An approximate vacuum state of temporal-gauge Yang–Mills theory in 2+1 dimensions

(in collaboration with Jeff Greensite)

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Introduction

- **Confinement** is the property of the vacuum of quantized non-abelian gauge theories. In the hamiltonian formulation in D=d+1 dimensions and temporal gauge:

$$\widehat{\mathcal{H}}\Psi_0[A] = E_0\Psi_0[A] \quad \ldots \quad \text{Schrödinger equation}$$

$$\widehat{\mathcal{H}} = \int d^dx \left\{ -\frac{1}{2}\frac{\delta^2}{\delta A_k^a(x)^2} + \frac{1}{4}F_{ij}^a(x)^2 \right\}$$

$$\left(\delta^{ac}\partial_k + g\epsilon^{abc}A_k^b\right)\frac{\delta}{\delta A_k^c}\Psi[A] = 0 \quad \ldots \quad \text{Gauß' law}$$

- At large distance scales one expects:

$$\Psi_0^{\text{eff}}[A] \approx \exp\left[-\mu \int d^d x\; F_{ij}^a(x) F_{ij}^a(x)\right]$$

- Halpern (1979), Greensite (1979)
- Greensite, Iwasaki (1989)
- Kawamura, Maeda, Sakamoto (1997)
- Karabali, Kim, Nair (1998)

- Property of **dimensional reduction**: Computation of a spacelike loop in d+1 dimensions reduces to the calculation of a Wilson loop in Yang-Mills theory in d Euclidean dimensions.

$$W(C) = \langle \mathrm{Tr}[U(C)] \rangle^{D=3+1} = \left\langle \Psi_0^{(3)} | \mathrm{Tr}[U(C)] | \Psi_0^{(3)} \right\rangle$$

$$\sim \langle \mathrm{Tr}[U(C)] \rangle^{D=2+1} = \left\langle \Psi_0^{(2)} | \mathrm{Tr}[U(C)] | \Psi_0^{(2)} \right\rangle$$

$$\sim \langle \mathrm{Tr}[U(C)] \rangle^{D=1+1} \quad \ldots$$ area law! (+ Casimir scaling)

Suggestion for an approximate vacuum wavefunctional

$$\Psi_0[A] = \exp\left[-\frac{1}{2} \int d^2x d^2y \; B^a(x) \left(\frac{1}{\sqrt{-\mathcal{D}^2 - \lambda_0 + m^2}} \right)^{ab}_{xy} B^b(y) \right]$$

$B^a(x) = F^a_{12}(x)$ ... the color magnetic field strength

$\mathcal{D}_k[A]$ ... the covariant derivative in the adjoint representation

$\mathcal{D}^2 = \mathcal{D}_k \cdot \mathcal{D}_k$ ... the covariant laplacian in the adjoint representation

$$\left(-\mathcal{D}^2\right)^{ab}_{xy} = \sum_{k=1}^{2} \left[2\delta^{ab}\delta_{xy} - \mathcal{U}^{ab}_k(x)\delta_{y,x+\hat{k}} - \mathcal{U}^{\dagger ba}_k(x-\hat{k})\delta_{y,x-\hat{k}} \right]$$

$$\mathcal{U}^{ab}_k(x) = \frac{1}{2}\mathrm{Tr}\left[\sigma^a U_k(x) \sigma^b U^\dagger_k(x) \right]$$

λ_0 ... the lowest eigenvalue of $(-\mathcal{D}^2)$

m ... a constant proportional to $g^2 \sim 1/\beta$

Support #1: Free-field limit (g→0)

$$\Psi_0[A] = \exp\left\{-\frac{1}{2}\int d^2x d^2y\, [\nabla \times A^a(x)] \left(\frac{\delta^{ab}}{\sqrt{-\nabla^2}}\right)_{xy} [\nabla \times A^b(y)]\right\}$$

For $g \to 0$:

- $m \sim g^2 \to 0$,
- $\mathcal{D}^2 \to \nabla^2$, $\lambda_0 \to 0$,

$\Psi_0[A]$ becomes the well-known vacuum wavefunctional of ED.

