Structure of Trapped Degenerate Fermi Gases

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Overview

- The World of Trapped Atomic Fermi Gases
- Theoretical Description of Trapped Degenerate Fermi Gases
  - The Many-Body Problem
  - Correlations & Effective Interaction
  - Mean-Field & Thomas-Fermi Approximation
  - Energy Functional
- Structure of Single- and Two-Component Fermi Gases
  - Energy Landscapes & Density Profiles
  - Mean-Field Induced Collapse
  - Component Separation
  - Phase Diagram

Trapped Degenerate Fermi Gas

Science 285 (1999) 1703

Onset of Fermi Degeneracy in a Trapped Atomic Gas

B. DeMarco and D. S. Jin*†

An evaporative cooling strategy that uses a two-component Fermi gas was employed to cool a magnetically trapped gas of 7×10^5 ^{40}K atoms to 0.5 of the Fermi temperature T_F. In this temperature regime, where the state occu- ... the lowest energies has increased from essentially zero ... nearly 60 percent, quantum degeneracy was ... ative cooling and as a modification of the t... the m... the tota... gas di... ics.

cooling a cloud of neutral ^{40}K atoms kept in a magnetic trap

^{40}K has fractional total spin: **fermion**

$$F = \underset{I}{4} \pm \underset{S}{1/2} = \tfrac{9}{2}, \tfrac{7}{2}$$

two-component mixture

$$\left|F = \tfrac{9}{2}, m_F = \tfrac{9}{2}\right\rangle$$
$$\left|F = \tfrac{9}{2}, m_F = \tfrac{7}{2}\right\rangle$$

$N \approx 10^5 ... 10^6$

$\rho \approx 10\,\mu\mathrm{m}^{-3}$

$\ell \approx 1\,\mu\mathrm{m}$

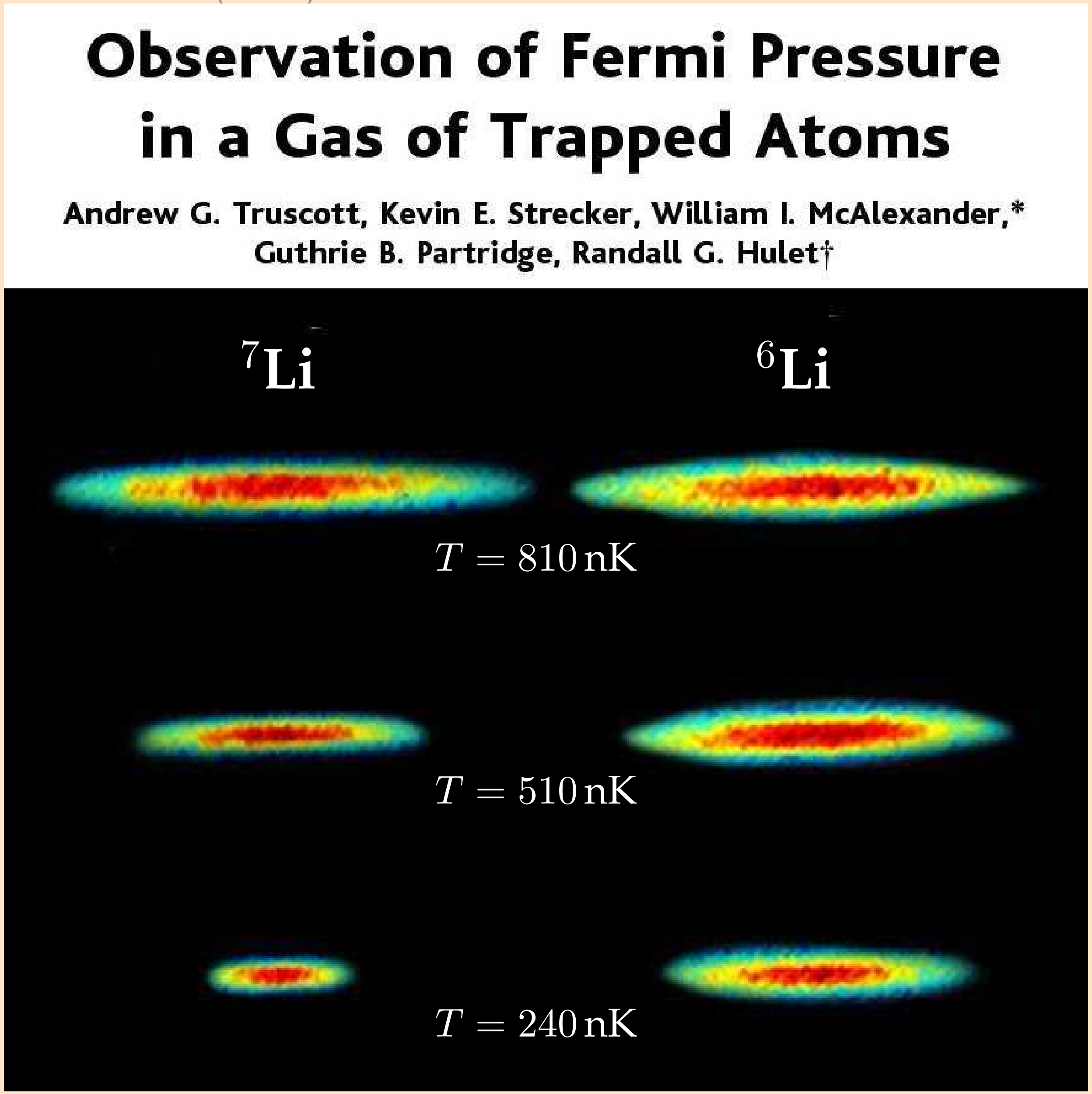
$\tau \approx 300\,\mathrm{s}$

$T \approx 300\,\mathrm{nK}$
$\approx 0.5\,\varepsilon_F$

Houston / Texas — March 2001

Degenerate Boson-Fermion Mixtures

Science 291 (2001) 2570


Observation of Fermi Pressure in a Gas of Trapped Atoms

Andrew G. Truscott, Kevin E. Strecker, William I. McAlexander,* Guthrie B. Partridge, Randall G. Hulet†

simultaneous trapping of

$^7\mathrm{Li} \;\rightarrow\; F = 2 \;\rightarrow\;$ **boson**

$^6\mathrm{Li} \;\rightarrow\; F = \frac{3}{2} \;\rightarrow\;$ **fermion**

evaporative cooling of the bosons $\rightarrow$ sympathetic cooling of the fermions

$N_\mathrm{B} \approx N_\mathrm{F} \approx 10^5 \ldots 10^6$

$T \approx 240\,\mathrm{nK}$
$\approx 0.25\,\varepsilon_F$

Fermion Experiments — Today

Two-Component Fermi Gases

09/1999	$^{40}\mathbf{K}$	$T = 0.5\varepsilon_F$	$N_F \sim 10^6$	JILA, Boulder/Colorado, B. DeMarco, D.S. Jin
11/2001	$^{6}\mathbf{Li}$	$T = 0.5\varepsilon_F$	$N_F \sim 10^5$	Duke Univ., Durham/North Carolina, S.R. Granade,..., J.E. Thomas

Binary Boson-Fermion Mixtures

03/2001	$^{7}\mathbf{Li}/^{6}\mathbf{Li}$	$T = 0.25\varepsilon_F$	$N_F \sim 10^5$	Rice Univ., Houston/Texas, A.G. Truscott,..., R.G. Hulet
07/2001	$^{7}\mathbf{Li}/^{6}\mathbf{Li}$	$T = 0.2\varepsilon_F$	$N_F \sim 10^4$	ENS, Paris F. Schreck,..., C. Salomon
08/2001	$^{87}\mathbf{Rb}/^{40}\mathbf{K}$	—	$N_F \sim 10^7$	JILA, Boulder/Colorado J. Goldwin,..., D.S. Jin
12/2001	$^{23}\mathbf{Na}/^{6}\mathbf{Li}$	$T = 0.5\,\varepsilon_F$	$N_F \sim 10^6$	MIT, Cambridge/Massachusetts Z. Hadzibabic,..., W. Ketterle

Theoretical Description of Trapped Degenerate (Fermi) Gases

- **The Many-Body Problem**
- **Correlations & Effective Interaction**
- **Mean-Field & Thomas-Fermi Approximation**
- **Energy Functional**

