Instanton constituents in the $O(3)$ model and Yang-Mills theory

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Part I: The $O(3)$ sigma model

a scalar field in 2D …

$$S = \int d^2x \, \frac{1}{2}(\partial_\mu \phi^a)^2 \qquad a = 1,2,3: \text{ global } O(3) \text{ symmetry}$$

… with a constraint

$$\phi^a \phi^a = 1 \qquad \text{(circumvent Derrick's theorem)}$$

nontrivial properties:

- asymptotic freedom
- dynamical mass gap
- instantons

condensed matter physics and toy model for gauge theories

Topology

finite action:

$$r \to \infty : \quad \phi^a \to \text{const.}$$

as a mapping:

$$\phi : \mathbb{R}^2 \cup \{\infty\} \simeq S_x^2 \longrightarrow S_c^2$$

winding number/degree: all such ϕ's are characterized by an integer Q

= how often S_c^2 is wrapped by S_x^2 through ϕ

(alternatively: how often any point ϕ_0 on S_c^2 is visited by ϕ)

here:

$$Q = \frac{1}{8\pi} \int d^2x \, \epsilon_{\mu\nu} \epsilon_{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \in \mathbb{Z}$$

topological quantum number = invariant under small deformations of ϕ
(not a Noether symmetry, since for every config. indep. of Lagrangian)

Bogomolnyi trick…

$$(\partial_\mu \phi^a \pm \epsilon_{\mu\nu}\epsilon_{abc}\phi^b\partial_\nu\phi^c)^2 = (\partial_\mu\phi^a)^2 \pm 2\epsilon_{\mu\nu}\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c + (\partial_\mu\phi^a)^2$$

…and bound:

$$S \geq 4\pi|Q|$$

where the equality holds iff

$$\partial_\mu\phi^a = \mp\,\epsilon_{\mu\nu}\epsilon_{abc}\phi^b\partial_\nu\phi^c$$ ‘selfduality equations’

first order (instead of second order in eqns. of motion)

classical solutions: solitons = instantons

= localised in both directions (see below)

Complex structure

introduce complex coordinates both in space and color space:

$$
\begin{aligned}
x_{1,2} &\quad\rightarrow\quad z^{(*)} = x_1 \pm i x_2 \\
\phi^a &\quad\rightarrow\quad u = \frac{\phi^1 + i\phi^2}{1-\phi^3}
\end{aligned}
\qquad
\begin{array}{lll}
\mathrm{N}: & \phi^a = (0,0,1) & u = \infty \\
\mathrm{S}: & \phi^a = (0,0,-1) & u = 0
\end{array}
$$

$\Rightarrow$ self-duality equations become Cauchy-Riemann conditions on u

any meromorphic function $u(z)$ is a solution

topological density

$$
q(x) = \frac{1}{\pi}\frac{1}{(1+|u|^2)^2}\left|\frac{\partial u}{\partial z}\right|^2
$$

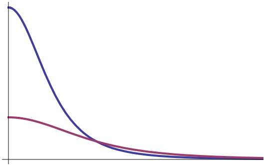
generalization:

$CP(N)$ models: more complex functions, again stable solutions

Charge 1 instantons

- simplest functions:

$$\left.\begin{array}{l} u(z) = \frac{\lambda}{z-z_0} \\[2em] u(z) = \frac{z-z_0}{\lambda} \end{array}\right\} \quad q(x) = \frac{1}{\pi}\frac{\lambda^2}{(|z-z_0|^2+\lambda^2)^2}$$



Belavin-Polyakov monopole

are $Q = 1$ instantons: location z_0, size λ

1 pole and 1 zero to cover S^2_c, one of them at infinity

- both, pole and zero, at finite z:

$$u(z) = \frac{z - z_I}{z - z_{II}}$$

$\rightsquigarrow$ constituents at $z = \{z_I, z_{II}\}$? 'instanton quarks'?

NO! same profile $q(x)$ as above $\Rightarrow$ one lump

with location $z_0 = (z_I + z_{II})/2$ and size $\lambda = |z_I - z_{II}|/2$

Finite temperature

= one compact direction, say: Im $z = x_2 \sim x_2 + \beta, \qquad \beta = 1/k_B T$

- instantons:

  use that higher charge solutions = products

$$u(z) = \prod_{k=1}^{Q} \frac{\lambda}{z - z_{0,k}} \qquad Q \text{ poles}$$

  and infinitely many copies: $z_{0,k} \equiv z_0 + k \cdot i\beta, \quad k \in \mathbb{Z}$

  $\Rightarrow$ infinite $u(z)$

- a regularized $u(z)$ is: Mittag-Leffler theorem

$$u(z) = \frac{\lambda}{\exp((z - z_0)\frac{2\pi}{\beta}) - 1} \qquad \text{has residues } \lambda \text{ at } z = z_0 + k \cdot i\beta$$