Support #2: Zero-mode, strong-field limit

- D. Diakonov (private communication to JG)

- Let's assume we keep only the zero-mode of the A-field, i.e. fields constant in space, varying in time. The lagrangian is

$$\mathcal{L} = \frac{1}{2}V\left[\sum_{k=1}^{2}\partial_t\vec{A}_k\cdot\partial_t\vec{A}_k - g^2(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)\right]$$

and the hamiltonian operator

$$\widehat{\mathcal{H}} = -\frac{1}{2V}\sum_{k=1}^{2}\frac{\partial^2}{\partial\vec{A}_k\cdot\partial\vec{A}_k} + \frac{1}{2}g^2V(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)$$

- The ground-state solution of the YM Schrödinger equation, up to 1/V corrections:

$$\Psi_0 = \exp[-VR_0] = \exp\left[-\frac{1}{2}gV\frac{(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)}{\sqrt{\vec{A}_1\cdot\vec{A}_1+\vec{A}_2\cdot\vec{A}_2}}\right]$$

- Now the proposed vacuum state coincides with this solution in the strong-field limit, assuming

$$|g\vec{A}_{1,2}| \gg m, \sqrt{\lambda_0}$$

- The covariant laplacian is then

$$\left(-\mathcal{D}^2\right)^{ab}_{xy} \approx g^2\delta_2(x-y)\left[(\vec{A}_1^2+\vec{A}_2^2)\delta^{ab} - A_1^a A_1^b - A_2^a A_2^b\right]$$
$$\equiv g^2\delta_2(x-y)\mathcal{M}^{ab}$$

and the eigenvalues of $\mathcal{M}$ are

$$\mu_1 = \tfrac{1}{2}\left(S - \sqrt{S^2-4C}\right), \qquad \mu_2 = \tfrac{1}{2}\left(S + \sqrt{S^2-4C}\right),$$
$$\mu_3 = S = \vec{A}_1^2 + \vec{A}_2^2, \qquad C = (\vec{A}_1 \times \vec{A}_2)\cdot(\vec{A}_1 \times \vec{A}_2)$$

- Then

$$\Psi_0 = \exp\left[-\frac{1}{2}\int d^2x d^2y\, B^a(x)\left(\frac{1}{\sqrt{-\mathcal{D}^2-\lambda_0+m^2}}\right)^{ab}_{xy} B^b(y)\right]$$

$$\approx \exp\left[-\frac{1}{2}gV\frac{(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)}{\sqrt{\mu_3-\mu_1+(m/g)^2}}\right]$$

$$\longrightarrow \exp\left[-\frac{1}{2}gV\frac{(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)}{\sqrt{\vec{A}_1\cdot\vec{A}_1+\vec{A}_2\cdot\vec{A}_2}}\right]$$

- The argument can be extended also to 3+1 dimensions.

Support #3: Dimensional reduction and confinement

- **What about confinement with such a vacuum state?**
- Define "slow" and "fast" components using a mode-number cutoff:

$$(-\mathcal{D}^2)^{ab}_{xy}\,\varphi^b_n(y) = \lambda_n\,\varphi^a_n(x)$$

$$B^a(x) = \sum_{n=0}^{\infty} b_n\,\varphi^a_n(x)$$

$$B^a_{\text{slow}}(x) = \sum_{n=0}^{n_{\text{max}}} b_n\,\varphi^a_n(x), \qquad \lambda_{n_{\text{max}}} - \lambda_0 \ll m^2$$

- Then:

$$\int d^2x d^2y\; B^a_{\text{slow}}(x)\left(\frac{1}{\sqrt{-\mathcal{D}^2 - \lambda_0 + m^2}}\right)^{ab}_{xy} B^b_{\text{slow}}(y)$$

$$\approx \frac{1}{m}\int d^2x\; B^a_{\text{slow}}(x)\; B^a_{\text{slow}}(x)$$

- Effectively for “slow” components

$$|\Psi_0|^2 \approx \exp\left[-\frac{1}{m}\int d^2x\; B^a_{\text{slow}}(x)\; B^a_{\text{slow}}(x)\right]$$

we then get the probability distribution of a 2D YM theory and can compute the string tension analytically (in lattice units):

$$\sigma(\beta) = \frac{3m(\beta)}{4\beta}$$

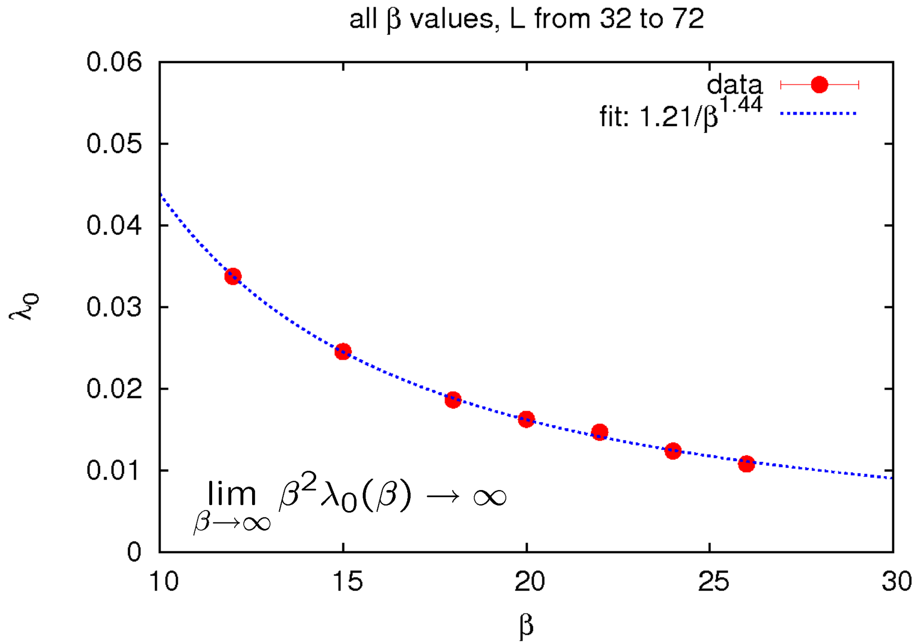
- **Non-zero value of *m* implies non-zero string tension σ and confinement!**
- Let’s revert the logic: to get σ with the right scaling behavior ~ $1/\beta^2$, we need to choose