Route Through the Many-Body Problem

Hamiltonian

$$\mathbf{H} = \sum_i U(\vec{\mathbf{x}}_i) + \frac{1}{2m}\sum_i \vec{\mathbf{p}}_i^{\,2} + \sum_{i<j} \mathbf{V}_{ij}$$

Model Space

mean-field states: antisym.
product of single-particle states

Energy Functional

energy expectation value as
functional of the density

Thomas-Fermi Approx.

neglect all gradients of the
density in the energy functional

Functional Variation

ground state density is obtained
by energy minimization

Short-Range Correlations

Interaction

many realistic two-body interactions show a strong short-range repulsion

(e.g. nucleon-nucleon & van der Waals interactions)

Correlations

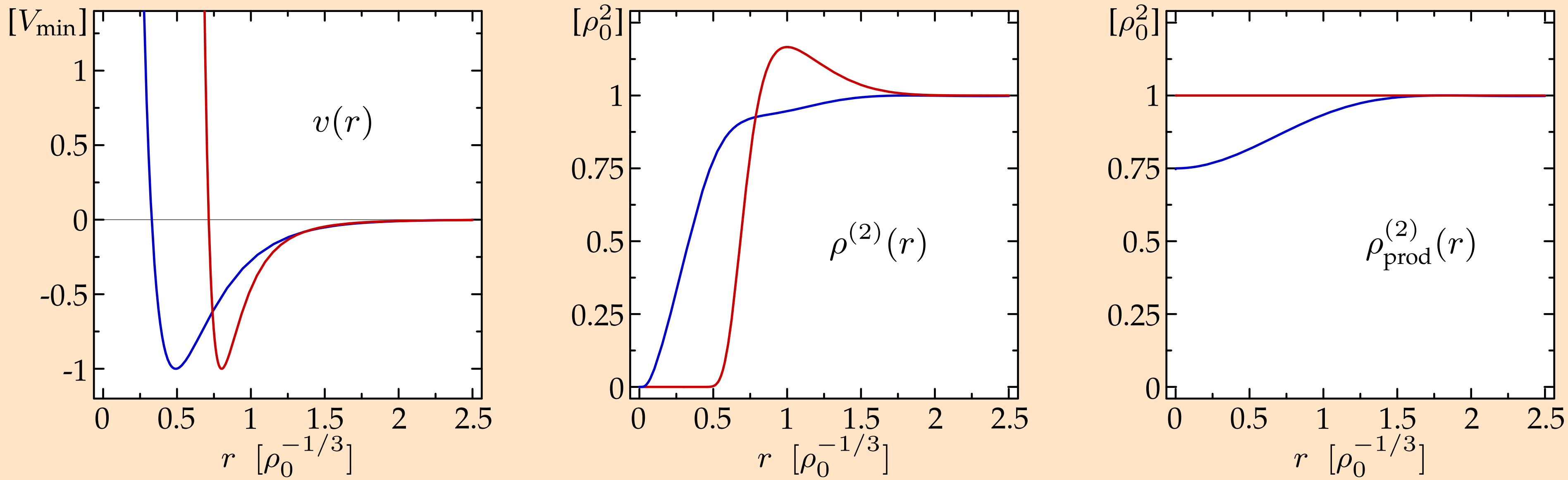
core induces strong short-range correlations in many-body state

(e.g. correlation hole in two-body density)

Product States

short-range correlations cannot be described by product-type states

(e.g. mean-field, superposition of few product states,...)



nuclear matter $\rho_0 = 0.17\,\text{fm}^{-3}$

liquid ^{4}He (bosonic) $\rho_0 = 0.022\,\text{Å}^{-3}$

Short-Range Correlations

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short-range correlations cannot be described by product-type states

(e.g. mean-field, superposition of few product states,...)

Effective Interaction

replace the full potential by a tamed effective interaction

Correlated States

include correlations in many-body model-space

Effective Contact Interaction

A Suitable Effective Interaction...

system is very **dilute** and **cold**

$\rho^{-1/3} \gg$ range of interaction

$q^{-1} \gg$ range of interaction

treat the many-body problem in a restricted **model-space** that does not contain correlations

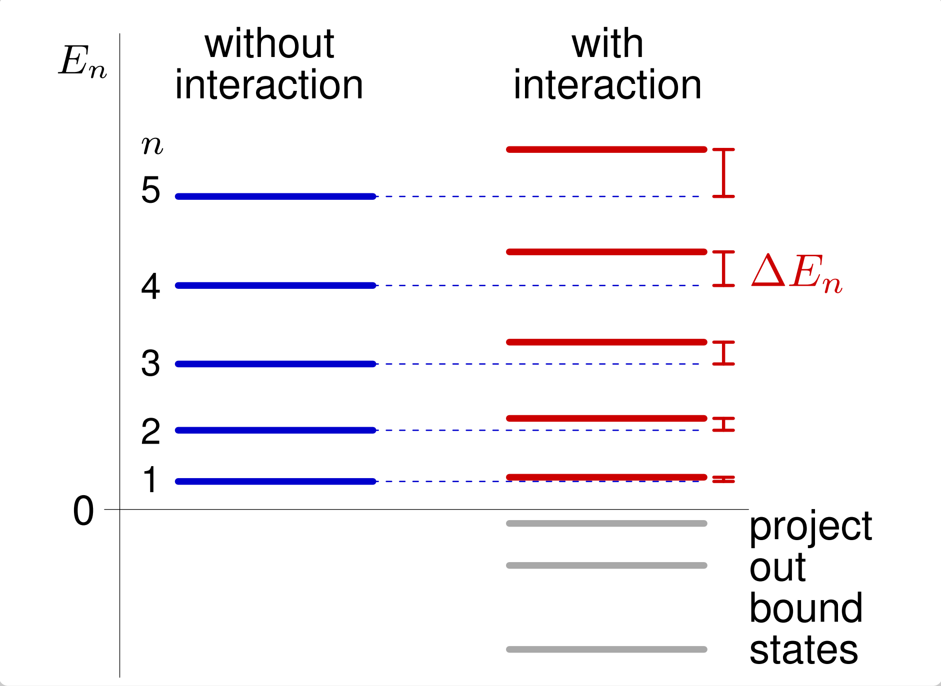
looking for the structure of **non-selfbound states** in an external potential

hermitean interaction operator that obeys standard symmetries (translation, rotation,...)

Effective Contact Interaction (ECI)

- zero-range potential (for each partial wave)
- expectation value in two-body model-states equals the energy shift induced by the full interaction

$$\left\langle \phi_n^{\text{mod}} \right| \mathbf{v}^{\text{ECI}} \left| \phi_n^{\text{mod}} \right\rangle \stackrel{!}{=} \Delta E_n$$



Effective Contact Interaction

Energy Shift

- relative two-body wave function w/o and with interaction (outside the range of $v(r)$)

$$\phi_{nlm}(\vec{r}) = R_{nl}(r) Y_{lm}(\vartheta, \varphi)$$

$$R_{nl}(r) \propto j_l(q_{nl} r)$$
$$\bar{R}_{nl}(r) \propto j_l(\bar{q}_{nl} r) - \tan \eta_l(\bar{q}_{nl})\, n_l(\bar{q}_{nl} r)$$

- auxiliary boundary condition $R_{nl}(\Lambda) = 0$ to obtain discrete momentum spectrum

$$q_{nl}\Lambda = \pi(n + \tfrac{l}{2})$$
$$\bar{q}_{nl}\Lambda = \pi(n + \tfrac{l}{2}) - [\eta_l(\bar{q}_{nl}) - \pi\, n_l^{\text{bound}}]$$

- momentum shift

$$\Delta q_{nl}\Lambda = (\bar{q}_{nl} - q_{nl})\Lambda$$
$$= -[\eta_l(q_{nl}) - \pi\, n_l^{\text{bound}}] =: -\hat{\eta}_l(q_{nl})$$

- relative energy shift

$$\frac{\Delta E_{nl}}{E_{nl}} = -\frac{2}{q_{nl}\Lambda}\, \hat{\eta}_l(q_{nl})$$

