E_{\text{HF}} + \lambda \Delta E^{(1)} + \lambda^2 \Delta E^{(2)} + \lambda^3 \Delta E^{(3)} + \dots$$

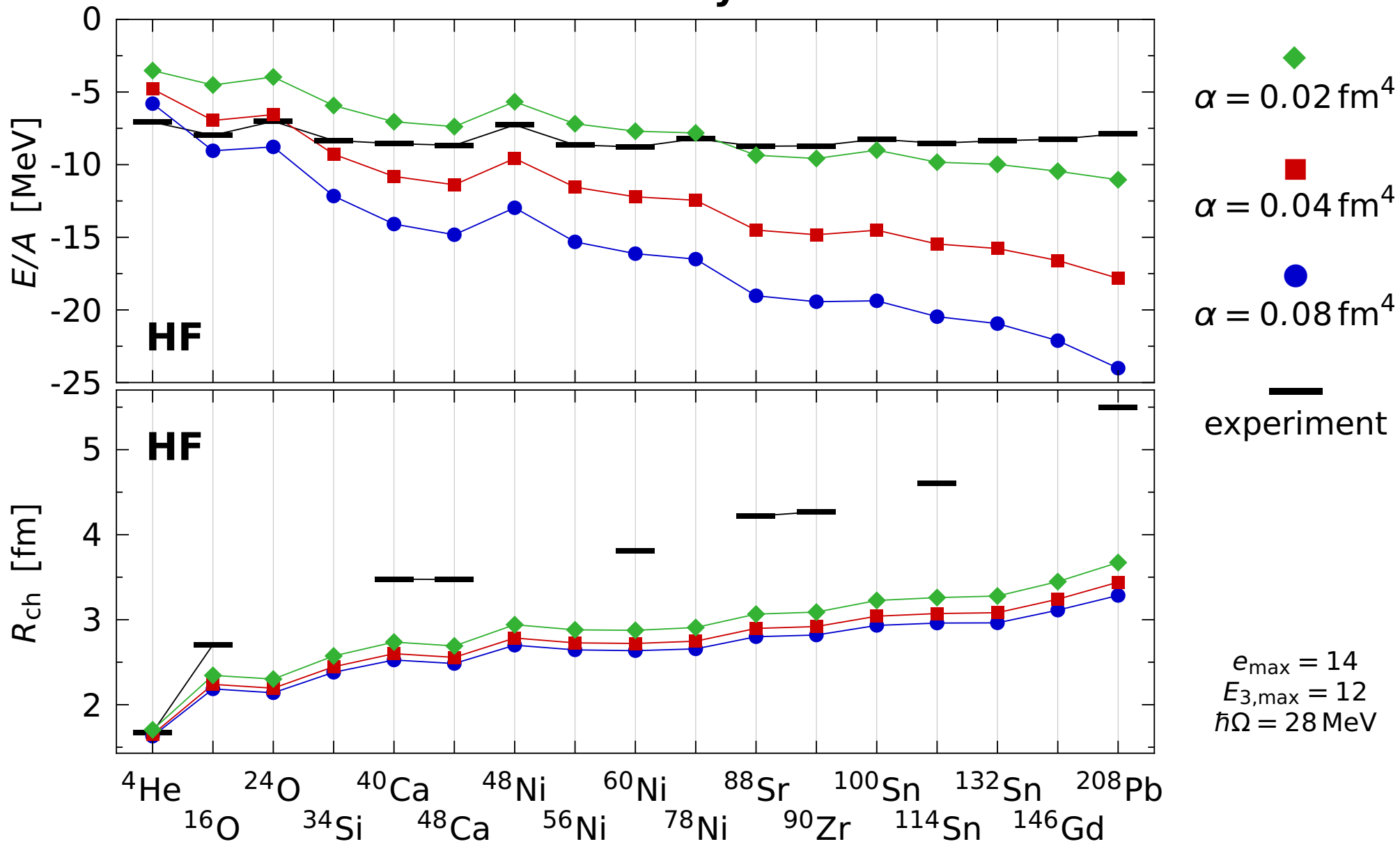
- **second-order energy correction** gives estimate for influence of correlations

$$\Delta E^{(2)} = -\frac{1}{4} \sum_{i,j}^{\text{occu.}} \sum_{k,l}^{\text{unoccu.}} \frac{|{}_a\langle \phi_k \phi_l | (T_{\text{int}} + V_{\text{NN}}) | \phi_i \phi_j \rangle_a|^2}{\epsilon_k + \epsilon_l - \epsilon_i - \epsilon_j}$$