$$m(\beta) = \tfrac{4}{3}\beta\sigma(\beta) \sim \beta^{-1} \sim g^2$$

Why $m_0^2 = -\lambda_0 + m^2$?

$$\Psi_0[A] = \exp\left[-\frac{1}{2}\int d^2x d^2y \; B^a(x)\left(\frac{1}{\sqrt{-\mathcal{D}^2 + m_0^2}}\right)^{ab}_{xy} B^b(y)\right]$$

- Samuel (1997)



Support #4: Non-zero m is energetically preferred

- Take m as a variational parameter and minimize <$\mathcal{H}$> with respect to m:

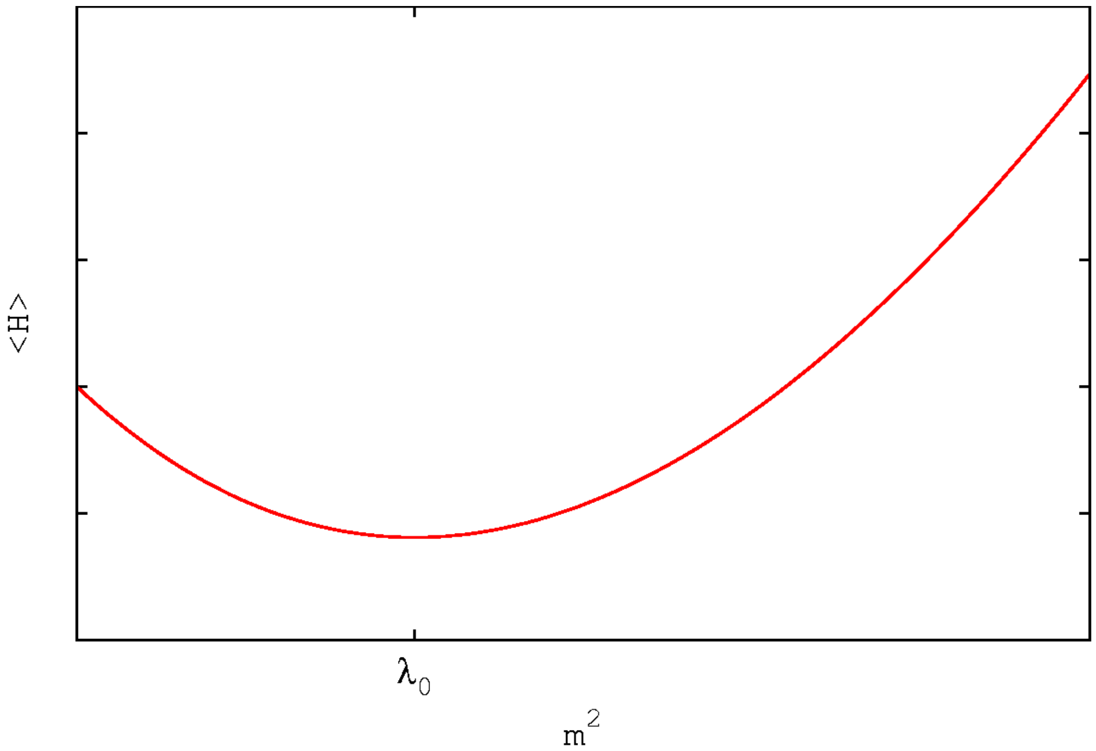
$$\hat{\mathcal{H}} = \int d^2x \left\{ -\frac{g^2}{2} \sum_{k=1}^{2} \frac{\delta^2}{\delta A_k^a(x)^2} + \frac{1}{2g^2} B^2(x) \right\}$$

$$\Psi_0[A] = \exp\left\{ -\frac{1}{2g^2} \int d^2x d^2y \; B^a(x) K_{xy}^{ab}[A] B^b(y) \right\}$$