Boundary conditions

S, Q and $q(x)$ are invariant under global $SO(3)$ rotations

an $SO(2)$ subgroup: $\phi \to \begin{pmatrix} \text{rotation} & \\ \text{with } \omega & \\ & 1 \end{pmatrix} \phi, \quad u \to e^{2\pi i\omega} u$

let the fields ϕ and u be periodic up to that $SO(2)$ subgroup:

$$u(z + i\beta) = e^{2\pi i\omega} u(z) \qquad \omega \in [0,1]$$

novel solution: FB '07

$$u(z) = \frac{e^{\omega(z-z_0)\frac{2\pi}{\beta}} \cdot \lambda}{\exp((z-z_0)\frac{2\pi}{\beta}) - 1}$$

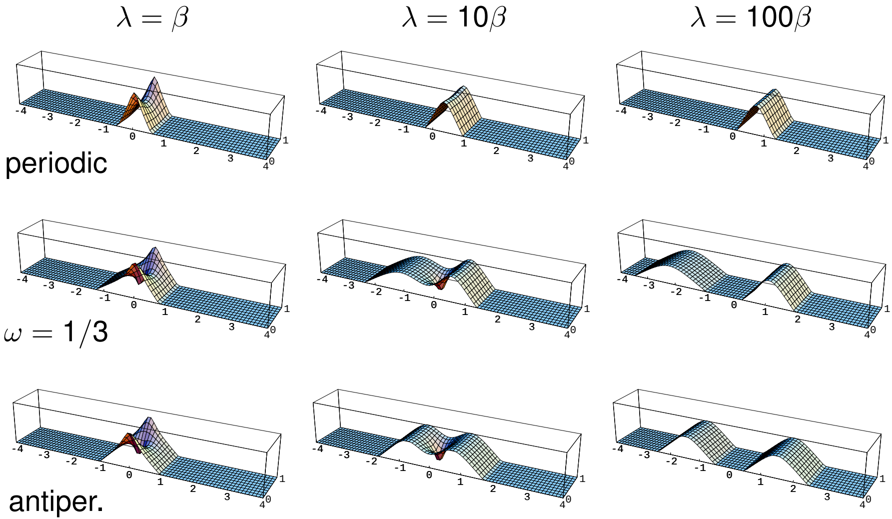
has residues $e^{2\pi i\omega k}\lambda$ at $z = z_0 + k \cdot i\beta$

'different orientation' of the instanton copies

$\Rightarrow$ nontrivial overlaps $\Rightarrow$ instanton constituents

Topological profiles

Topological density $\ln q(x)$ of finite temperature instantons:



($z_0 = 0$, cut off below e^{-5})

‘Dissociation’

for large size λ: 2 lumps with action ω and $\bar{\omega} = 1 - \omega$

why? rewrite:

$$u(z) = \frac{1}{\exp(-\omega(z - z_1)\frac{2\pi}{\beta}) - \exp(\bar{\omega}(z - z_2)\frac{2\pi}{\beta})}$$

locations: $z_1 = z_0 - \beta\,\frac{\ln\lambda}{2\pi\omega}, \quad z_2 = z_0 + \beta\,\frac{\ln\lambda}{2\pi\bar{\omega}}$

transmutation of λ: instanton size $\rightarrow$ constituent distance

$(z_2 - z_1 \sim \ln\lambda)$

really locations of topological lumps?

YES: corrections of the second term at $z = z_1$ are exp. small
therefore consider only one exponential

Individual constituents

$$u(z) = \exp(\omega z\,\frac{2\pi}{\beta}) \qquad [z_1 = 0]$$

- fulfils the phase boundary condition
- has an exponentially localised profile

$$q(x) = \frac{\pi\omega^2}{\beta^2 \cosh^2(\omega\,\mathrm{Re}\,z\,\frac{2\pi}{\beta})} \qquad \text{width } \sim \beta$$

- $|u|$ and hence $q(x)$ are static (Im z-indep.)
- has fractional charge $Q = \omega \quad \rightleftharpoons \quad$ covers fraction ω of $\mathbb{C}$
- the other constituent: same with ω replaced by $\bar{\omega}$
- both can occur in isolation; put together in the instanton
- can be seen on the lattice by cooling — Wipf, Wozar

• topological description

possible values for Q: $0, 1, \ldots$ $\omega, 1+\omega, \ldots$ $1-\omega, 2-\omega, \ldots$

instantons constituents anticonstituents

never a 'two-constituent' with $Q = 2\omega$ (always $-\omega$ inbetween)

• why instanton quarks not visible for zero temperature, i.e. on $\mathbb{R}^2$?

$\beta \to \infty$: constituents large and overlap!

no other scale competing with their distance

Outlook:

• fermionic zero modes Brendel,FB

• semiclassics

• $CP(N)$, $N \to \infty$

• relevance of the constituents for the dynamics of the $O(3)$ model?!