- higher orders or partial all-order summations can be evaluated
- eventually, at high-order (with appropriate resummations) perturbation theory will **recover the exact result**

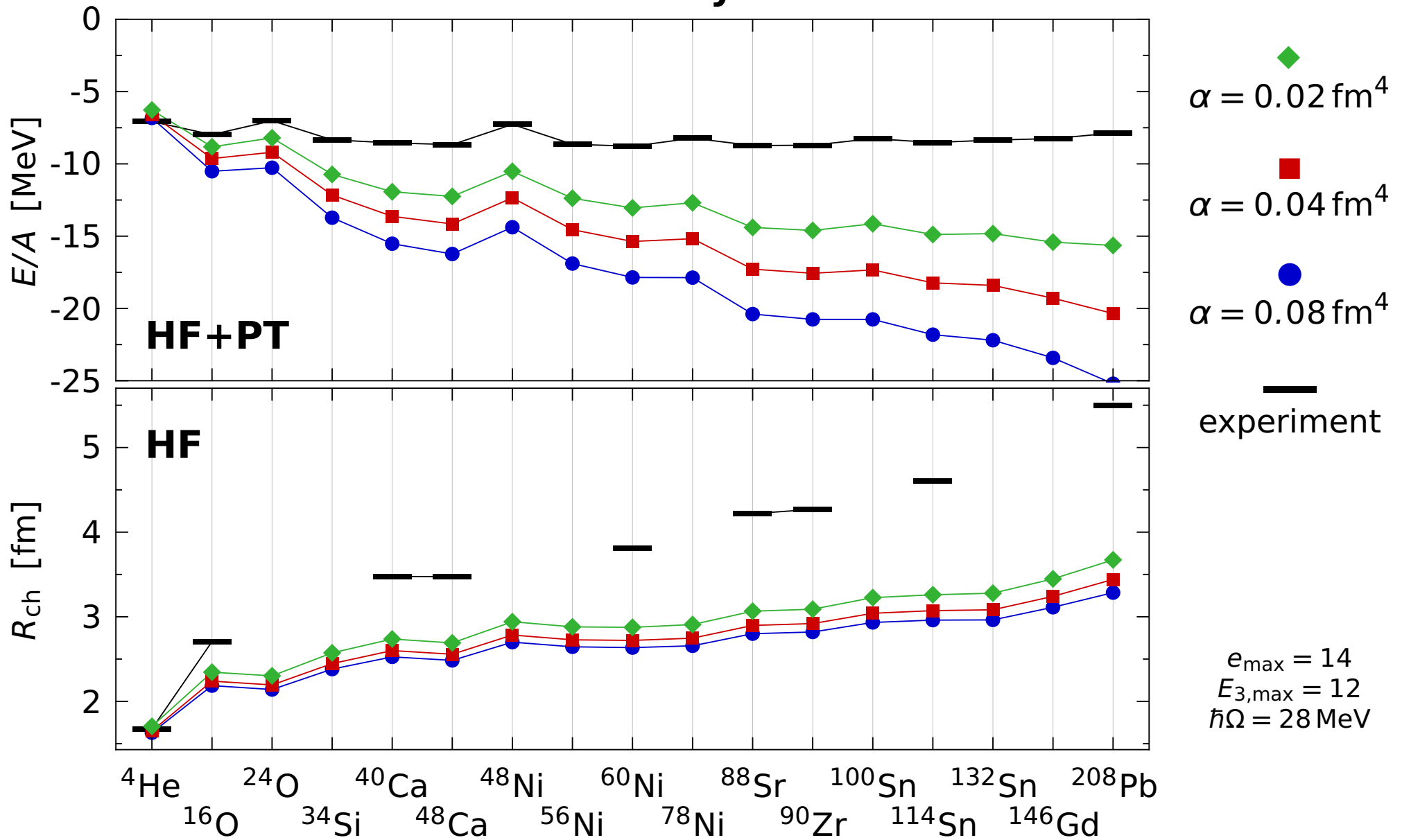
Systematics: E/A and R_{ch}

NN-only



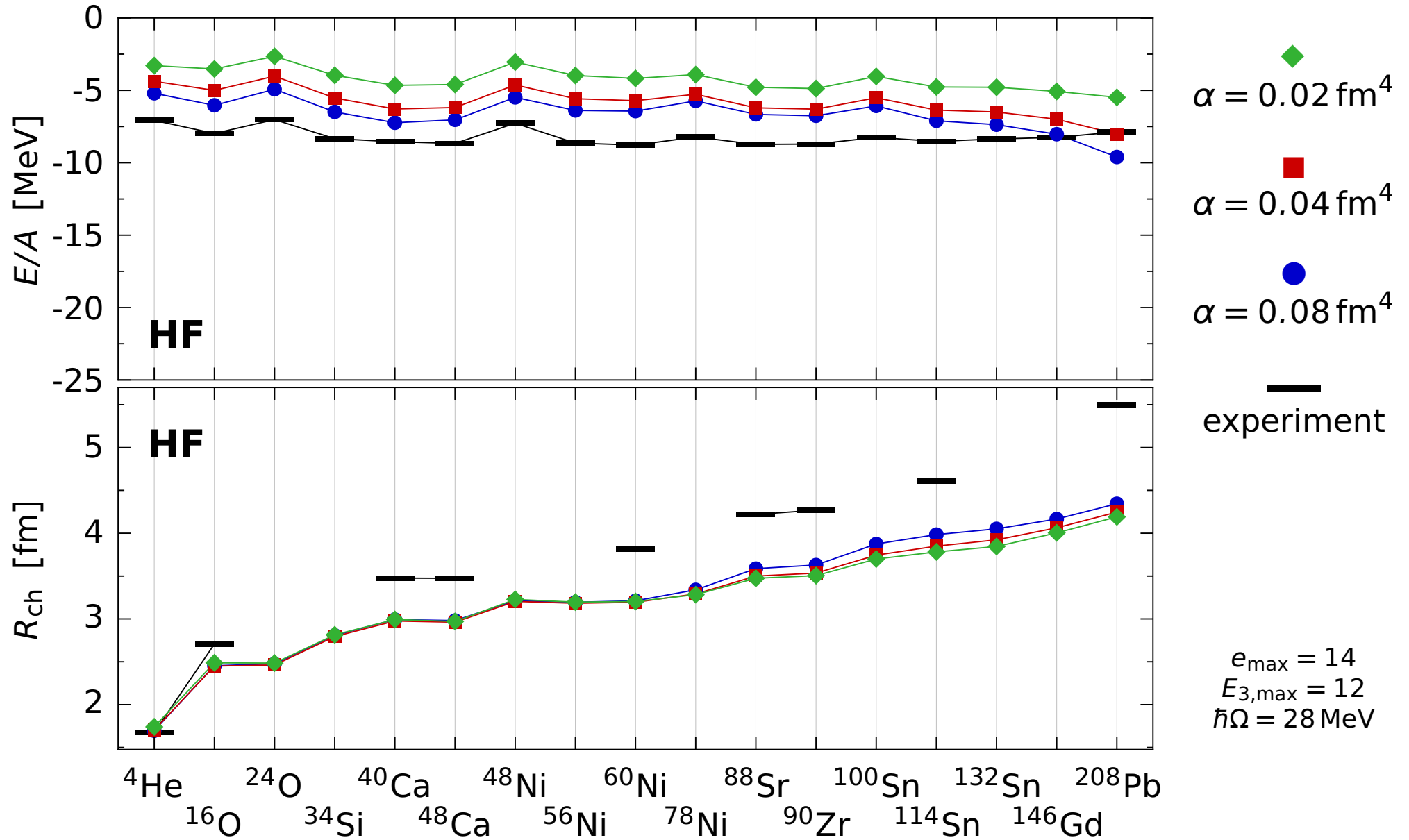