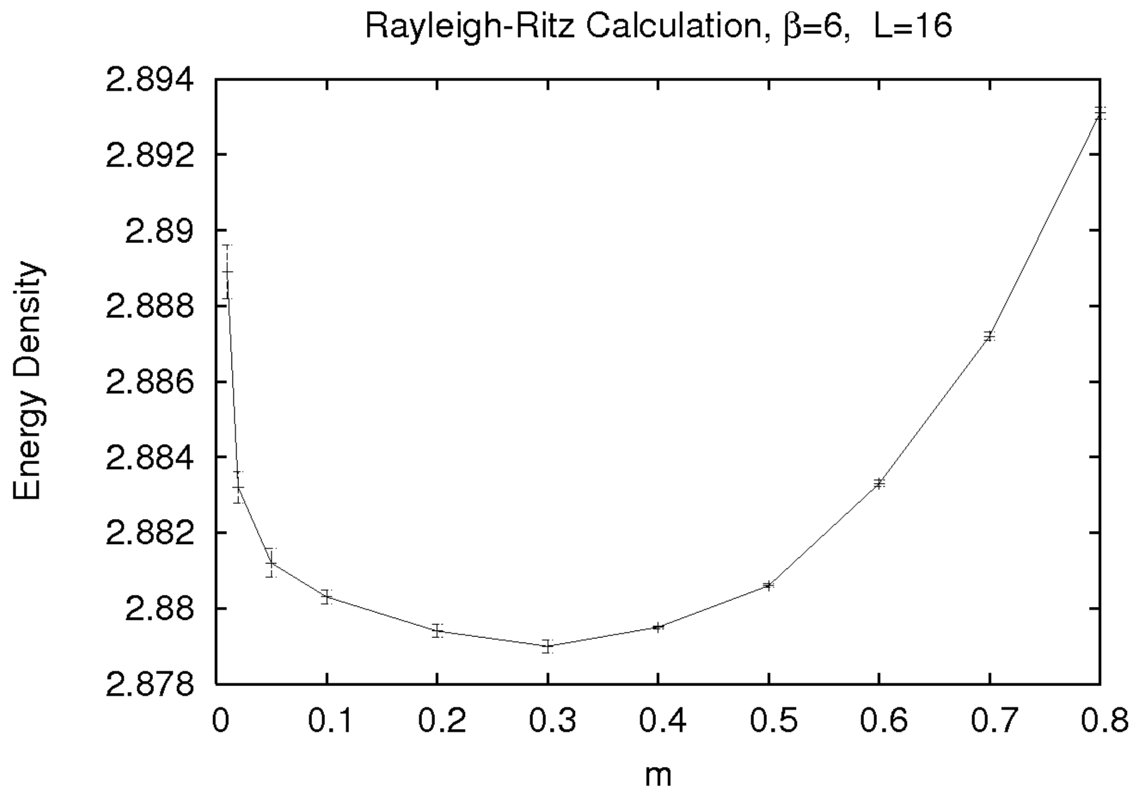
- Assuming the variation of K with A in the neighborhood of thermalized configurations is small, and neglecting therefore functional derivatives of K w.r.t. A one gets:

$$\langle \hat{\mathcal{H}} \rangle = \tfrac{1}{2} \left\langle \mathrm{Tr}\left[\sqrt{-\mathcal{D}^2 - \lambda_0 + m^2} \right] + \tfrac{1}{2} \mathrm{Tr}\left[\frac{\lambda_0 - m^2}{\sqrt{-\mathcal{D}^2 - \lambda_0 + m^2}} \right] \right\rangle$$

$$= \tfrac{1}{2} \left\langle \sum_n \left(\sqrt{\lambda_n(m) - \lambda_0(m) + m^2} + \tfrac{1}{2} \frac{\lambda_0(m) - m^2}{\sqrt{\lambda_n(m) - \lambda_0(m) + m^2}} \right) \right\rangle$$



- **Abelian free-field limit:** minimum at $m^2 = \lambda_0 \to 0$.



- **Non-abelian case:** Minimum at non-zero m^2 (~ 0.3), though a higher value (~ 0.5) would be required to get the right string tension.
- Could (and should) be improved!

Support #5: Calculation of the mass gap

- To extract the mass gap, one would like to compute

$$\mathcal{G}(x-y) = \left\langle (B^aB^a)_x (B^bB^b)_y \right\rangle - \left\langle (B^aB^a)_x \right\rangle^2$$

in the probability distribution:

$$P[A] = |\Psi_0[A]|^2 = \mathcal{N} \exp\left[-\int d^2x d^2y \ B^a(x) K^{ab}_{xy}[A] B^b(y) \right]$$

$$K^{ab}_{xy}[A] = \left(\frac{1}{\sqrt{-\mathcal{D}^2[A] - \lambda_0 + m^2}} \right)^{ab}_{xy}$$

- Looks hopeless, K[A] is highly non-local, not even known for arbitrary fields.
- But if - after choosing a gauge - K[A] does not vary a lot among thermalized configurations … then something can be done.