• realisation in condensed matter: strip of ... ?!

pure Yang-Mills theory in (Euclidean) 4D:

$$S = \int \frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu}^2 \geq |Q| = |\int \frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu}|$$

dual field strength $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^a_{\rho\sigma} \quad (\vec{E}^a \rightleftharpoons \vec{B}^a)$

integer Q: topological charge/instanton number

topology:

$$A_\mu \overset{r\to\infty}{\to} i\Omega^{-1}\partial_\mu\Omega \qquad \ldots\text{pure gauge}$$

$$Q = \deg(\Omega : S^3_{r\to\infty} \to SU(N)) \qquad \ldots\text{winding number}$$

Instantons

(anti)selfdual: $F^a_{\rho\sigma} = \pm \tilde{F}^a_{\mu\nu}$ first order, nonlinear

charge 1: axially symmetric ansatz and solution BPST

$$A^a_\mu = \eta^a_{\mu\nu} \frac{2x_\nu}{x^2+\rho^2} \qquad \mathrm{tr} F^2 = \frac{\rho^4}{(x^2+\rho^2)^4} \qquad \eta^a_{\mu\nu} \in \{-1, 0, 1\}$$

size ρ
localized in space and time
algebraic decay, similar to $O(3)$ instantons on $\mathbb{R}^2$

physics: instanton liquid model from semiclassical path integral

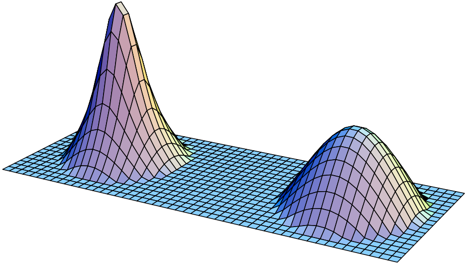
- chiral symmetry breaking
- axial anomaly
- topological susceptibility
- confinement?

- use higher charge solutions of same color orientation CFTW

$\Rightarrow$ first calorons Harrington-Shepard (1978)

- more general: ADHM formalism and Nahm transform

$\Rightarrow$ calorons of nontrivial holonomy Kraan,van Baal; Lee, Lu (1998)



space-space plot of action density for $SU(2)$, intermediate holonomy

$\Rightarrow$ 2 lumps, almost static

N_c for gauge group $SU(N_c)$, like quarks in baryons

magnetic monopoles of opposite magnetic charge

in fact dyons with same electric as magn. charge (selfdual)

Role of the holonomy

relative gauge orientation of instanton copies in the ADHM constr.

$\Rightarrow A_\mu$ periodic up to a gauge transformation $e^{2\pi i\omega\sigma_3}$ (cf. $O(3)$)

gauge theory: compensated by time-dependent transf. $e^{2\pi i\omega\sigma_3 x_0}$

$\Rightarrow$ introduces an asymptotic gauge field A_0

$\Rightarrow$ asymptotic Polyakov loop = holonomy

$$\mathcal{P}(\vec{x}) \equiv \mathcal{P}\exp\left(i\int_0^\beta dx_0\, A_0\right) \to e^{2\pi i\omega\sigma_3} \equiv \mathcal{P}_\infty$$

(indep. of direction if magnetically neutral)

acts like a Higgs field, in the group: vev ω, direction σ_3

- monopoles have masses $2\omega/\beta$ and $2\bar{\omega}/\beta$, $\qquad 2\bar{\omega} = 1 - 2\omega$
- $A_\mu^{a=3}$: power law decay (massless 'photon'),
  $A_\mu^{a=1,2}$: exponential decay (massive 'W-bosons')

Interesting features of the caloron

- $A_\mu^{a=3}$ in the far-field limit: dipole from monopole/antimonopole
- $A_\mu^{a=1,2}$ finetuned to avoid Dirac strings
- Polyakov loop in the bulk: $\mathcal{P}(\vec{x}) = \pm\mathbb{1}_2$ at the monopoles
  Higgs field vanishes = 'false vacuum'
  necessary for top. reasons — Ford et al.; Reinhardt; Jahn et al.
- index theorem valid — Nye, Singer
  but 1 zero mode for 2 monopoles? $\Rightarrow$ localised depending on bc.s:

$$\psi(x_0 + i\beta) = e^{2\pi i z}\psi(x_0) \qquad (A_\mu \text{ still periodic})$$

  $z \in \{-\omega, \omega\}$ incl. periodic: localised at monopole — Garcia Perez et al.
  $z \in \{\omega, 1-\omega\}$ incl. antiperiodic: localised at antimonopole

  a zero in their profiles at the 'other' monopole, topological — FB

- can be studied on the lattice by cooling — Ilgenfritz et al., FB et al.

Calorons and the dynamics of YM theories

- eff. potential at 1-loop: triv. holonomy favored!

Gross, Pisarski, Jaffe; Weiss

overruled by caloron gas contribution: Diakonov et al.

$\Rightarrow$ minima at $\mathcal{P} = \pm\mathbb{1}_2$ become unstable for low enough temperature

$\Rightarrow$ onset of confinement

- gas of calorons and anticalorons put on the lattice: Gerhold et al.

superposition problem solved for fixed holonomy

$\Rightarrow$ linearly rising interquark potential just for nontrivial holonomy!

- confinement from a gas of purely selfdual dyons Diakonov,Petrov
  unphysical...

$\rightsquigarrow$ confinement in reach of instantons!? stay tuned!