Systematics: E/A and R_{ch}

NN-only



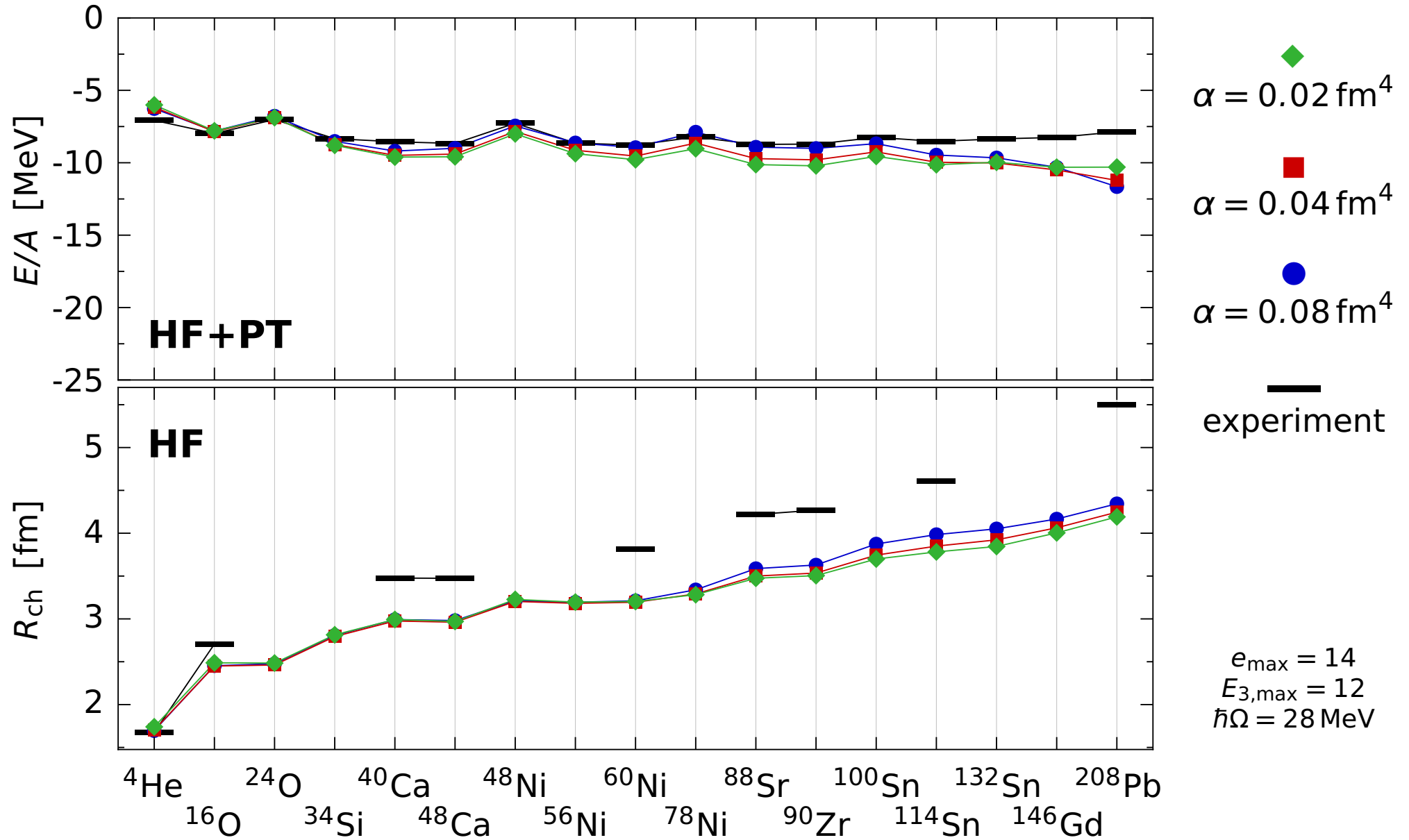
Systematics: E/A and R_{ch}

NN + 3N-induced



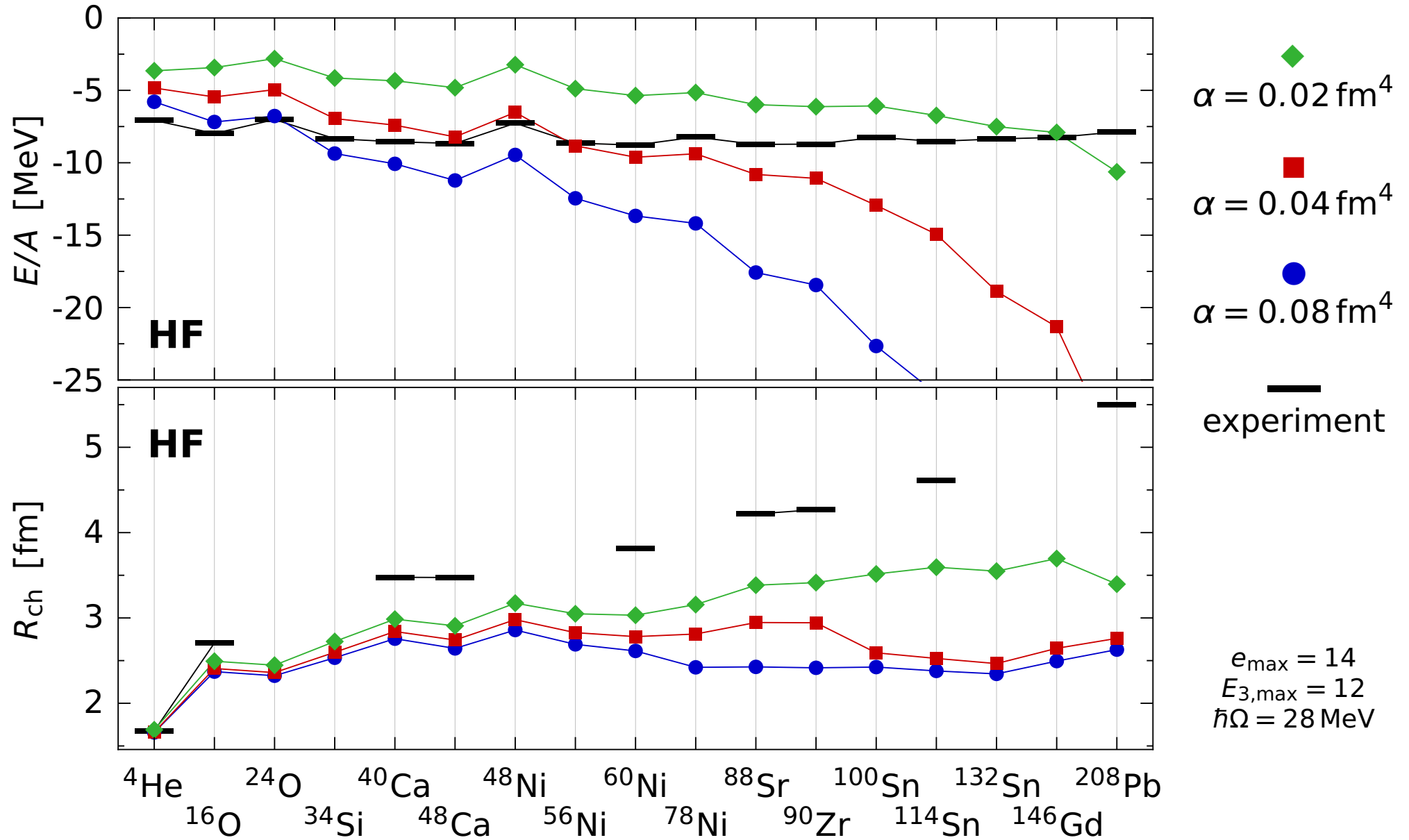
Systematics: E/A and R_{ch}

NN + 3N-induced



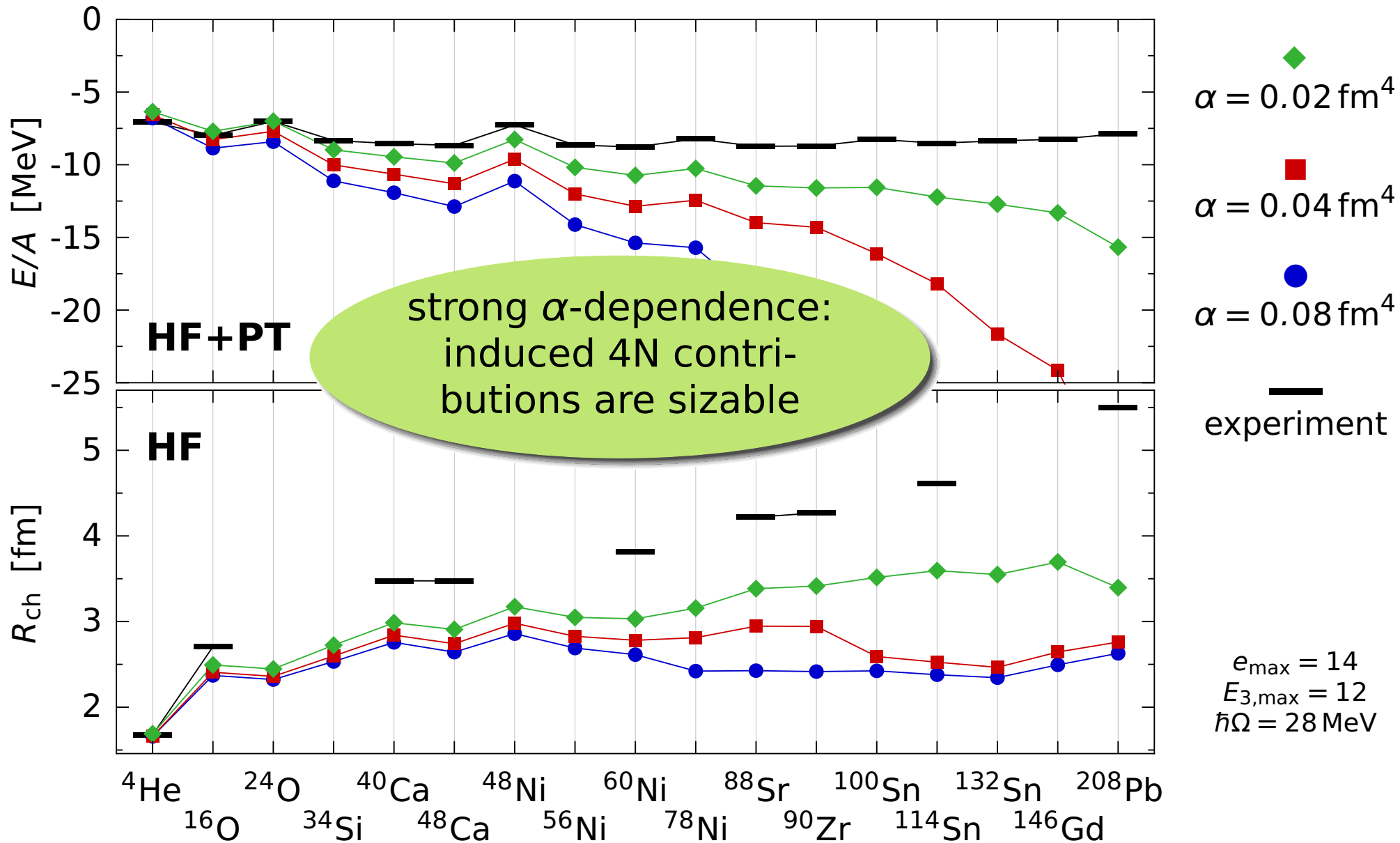
Systematics: E/A and R_{ch}

NN + 3N-full



Systematics: E/A and R_{ch}

NN + 3N-full



Conclusions

Conclusions

- new era of **ab-initio nuclear structure and reaction theory** connected to QCD via chiral EFT
 - chiral EFT as universal starting point... some formal issues remain
- consistent **inclusion of 3N interactions** in similarity transformations & many-body calculations
 - breakthrough in computation & handling of 3N matrix elements
- **innovations in many-body theory**: extended reach of exact methods & improved control over approximations
 - versatile toolbox for different observables & mass ranges
- many **exciting applications** ahead...

Epilogue

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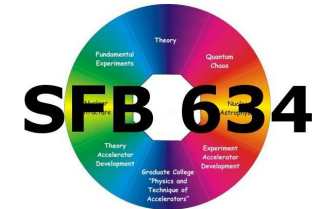
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GSI Helmholtzzentrum



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