Numerical simulation of $|\Psi_0|^2$

- Define:

$$\mathcal{P}[A; K[A']] = \mathcal{N} \exp\left[-\int d^2x d^2y \; B^a(x; A) K^{ab}_{xy}[A'] B^b(y; A) \right]$$

- Hypothesis:

$$P[A] = \mathcal{P}[A; K[A]] \approx \mathcal{P}[A; \langle K \rangle] \approx \int dA' \; \mathcal{P}[A; K[A']] P[A']$$

- Iterative procedure:

$$P^{(1)}[A] = \mathcal{P}[A; K[0]]$$

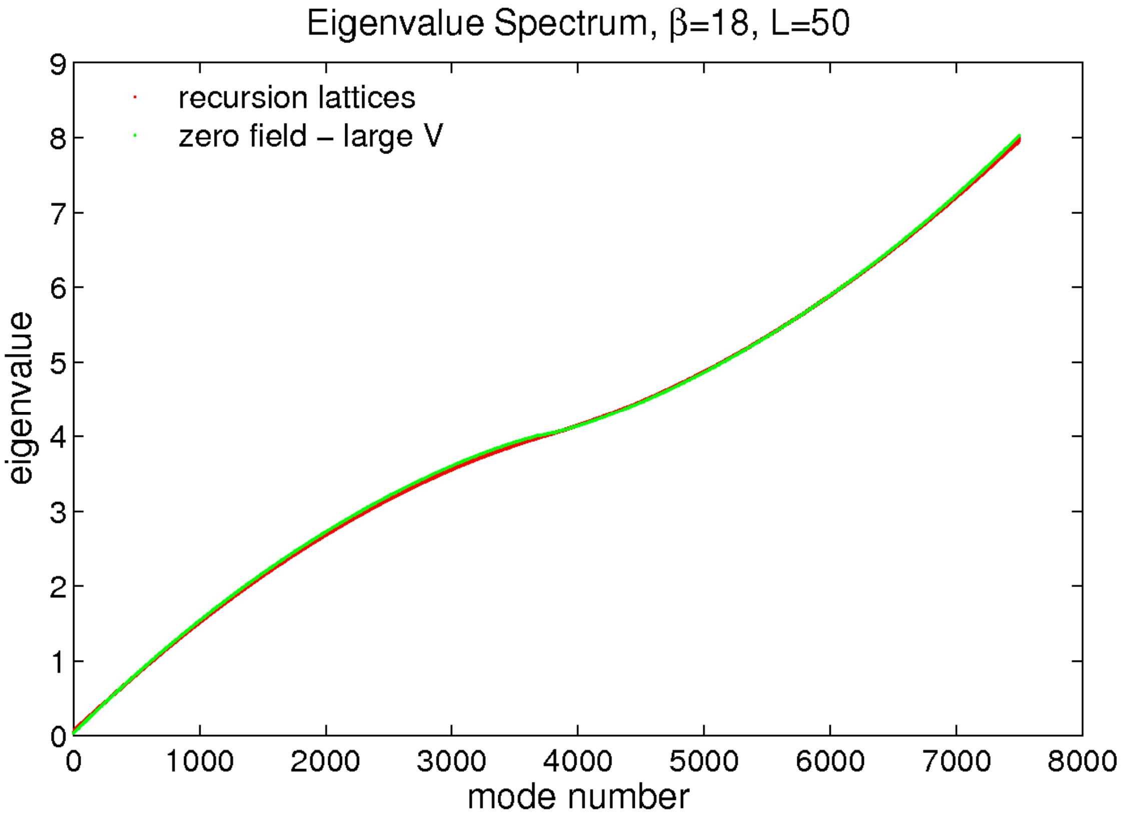
$$P^{(k+1)}[A] = \int dA' \; \mathcal{P}[A; K[A']] P^{(k)}[A']$$

