Besprechung Vorlesungsevaluation

Review: Angular momentum operator $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$

in coordinate space $\hat{L}_j = \frac{\hbar}{i} \varepsilon_{jkl} r_k \frac{\partial}{\partial r_l}$

$\Rightarrow [\hat{L}_j, \hat{L}_k] = i\hbar \varepsilon_{jkl} \hat{L}_l$ angular momentum operator $\hat{L}_j$ has closed algebra

→ Pause $[\hat{L}_x, \hat{L}_y] = \cdots$

$[\hat{\vec{L}}^2, \hat{L}_j] = 0$

$\Rightarrow$ simultaneous eigenstates of $\hat{\vec{L}}^2$ and one $\hat{L}_j$, choose $\hat{L}_z$

For spherically symmetric potential $\hat{V}(|\hat{\vec{r}}|)$

$\Rightarrow [\hat{H}, \hat{L}_j] = 0$ $\qquad \hat{H} = \frac{\hat{\vec{P}}^2}{2m} + \hat{V}(|\vec{r}|, |\vec{p}|, ..)$

$\Rightarrow$ simultaneous eigenstates of $\hat{\vec{L}}^2$, $\hat{L}_j$ ($j = z$) and $\hat{H}$

⊛ quantum numbers of these ($\hat{\vec{L}}^2 \to l$, $\hat{L}_j \to m$, $\hat{H} \to E$) characterize QM state in 3d

(vs. in 1d only quantum number n of $\hat{H} \to E_n$)

$\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y$ defined ladder operators regarding $\hat{L}_z$

$\Rightarrow$ simultaneous eigenstates $|l, m\rangle$ with $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
and $m = \underbrace{-l, -l+1, \ldots l-1, l}_{2l+1 \text{ values}}$

⊛ $\hat{H}|\Psi\rangle = E|\Psi\rangle \Rightarrow |\Psi\rangle = |E, l, m\rangle$ nicht rotinv. $|\Psi\rangle = |E\rangle$

$$\hat{\vec{L}}^2 |l,m\rangle = l(l+1)\hbar^2 |l,m\rangle$$

l: angular momentum quantum number

$$\hat{L}_z |l,m\rangle = m\hbar |l,m\rangle$$

m: magnetic quantum #

$$\hat{L}_\pm |l,m\rangle = \sqrt{l(l+1) - m(m\pm 1)}\,\hbar\, |l, m\pm 1\rangle$$

↳ raises m by $+1$
lowers m by -1, l unchanged

Example: Consider $l = 2$

Possible states: $|2,2\rangle \quad |2,1\rangle \quad |2,0\rangle \quad |2,-1\rangle \quad |2,-2\rangle$

$l \; m$

$\hat{L}_-$ $\qquad$ $\hat{L}_-$ $\rightarrow 0$

$0 \leftarrow \hat{L}_+$ $\qquad$ $\hat{L}_+$

$2l+1 = 5$

Expectation value of $\hat{\vec{L}}^2$: $\langle 2,m | \hat{\vec{L}}^2 | 2,m\rangle = 2(2+1)\hbar^2 = 6\hbar^2$

$\Delta \vec{L}^2 = 0$ because eigenstate

Expectation value of $\hat{L}_z$: $\langle 2,m | \hat{L}_z | 2,m\rangle = m\hbar$, $\Delta L_z = 0$

Consider $|2,2\rangle$: Expectation values of $\hat{L}_x$ and $\hat{L}_y$:

$$\langle 2,2 | \hat{L}_x | 2,2\rangle = \frac{1}{2} \langle 2,2 | \cancel{\hat{L}_+}^{\,0} + \cancel{\hat{L}_-}^{\,0} | 2,2\rangle = 0$$

$\hat{L}_y$ $\qquad$ $\frac{1}{2i}$ $\qquad$ $-$ $\qquad$ $= 0$

$$\langle 2,2 | \hat{L}_x^2 | 2,2\rangle = \frac{1}{4} \langle 2,2 | \cancel{\hat{L}_+^2}^{\,0} + \hat{L}_+\hat{L}_- + \hat{L}_-\cancel{\hat{L}_+}^{\,0} + \cancel{\hat{L}_-^2}^{\,0} | 2,2\rangle$$