Interaction Operator

- ansatz for a nonlocal contact interaction for the lth partial wave

$$\mathbf{v}_l^{\text{ECI}} = (\vec{\mathbf{q}} \cdot \tfrac{\vec{\mathbf{r}}}{\mathbf{r}})^l\, g_l\, \delta^{(3)}(\vec{\mathbf{r}})\, (\tfrac{\vec{\mathbf{r}}}{\mathbf{r}} \cdot \vec{\mathbf{q}})^l$$

$$= \int d^3r\, |\vec{r}\rangle \frac{\overleftrightarrow{\partial}^l}{\partial r^l}\, g_l\, \delta^{(3)}(\vec{r})\, \frac{\overrightarrow{\partial}^l}{\partial r^l} \langle \vec{r}|$$

- expectation value in non-interacting two-body states

$$\langle \phi_{nlm} |\, \mathbf{v}_l^{\text{ECI}}\, | \phi_{nlm} \rangle \stackrel{!}{=} \Delta E_{nl}$$

- interaction strengths g_l determined by $\hat{\eta}_l(q)$

$$g_l = -\frac{4\pi}{2m_{\text{red}}} \left[\frac{(2l+1)!!}{l!}\right]^2 \frac{\hat{\eta}_l(q)}{q^{2l+1}}$$

- parametrization of $\hat{\eta}_l(q)$ in terms of the scattering lengths a_l for $|q\, a_l| \ll 1$

$$g_l = \frac{4\pi}{2m_{\text{red}}} \frac{(2l+1)}{(l!)^2}\, a_l^{2l+1} + \mathcal{O}(q^2)$$

Trapped Degenerate Fermi Gas

- trapped gas of Ξ distinguishable fermionic species ($\xi = 1, ..., \Xi$) interacting via the s- and p-wave contact interaction
- for simplicity: equal trapping potentials and s- and p-wave scattering lengths, a_0 and a_1, for all components

Hamiltonian

$$\mathbf{H} = \underbrace{\sum_i U(\vec{\mathbf{x}}_i)}_{\text{trap}} + \underbrace{\frac{1}{2m}\sum_i \vec{\mathbf{p}}_i^2}_{\text{kinetic}} + \underbrace{\frac{4\pi a_0}{m}\sum_{i<j}\delta^{(3)}(\vec{\mathbf{r}}_{ij})}_{\text{s-wave}} + \underbrace{\frac{12\pi a_1^3}{m}\sum_{i<j}\Big(\vec{\mathbf{q}}_{ij}\cdot\frac{\vec{\mathbf{r}}_{ij}}{\mathbf{r}_{ij}}\Big)\,\delta^{(3)}(\vec{\mathbf{r}}_{ij})\,\Big(\frac{\vec{\mathbf{r}}_{ij}}{\mathbf{r}_{ij}}\cdot\vec{\mathbf{q}}_{ij}\Big)}_{\text{p-wave}}$$

Mean-Field States (homogeneous)

- N-body state $|\Psi\rangle$ is an antisymmetrized product of single-particle momentum eigenstates $|\vec{k}_i, \xi_i\rangle$

$$|\Psi\rangle = \mathcal{A}\left(|\vec{k}_1, \xi_1\rangle \otimes \cdots \otimes |\vec{k}_N, \xi_N\rangle\right)$$

- for each component ξ all momenta $|\vec{k}|$ up to the **Fermi momentum** κ_ξ appear

Thomas-Fermi Approximation

- energy density of the trapped gas is locally given by the energy density of the homogeneous system

$$\mathcal{E}_{\text{hom}}(\kappa_1, ..., \kappa_\Xi) = \frac{1}{V}\langle\Psi|\,\mathbf{H}_{\text{hom}}\,|\Psi\rangle$$

- i.e. the Fermi momenta κ_ξ are replaced by **local Fermi momenta** $\kappa_\xi(\vec{x})$

Energy-Density for Trapped Fermions

Single-Component System

$$\mathcal{E}_1[\kappa](\vec{x}) =$$
$$= \frac{1}{6\pi^2}\, U(\vec{x})\, \kappa^3(\vec{x})$$
$$+ \frac{1}{20\pi^2 m}\, \kappa^5(\vec{x})$$
✗
$$+ \frac{a_1^3}{30\pi^3 m}\, \kappa^8(\vec{x})$$

trap

kinetic

s-wave

p-wave

Two-Component System

$$\mathcal{E}_2[\kappa_1, \kappa_2](\vec{x}) =$$
$$= \frac{1}{6\pi^2} U(\vec{x})\, \left[\kappa_1^3(\vec{x}) + \kappa_2^3(\vec{x})\right]$$
$$+ \frac{1}{20\pi^2 m}\, \left[\kappa_1^5(\vec{x}) + \kappa_2^5(\vec{x})\right]$$
$$+ \frac{a_0}{9\pi^3 m}\, \kappa_1^3(\vec{x})\, \kappa_2^3(\vec{x})$$
$$+ \frac{a_1^3}{30\pi^3 m}\, \Big[\kappa_1^8(\vec{x}) + \kappa_2^8(\vec{x}) +$$
$$+ \tfrac{1}{2}\kappa_1^3(\vec{x})\, \kappa_2^5(\vec{x}) + \tfrac{1}{2}\kappa_1^5(\vec{x})\, \kappa_2^3(\vec{x})\Big]$$

- energy expectation value

$$E_\Xi[\kappa_1, ..., \kappa_\Xi] = \int d^3x\; \mathcal{E}_\Xi[\kappa_1, ..., \kappa_\Xi](\vec{x})$$

- density

$$\rho_\xi(\vec{x}) = \frac{1}{6\pi^2}\, \kappa_\xi^3(\vec{x})$$

- particle number

$$N[\kappa_\xi] = \frac{1}{6\pi^2} \int d^3x\; \kappa_\xi^3(\vec{x})$$

Ground State — Variationally

Functional Variation

minimization of the energy $E_\Xi[\kappa_1, ..., \kappa_\Xi]$ for fixed numbers of particles $N_1, ..., N_\Xi$ gives the ground state density profile

- **chemical potentials**: implement constraints on the particle numbers via a set of Lagrange multipliers $\mu_1, ..., \mu_\Xi$
- unconstraint minimization of the transformed energy functional

$$F_\Xi[\kappa_1, ..., \kappa_\Xi] = E_\Xi[\kappa_1, ..., \kappa_\Xi] - \sum_{\xi=1}^{\Xi} \mu_\xi N[\kappa_\xi]$$
$$= \int \mathrm{d}^3x \; \mathcal{F}_\Xi[\kappa_1, ..., \kappa_\Xi](\vec{x})$$

- stationary points of the energy density are solutions of the Euler-Lagrange equations

$$\frac{\partial}{\partial \kappa_\xi(\vec{x})} \mathcal{F}_\Xi[\kappa_1, ..., \kappa_\Xi](\vec{x}) = 0 \;, \quad \forall \xi$$

- since $\mathcal{F}_\Xi$ is local (does not depend on gradients) the ground state has to minimize $\mathcal{F}_\Xi$ for each $\vec{x}$

Recipe

ground-state densities at some $\vec{x}$ are given by the minimum of the transformed energy density $\mathcal{F}_\Xi[\kappa_1, ...\kappa_\Xi](\vec{x})$ for this $\vec{x}$

Structure of a Trapped Degenerate Two-Component Fermi Gas

- **Energy Landscapes & Density Profiles**
- **Mean-Field Induced Collapse**
- **Component Separation**
- **Phase Diagram**