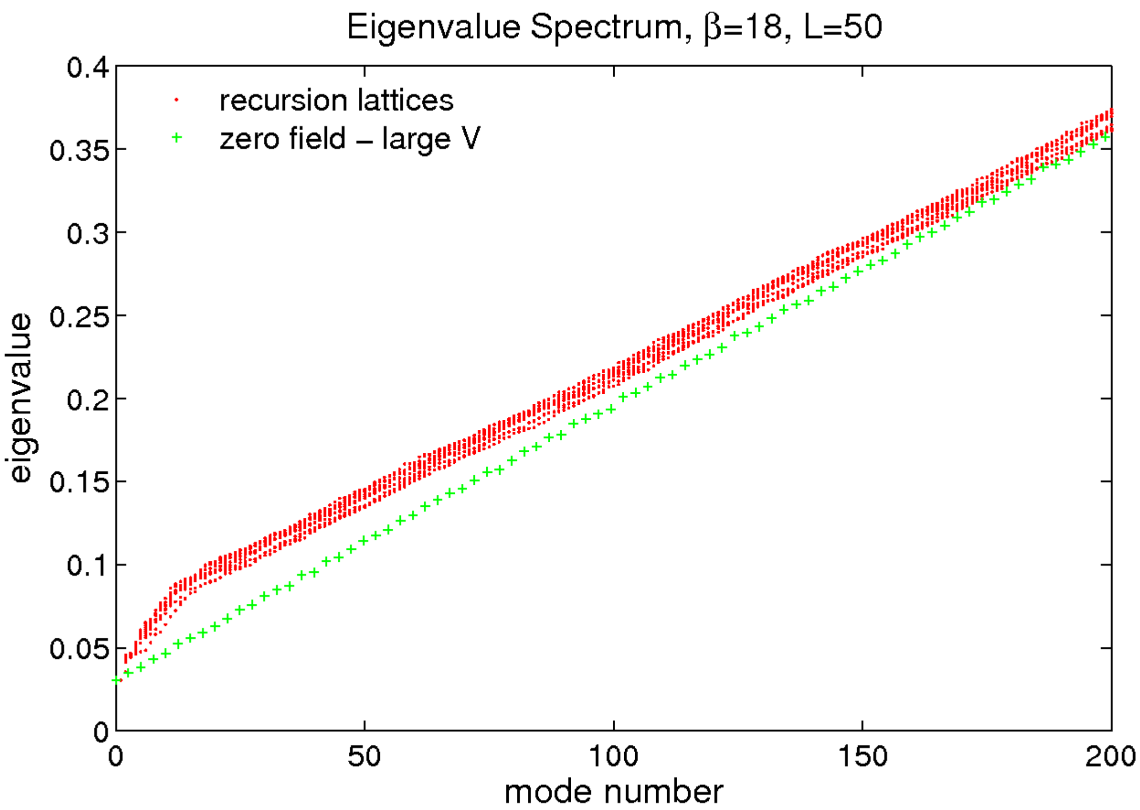
- Practical implementation:
  choose e.g. axial A_1=0 gauge, change variables from A_2 to B. Then
  1. given A_2, set $A_2'=A_2$,
  2. the probability $\mathcal{P}$[A;K[A′]] is gaussian in B, diagonalize K[A′] and generate new B-field (set of Bs) stochastically;
  3. from B, calculate A_2 in axial gauge, and compute everything of interest;
  4. go back to the first step, repeat as many times as necessary.
- All this is done on a lattice.
- Of interest:
  - Eigenspectrum of the adjoint covariant laplacian.
  - Connected field-strength correlator, to get the mass gap:

$$\mathcal{G}(x-y) \sim G(x-y) = \langle (K^{-1})^{ab}_{xy} (K^{-1})^{ba}_{yx} \rangle$$

- For comparison the same computed on 2D slices of 3D lattices generated by Monte Carlo.

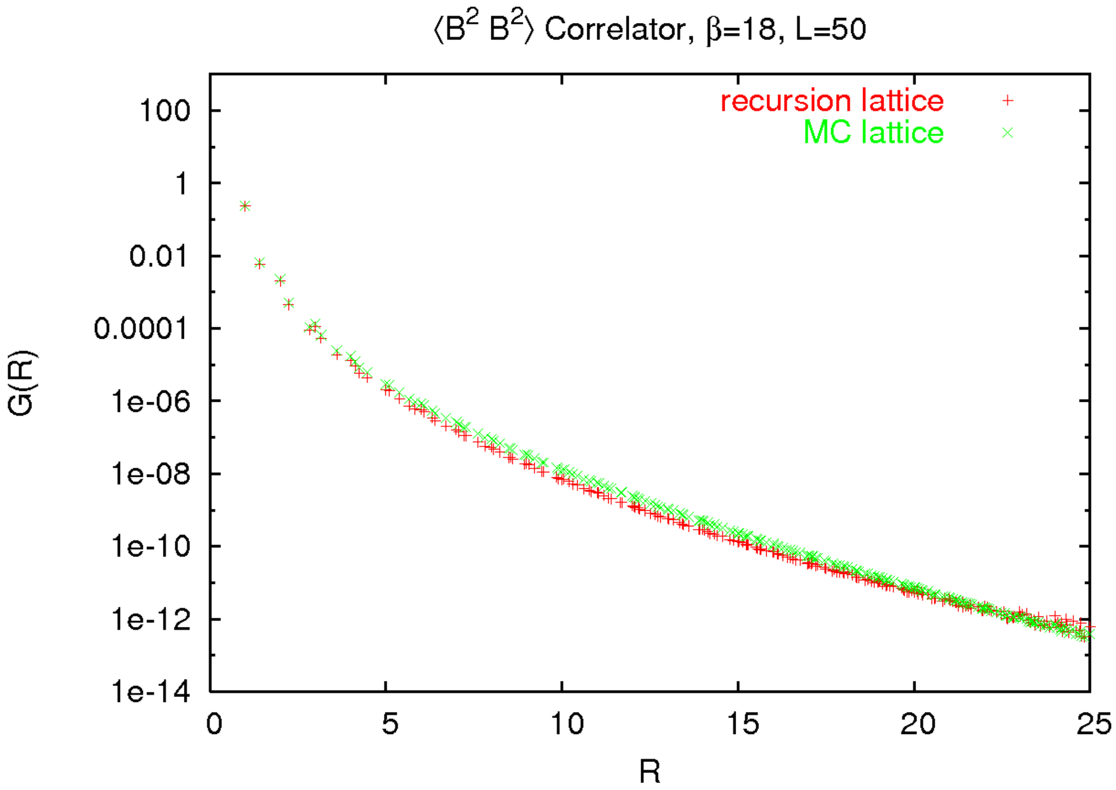
Eigenspectrum of the adjoint covariant laplacian