$\hat{L}_y^2$ $\qquad$ $-\frac{1}{4}$ $\qquad$ $-$ $\qquad$ $-$

$$= \frac{1}{4} \langle 2,2 | \underbrace{\hat{L}_+ \hat{L}_- | 2,2\rangle}_{\sim |2,2\rangle}$$

$$\langle 2,2 | \hat{L}_x^2 | 2,2\rangle = \frac{1}{4} \sqrt{6 - 2\cdot 1}\,\hbar \sqrt{6 - 1\cdot 2}\,\hbar \,\langle 2,2 | 2,2\rangle = \hbar^2$$

$\hat{L}_y^2$ $\qquad$ $= \hbar^2$

$$(\Delta A)^2 = \langle \hat{A}^2 \rangle - (\langle \hat{A} \rangle)^2$$

$$(\Delta L_z)^2 = \langle \hat{L}_z^2 \rangle - (\langle \hat{L}_z \rangle)^2$$

$$= \langle 2m | \hat{L}_z^2 | 2m \rangle - (\underbrace{\langle 2m | \hat{L}_z | 2m \rangle}_{m\hbar})^2$$

$$= \langle 2m | (m\hbar)^2 | 2m \rangle - (m\hbar)^2$$

$$= 0$$

$$(\Delta L_x)^2 = \langle \hat{L}_x^2 \rangle - (\underbrace{\langle \hat{L}_x \rangle}_{=0})^2$$

$$\Delta L_x = \sqrt{\langle \hat{L}_x^2 \rangle} = \hbar$$

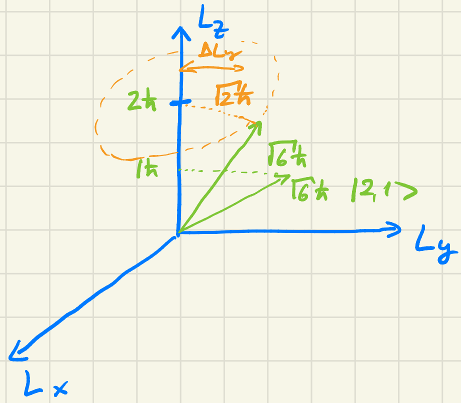
$|2,2\rangle$

→ Visualization of angular momentum vector $|\vec{L}| \to \sqrt{6}\hbar$

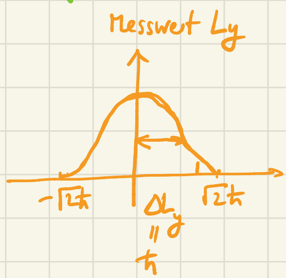
$L_z \to 2\hbar$

$\Delta L_x = \hbar$

$\Delta L_y = \hbar$



scharf messbar



Vorlesung heute:

8. Drehimpulsoperator und sphärisch symmetrische Potentiale

- Explizite Eigenfunktionen im Ortsraum
- Eigenschaften der Eigenfunktionen
- Sphärisch symmetrische Potentiale und radiale Schrödingergleichung

Explizite Eigenfunktionen im Ortsraum

definiert durch $\hat{L}_\pm |\ell, \pm\ell\rangle = 0$ vgl. $\hat{a}|0\rangle = 0$ für ld HO

und Anwendung von $(\hat{L}_\pm)^k$ gibt $|\ell, \pm\ell \mp k\rangle$

sphärische Symmetrie → <u>Polarkoordinaten</u>

$$x = r\sin\theta\cos\varphi \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\theta\sin\varphi \qquad \varphi = \arctan(y/x)$$
$$z = r\cos\theta \qquad \theta = \arctan\left(\sqrt{x^2+y^2}/z\right)$$

explizite Rechnung für $\hat{L}_z = \frac{\hbar}{i}\left(x\partial_y - y\partial_x\right)$

Kettenregel:

$$x\cdot\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}$$

$$= \frac{2yx}{2r}\frac{\partial}{\partial r} + \frac{x^2}{x^2+y^2}\frac{\partial}{\partial \varphi} + \frac{yzx}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta}$$

$$y\,\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

$$= \frac{2xy}{2r}\frac{\partial}{\partial r} - \frac{y^2}{x^2+y^2}\frac{\partial}{\partial \varphi} + \frac{xzy}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \hat{L}_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}$$