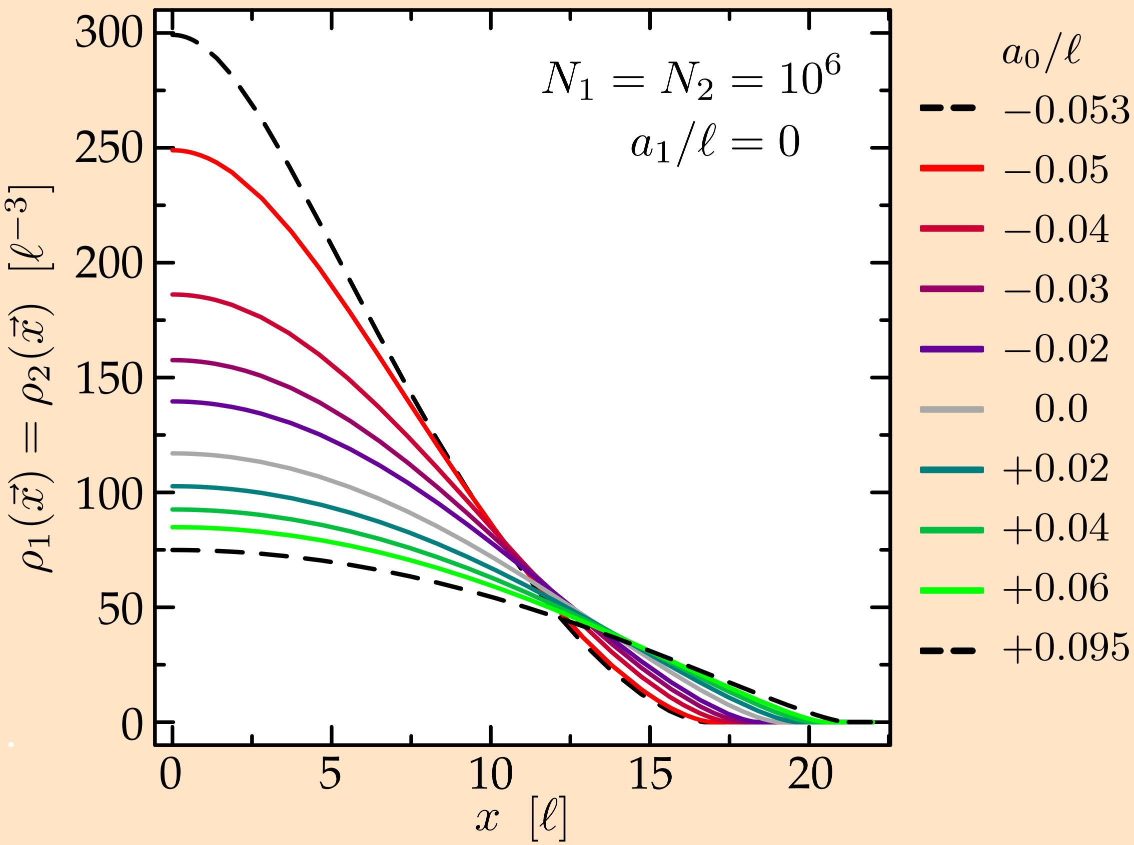
Density Profiles

- assume a spherical symmetric parabolic trapping potential

$$U(\vec{x}) = \frac{m\omega^2}{2}\, x^2 = \frac{1}{2m\ell^4}\, x^2$$

- determine the densities for μ_1, μ_2 chosen such that the desired particle numbers are reproduced

- $a_0 > 0$: repulsive interactions flatten the density profile
- $a_0 < 0$: attractive interactions enhance the central density
- outside a certain range of scattering lengths a_0 no solutions of this type exist anymore



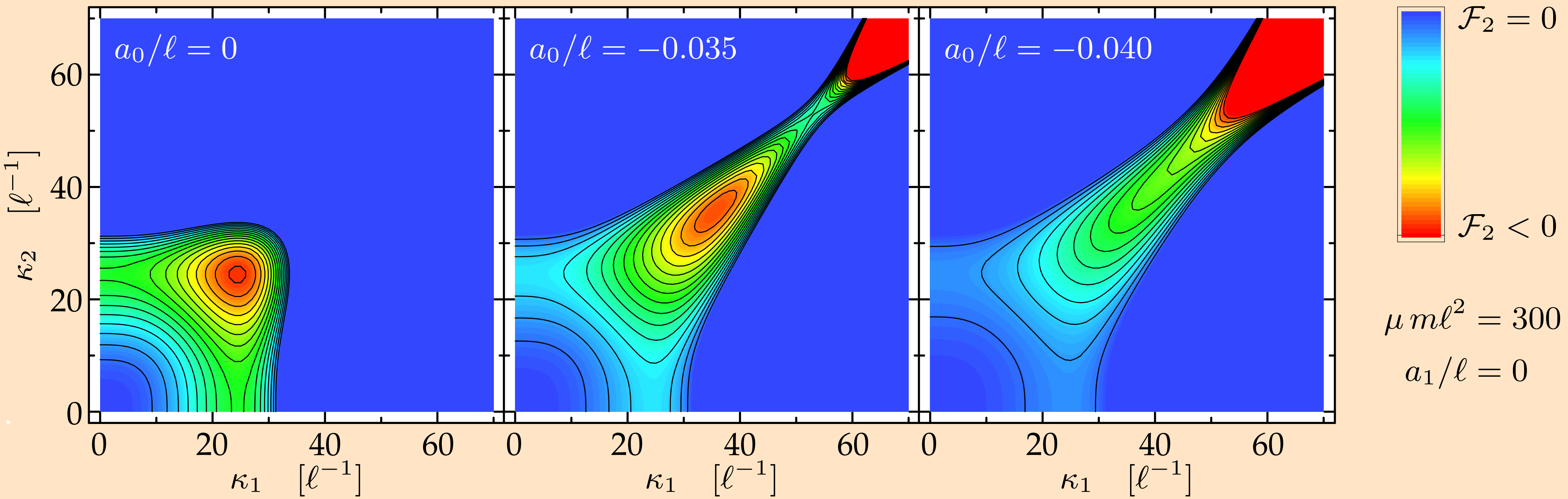
for a typical trap with $\ell = 1\,\mu\mathrm{m}$:

$$a_0 = 200\, a_{\mathrm{Bohr}} \quad \rightarrow \quad a_0/\ell = 0.01$$

$$a_0 = 2000\, a_{\mathrm{Bohr}} \quad \rightarrow \quad a_0/\ell = 0.1$$

Energy-Density Landscape: $a_0 < 0$

$\mathcal{F}_2[\kappa_1, \kappa_2](\vec{x} = 0)\,, \quad \mu_1 = \mu_2$

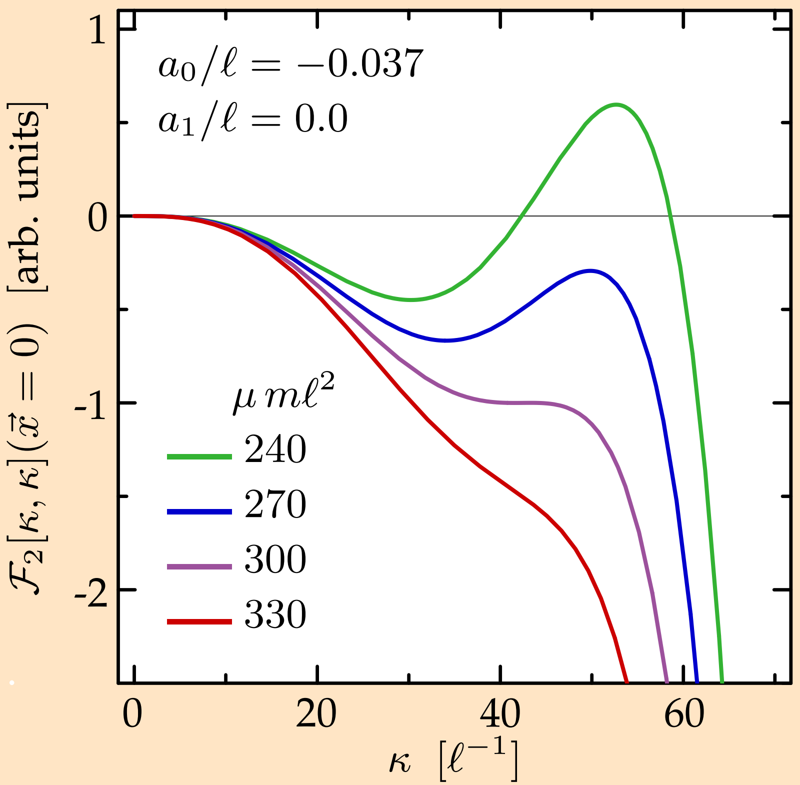


- minimum of $\mathcal{F}_2$ is only local for attractive interactions ($a_0 < 0$ or $a_1 < 0$)
- NB: physically the state is metastable for all signs of the scattering lengths
- local minimum vanishes if the attractive s-wave interaction exceeds a critical strength

attractive interactions can induce a **collapse** of the Fermi gas towards high densities