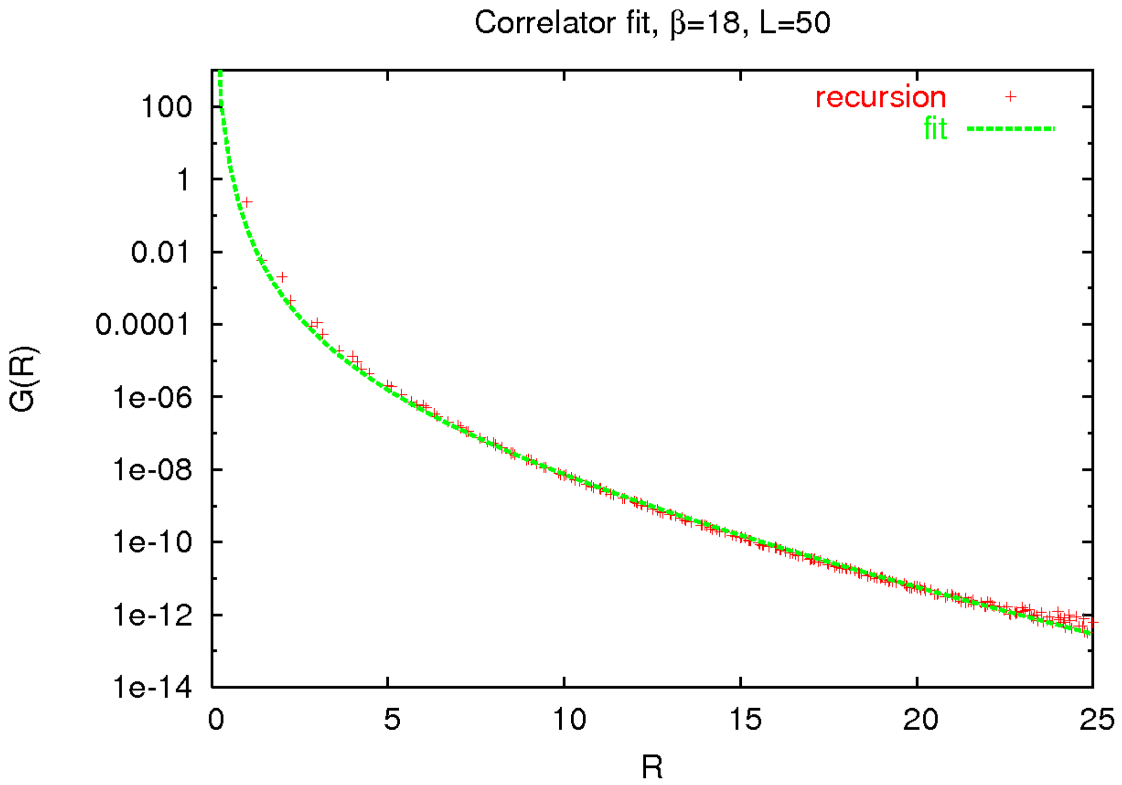


Eigenvalue Spectrum, β=18, L=50
recursion lattices
zero field – large V
eigenvalue
mode number
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
50
100
150
200