$\rightarrow$ wie Impulsop. $= \frac{\hbar}{i}\cdot$ Ableitung in φ-Richtung

Auf gleiche Weise

$$\hat{L}_x = \frac{\hbar}{i}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_y = \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_\pm = \hbar\, e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$\hat{\vec{L}}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

Drehimpulsoperator hängt nicht mehr von r ab! vgl. $[\hat{L}_j, |\vec{r}|] = 0$

$\Rightarrow$ Expliziten Eigenfunktionen sind r unabhängig

$$\boxed{\langle \theta, \varphi | \ell, m \rangle \equiv Y_\ell^m(\theta,\varphi)} = \Phi(\varphi)\,\Theta(\theta)$$

Separationsansatz, da $\hat{L}_z$ nur von $\frac{\partial}{\partial \varphi}$ abhängt

Bestimmung der φ Abhängigkeit aus

$$\hat{L}_z |\ell, m\rangle = m\hbar\, |\ell, m\rangle$$

$\langle \theta, \varphi|$

$$\Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \underbrace{Y_\ell^m(\theta,\varphi)}_{=\Phi\cdot\Theta} = m\hbar\, Y_\ell^m(\theta,\varphi)$$

$$\Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\Phi(\varphi) = m\hbar\,\Phi(\varphi) \quad \Rightarrow \quad \boxed{\Phi_m(\varphi) = e^{im\varphi}}$$

Wellenfunktion soll eindeutig sein, d.h. $\Phi_m(\varphi + 2\pi) = \Phi_m(\varphi)$

$\Rightarrow$ m ganzzahlig und ℓ ganzzahlig für Bahndrehimpuls

(halbzahlige ℓ-Werte nur für Spin / intrinsischen Drehimpuls)

Bestimmung der θ-Abhängigkeit aus

$$\hat{\vec{L}}^2 |\ell, m\rangle = \ell(\ell+1)\hbar^2 |\ell, m\rangle$$

$\langle \theta, \varphi|$

$$\Rightarrow -\cancel{\hbar^2}\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{\sin^2\theta}\left(-m^2\right)\right)\Theta(\theta)\,\cancel{\Phi_m(\varphi)} =$$

$$= \ell(\ell+1)\cancel{\hbar^2}\,\Theta(\theta)\,\cancel{\Phi_m(\varphi)}$$

$\vdots$

$$\Longleftrightarrow \quad \left(-\partial_\theta^2 - \cot\theta\,\partial_\theta + \frac{m^2}{\sin^2\theta}\right)\Theta_\ell^m(\theta) = \ell(\ell+1)\,\Theta_\ell^m(\theta)$$

Einfache Fälle $\quad m=\ell \;,\quad \hat{L}_+\,|\ell,\ell\rangle = 0$

$\langle\theta,\varphi|$

$$\Rightarrow \quad \cancel{\hbar}\,\cancel{e^{i\varphi}}\left(\partial_\theta + i\cot\theta\,\partial_\varphi\right)\Theta_\ell^\ell(\theta)\,\underbrace{\Phi_\ell(\varphi)}_{e^{i\ell\varphi}} = 0$$

$$\Rightarrow \quad \left(\partial_\theta - (\cot\theta)\cdot\ell\right)\Theta_\ell^\ell(\theta) = 0$$

$$\underbrace{\frac{\partial}{\partial\theta}\Theta_\ell^\ell}_{\ell(\sin\theta)^{\ell-1}\cdot\cos\theta} = \ell(\cot\theta)\,\Theta_\ell^\ell \quad \Rightarrow \quad \Theta_\ell^\ell(\theta) \sim (\sin\theta)^\ell$$

$$\ell(\sin\theta)^{\ell-1}\cdot\cos\theta = \ell\cot\theta\,(\sin\theta)^\ell \;\checkmark$$

$$\Rightarrow \quad Y_\ell^\ell(\theta,\varphi) = \text{const.}\,(\sin\theta)^\ell\,e^{i\ell\varphi}$$

.... Y_ℓ^m Kugelflächenfkten: Satz von vollst. Funktion auf Einheitskugel