Collapse — Stability Condition

$$\mathcal{F}_2[\kappa,\kappa](\vec{x}) = \frac{1}{3\pi^2}\left[U(\vec{x}) - \mu\right] \kappa^3(\vec{x}) + \frac{1}{10\pi^2 m}\kappa^5(\vec{x}) + \frac{a_0}{9\pi^3 m}\kappa^6(\vec{x}) + \frac{a_1^3}{10\pi^3 m}\kappa^8(\vec{x})$$



- onset of instability is indicated by the appearance of a saddle point in the energy density, i.e., a vanishing first and second derivative

- **stability condition**: metastable states exist only for

$$\mu < \mu_{\mathrm{cr}}(a_0, a_1) \quad \text{and} \quad \kappa(\vec{x}) < \kappa_{\mathrm{cr}}(a_0, a_1)$$

- the critical Fermi momentum and the critical chemical potential are given by

$$-2\, a_0\, \kappa_{\mathrm{cr}} - 4\,(a_1\, \kappa_{\mathrm{cr}})^3 = \pi$$

$$m\, \mu_{\mathrm{cr}} = \frac{1}{2}\,\kappa_{\mathrm{cr}}^2 + \frac{2\, a_0}{3\pi}\,\kappa_{\mathrm{cr}}^3 + \frac{8\, a_1^3}{15\pi}\,\kappa_{\mathrm{cr}}^5$$

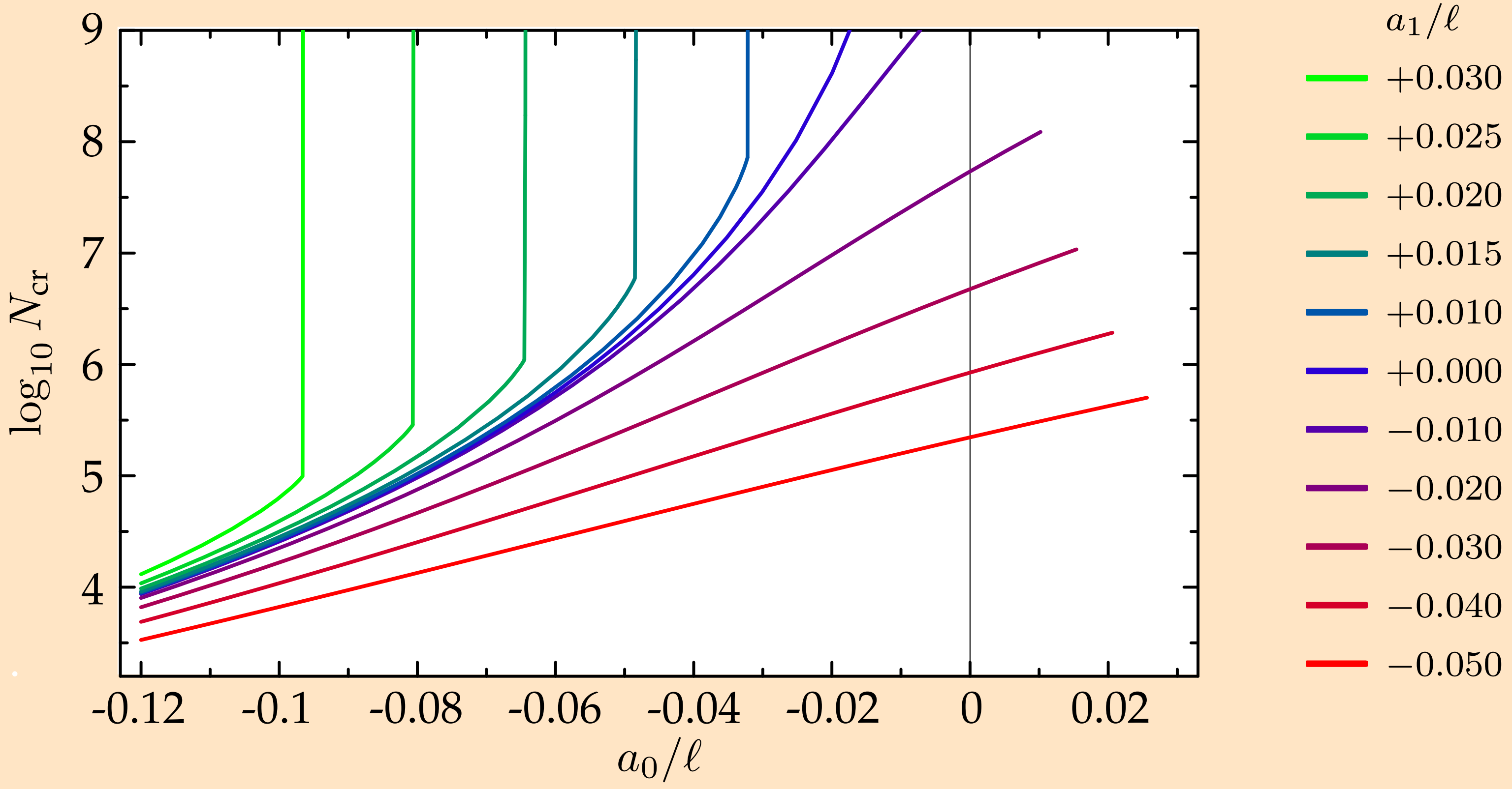
Collapse — Critical Particle Number

- assume parabolic trapping potential with mean oscillator length ℓ
- obtain the density profile for the critical chemical potential $\mu_{\rm cr}$ and calculate $N_{\rm cr}$

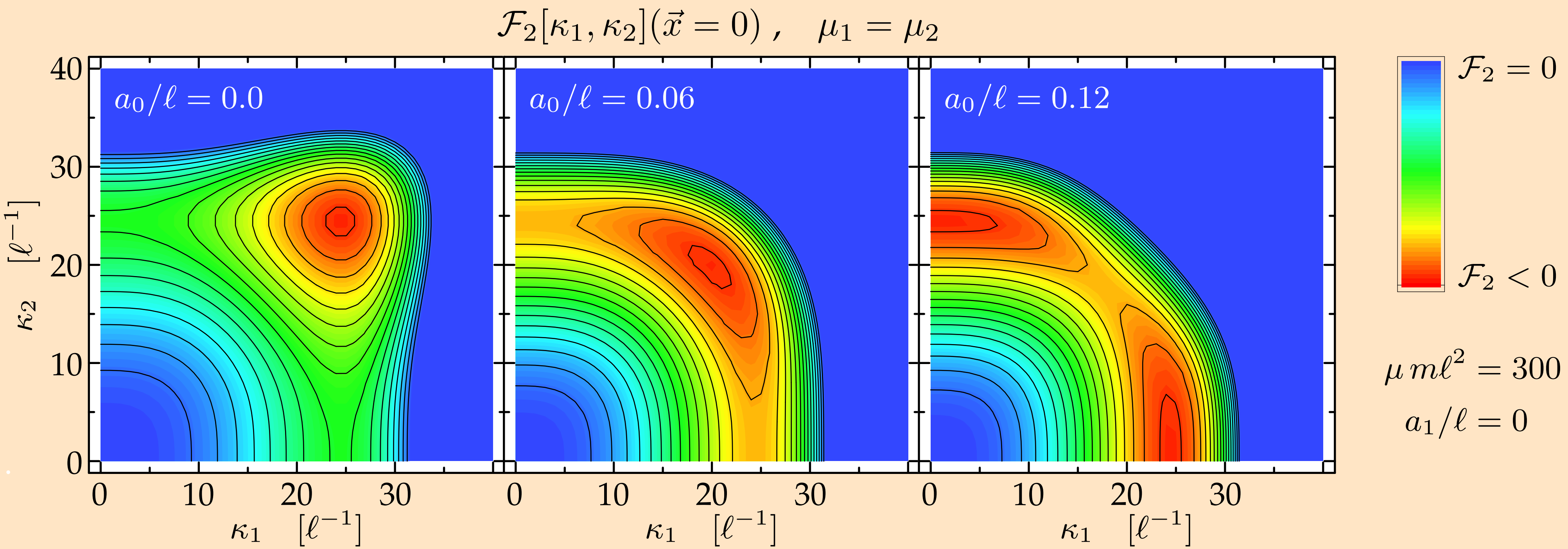
abs. stabilization due to p-wave repulsion $a_1/|a_0| > 2/(3\pi^{2/3})$