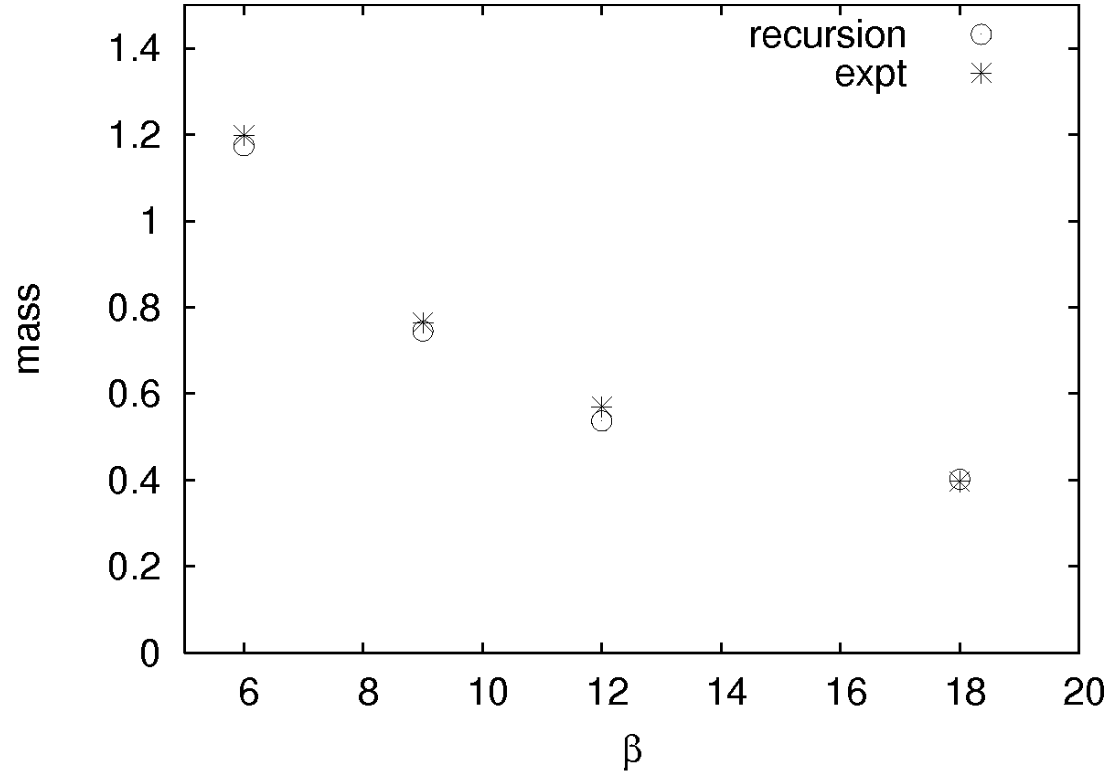
Mass gap



$$\mathcal{G}(x-y) \sim G(x-y) = \langle (K^{-1})^{ab}_{xy} (K^{-1})^{ba}_{yx} \rangle$$



$$G_0(x) = \delta^{ab}\delta^{ba}\left[\left(\sqrt{-\nabla^2+\mu^2}\right)_{xy}\right]^2 \sim (1+\mu R)^2\,\frac{e^{-2\mu R}}{R^6}, \qquad M = 2\mu$$



$$G_0(x) = \delta^{ab}\delta^{ba}\left[\left(\sqrt{-\nabla^2+\mu^2}\right)_{xy}\right]^2 \sim (1+\mu R)^2\,\frac{e^{-2\mu R}}{R^6}, \qquad M = 2\mu$$

Summary (of apparent pros)

- Our simple approximate form of the confining YM vacuum wavefunctional in 2+1 dimensions has the following properties:
  - It is a solution of the YM Schrödinger equation in the weak-coupling limit …
  - … and also in the zero-mode, strong-field limit.
  - Dimensional reduction works: There is confinement (non-zero string tension) if the free mass parameter m is larger than 0.
  - $m > 0$ seems energetically preferred.
  - If the free parameter m is adjusted to give the correct string tension at the given coupling, then the correct value of the mass gap is also obtained.

Open questions (or contras?)

- Can one improve (systematically) our vacuum wavefunctional Ansatz?
- Can one make a more reliable variational estimate of m?
- Comparison to other proposals?
  - Karabali, Kim, Nair (1998)
  - Leigh, Minic, Yelnikov (2007)
- What about N-ality?
- Knowing the (approximate) ground state, can one construct an (approximate) flux-tube state, estimate its energy as a function of separation, and get the right value of the string tension?
- How to go to 3+1 dimensions?
  - Much more challenging (Bianchi identity, numerical treatment very CPU time consuming).
  - The zero-mode, strong-field limit argument valid (in certain approximation) also in D=3+1.

Thank you for attention, and **Axel and Christian for invitation** to this beautiful place and workshop which has been such a refreshment in the hard life of a theorist!