p-wave attraction lowers critical particle number substantially

p-wave induced collapse and interference with separation



Energy-Density Landscape: $a_0 > 0$



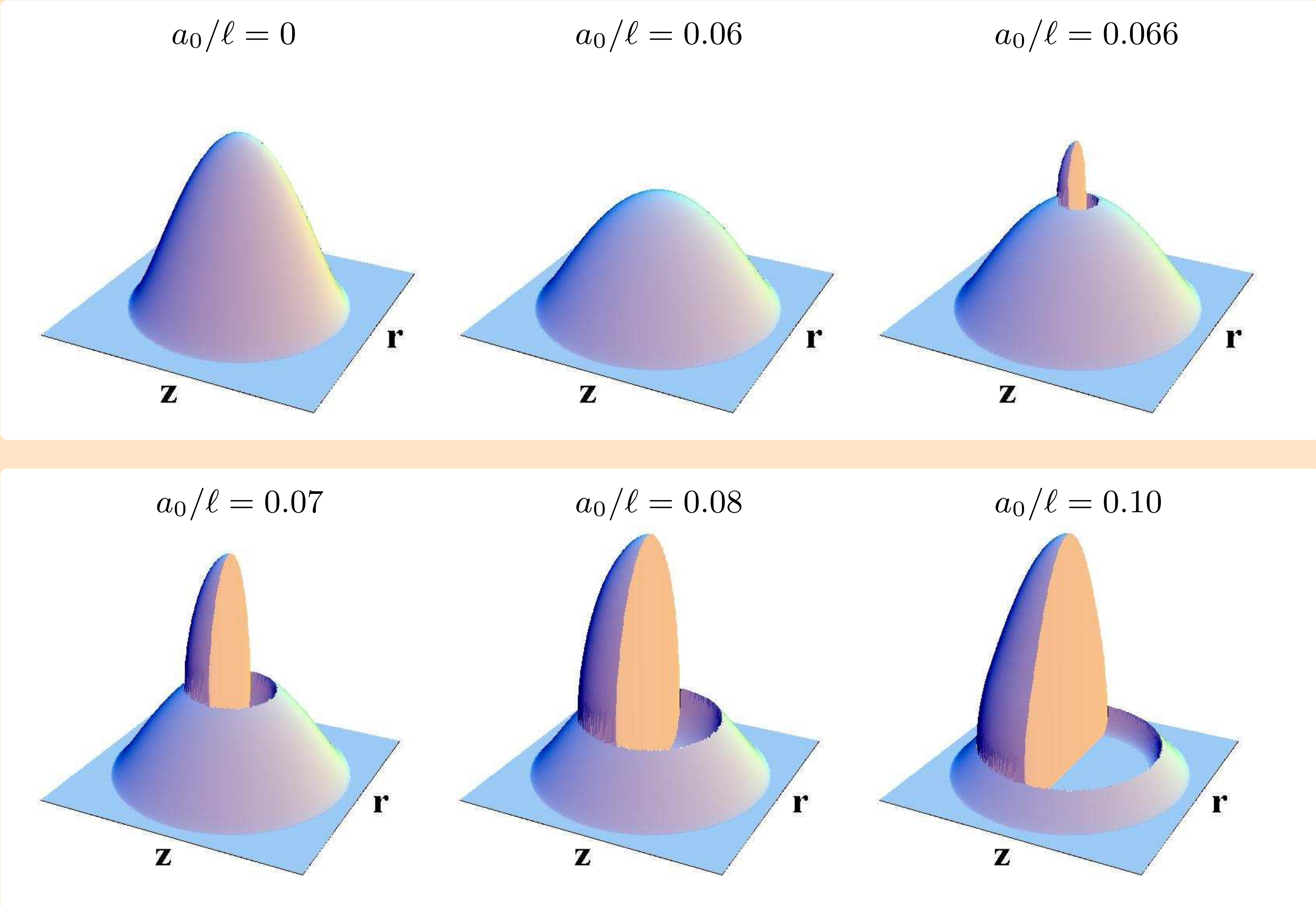
- **overlapping configuration**: for moderate repulsive s-wave interactions a unique minimum exists at $\kappa_1 = \kappa_2$
- **separation**: beyond a critical interaction strength two separate minima emerge at

$$\kappa_1 = 0,\ \kappa_2 > 0 \qquad \text{and} \qquad \kappa_1 > 0,\ \kappa_2 = 0$$

repulsive interactions can induce a spatial **separation** of the two components

Separation — Density Distributions

$$\rho_1(r, z) = \rho_2(r, -z)$$



$N_1 = N_2 = 10^7$

$a_1/\ell = 0$

Two-Component Fermi Gas

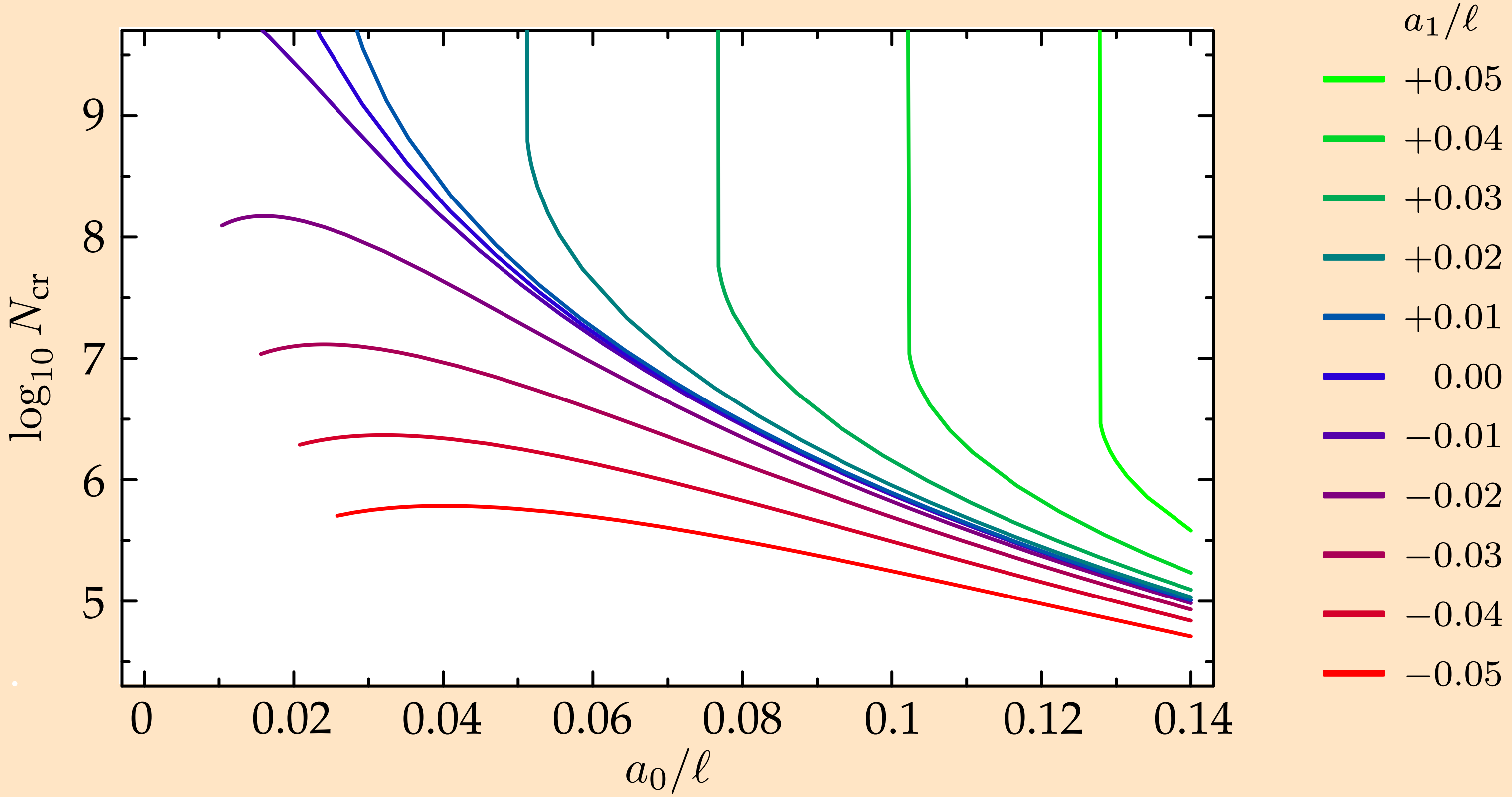
Separation — Critical Particle Number

- assume parabolic trapping potential with mean oscillator length ℓ
- obtain the density profile for the critical chemical potential μ_{cr} and calculate N_{cr}

interference with collapse induced by p-wave attraction

p-wave attraction lowers critical particle number substantially

abs. stabilization due to p-wave repulsion $a_1/a_0 > 2^{4/3}/(3\pi^{2/3})$

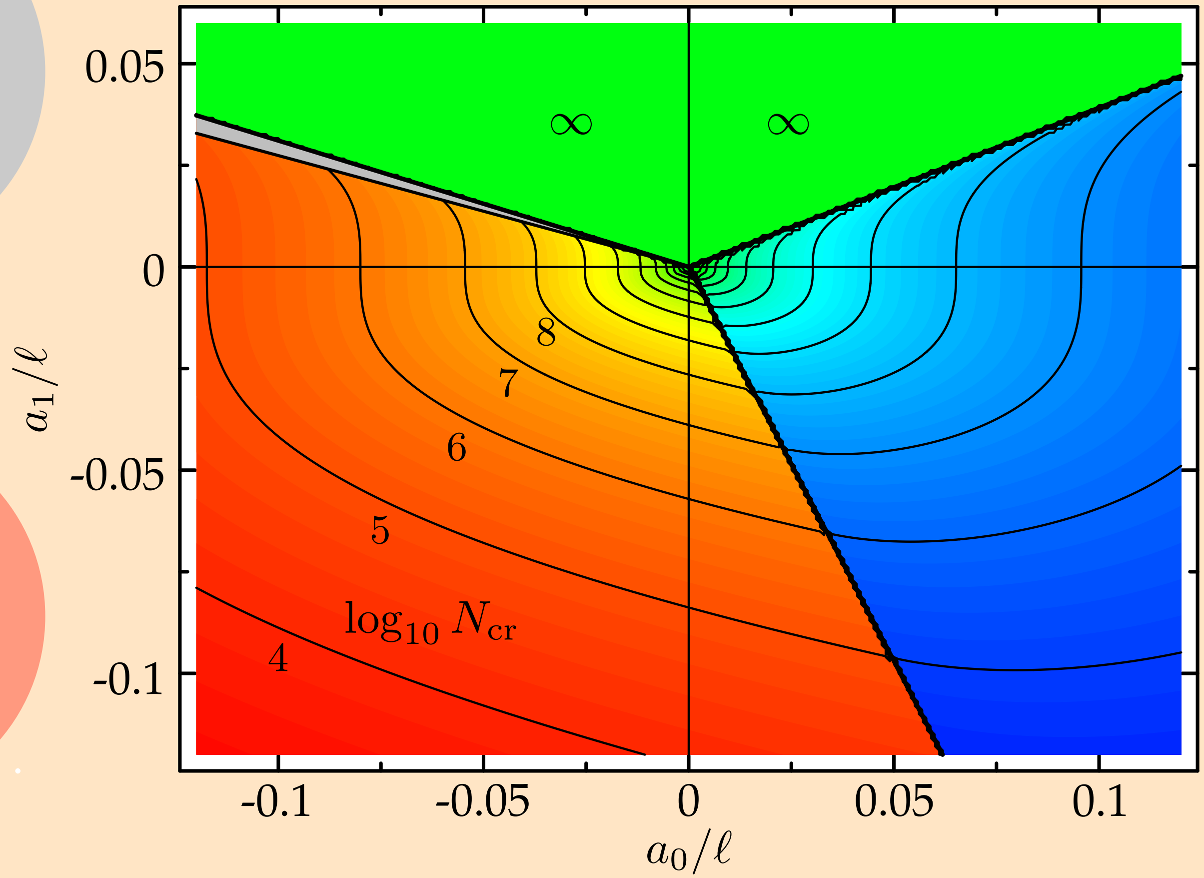


Stability Map

overlapping conf.
is stable for all
particle numbers

p-wave
stabilized
high-density
phase above
N_{cr}

mean-field
collapse
above critical
particle
number



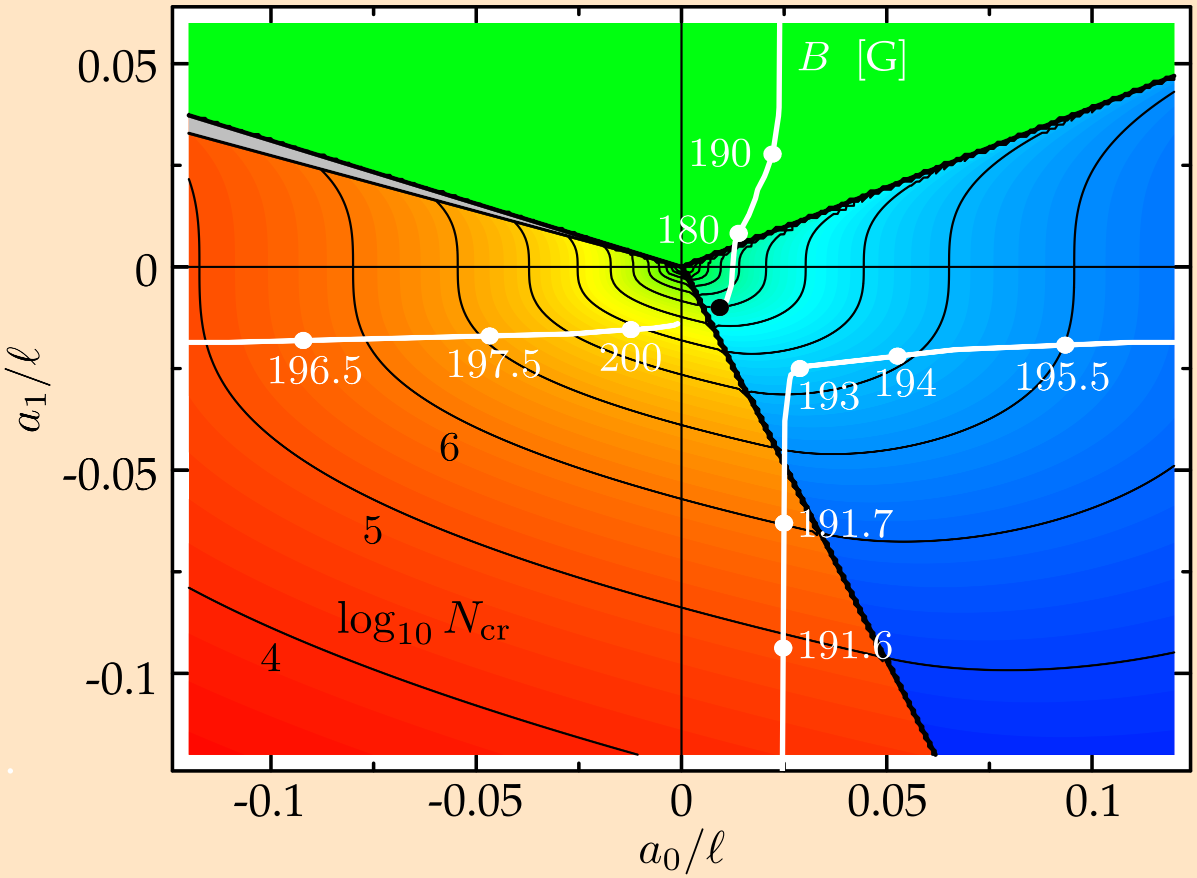
components
separate
above critical
particle
number

Stability Map & Feshbach Resonances

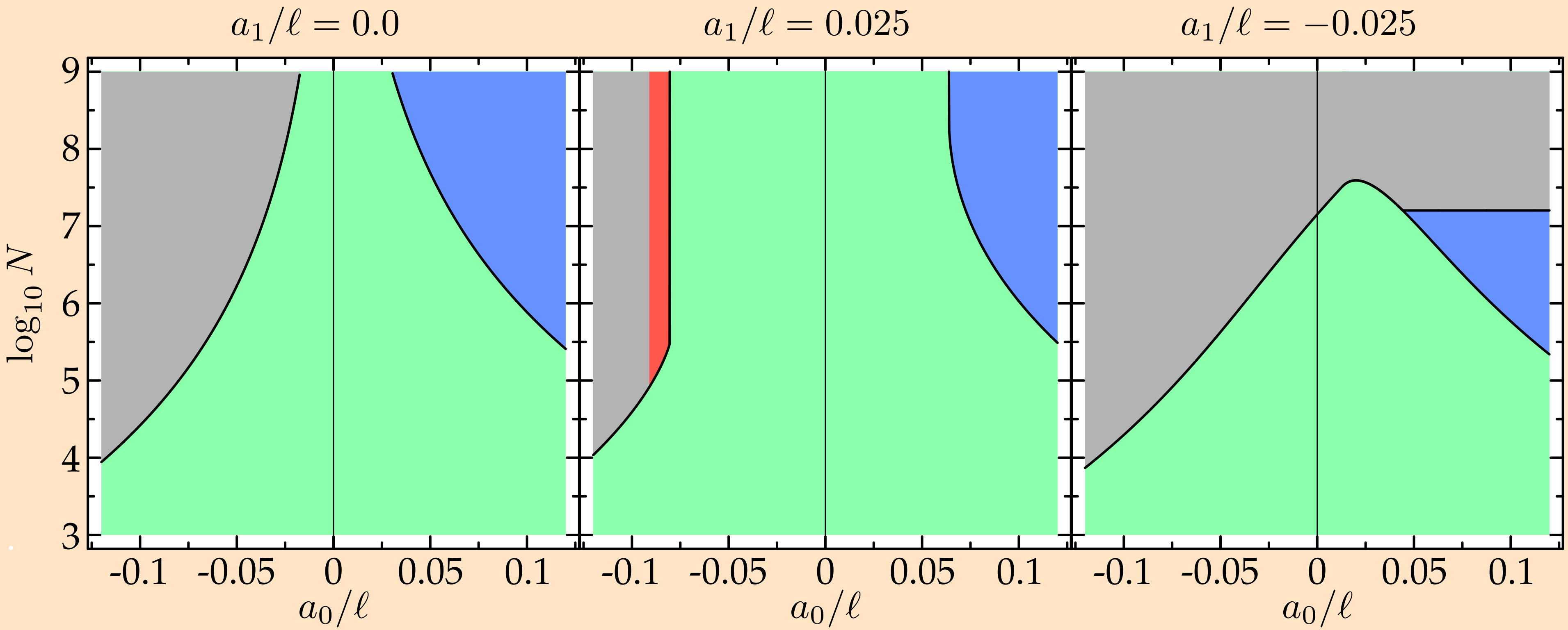
- **Feshbach resonances** allow to tune the strength of the atom-atom interaction (scattering lengths) via an external magnetic field

- simultaneous s- and p-wave Feshbach resonance predicted for a two-component ^{40}K system with $F = \frac{9}{2}$, $m_F = -\frac{9}{2}, -\frac{7}{2}$

  [J. Bohn, Phys. Rev. A61 (2000) 053409]



Two-Component Fermi Gas
Phase Diagram



identical overlapping density distributions

spatial separation of the components

unstable against collapse towards high densities

p-wave stabilized high-density phase

What's about the Single-Component Fermi Gas?

- **Density Profiles**
- **p-Wave Induced Collapse**

Density Profiles & Collapse

$$\mathcal{F}_1[\kappa](\vec{x}) = \frac{1}{6\pi^2}\left[U(\vec{x}) - \mu\right]\kappa^3(\vec{x}) + \frac{1}{20\pi^2 m}\kappa^5(\vec{x}) + \frac{a_1^3}{30\pi^3 m}\kappa^8(\vec{x})$$

- the p-wave interaction can have a strong effect on the density profile
- attractive p-wave interactions can cause a collapse of the single component gas
- **stability conditions** for the existence of a metastable state

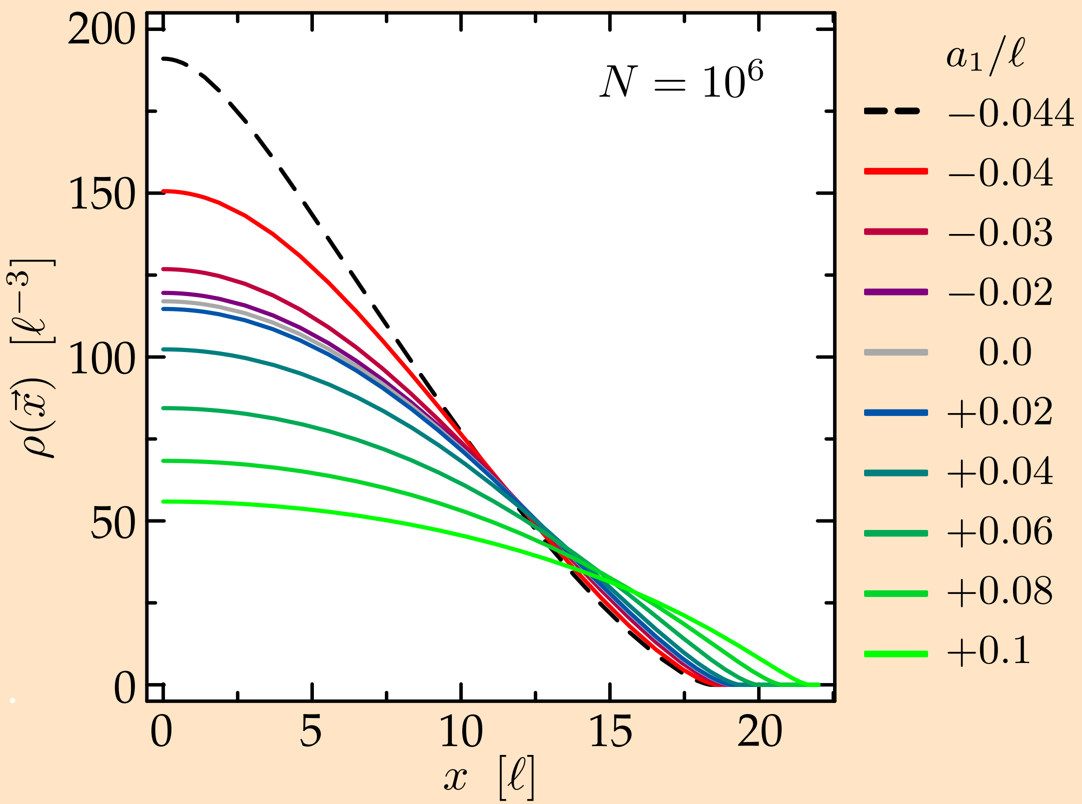
$$\kappa_{\text{cr}} = \frac{\sqrt[3]{3\pi}}{2|a_1|} \qquad \mu_{\text{cr}} = \frac{3(3\pi)^{2/3}}{40m|a_1|^2}$$

$$N_{\text{cr}} = \left(0.445\frac{\ell}{|a_1|}\right)^6$$

- p-wave attraction can cause a severe limitation of the fermion number

$$a_1/\ell = -0.01 \quad\rightarrow\quad N_{\text{cr}} = 7.8\times 10^9$$

$$a_1/\ell = -0.1 \quad\rightarrow\quad N_{\text{cr}} = 7800$$



Summary

Strategy

- developed a simple framework to describe interacting degenerate quantum gases
- effective contact interaction + mean-field states + Thomas-Fermi approximation $\rightarrow$ energy functional
- investigated the influence of s- and p-wave interactions on structure and stability of degenerate Fermi gases

Results

- s- and p-wave interactions have strong influence on the density profiles and the stability of the gas
- **collapse**: attractive interactions can induce a collapse of the dilute gas towards high densities
- **separation**: repulsive interactions can cause a spatial separation of the different components
- in all cases a complex interplay between s- and p-wave interactions is observed

...have a look at

http://theory.gsi.de/~trap