QCD thermodynamics at weak coupling

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Motivation

Why thermal QCD?

- study confinement and chiral symmetry breaking [cf other talks and poster session]
- phenomenologically relevant for cosmology [$\rightarrow$ see below]
- phenomenologically relevant for RHIC, LHC [$\rightarrow$ see below]
- theoretical limit tractable with analytic methods [this talk]
  - ▷ *goal: no models - stay within QCD!*

Why weak-coupling methods?

- need to complement / understand other approaches (mainly LAT)

	Lattice	**weak-coupling QCD**
Temperature	$T \sim T_c$	$T \gg T_c$
Chemical Potentials	$\mu \lesssim T$	any
Quark masses	close to physical	only at low orders
Dynamic quantities	very limited	yes
Cost	up to ∞	modest
Physical picture	''black box''	sometimes

- can be systematically improved

Motivation

Focus on equilibrium thermodynamics of QCD

- structure of QCD phase diagram
  location of transition line; critical point; properties of transition; ...
- equation of state (EoS) of QGP
- properties of QGP: correlation lengths, spectral functions, ...

Interplay of methods

- QGP is strongly coupled system near $T_c \Rightarrow$ need e.g. LAT
- asymptotic freedom at high $T \Rightarrow$ weak-coupling approach
- cave: strict loop expansion not well-defined
  IR divergences at higher orders [Linde 1979; Gross/Pisarski/Yaffe 1981]

Discuss

- effective theories
- spatial string tension
- basic thermodynamic observable: pressure $p(T)$ [$\Leftarrow$ main playground]
- quark mass effects on EoS

Motivation

$p(T)$ important for cosmology:

- cooling rate of the universe

$$\partial_t T = -\frac{\sqrt{24\pi}}{m_{\mathsf{Pl}}} \frac{\sqrt{e(T)}}{\partial_T \ln s(T)}$$

- with entropy $s = \partial_T \, p$ and energy density $e = Ts - p$
- $\Rightarrow$ cosmol. relics (dark matter, background radiation etc.) originate when an interaction rate $\tau(T)$ gets larger than the age of the universe $t(T)$.
  - ▷ *Ex.: ''sterile'' ν_R with $m_\nu \sim$ keV can be warm dark matter, and decouple around $T \sim 150$ MeV* [Abazajian, Fuller 02; Asaka, Shaposhnikov 05]

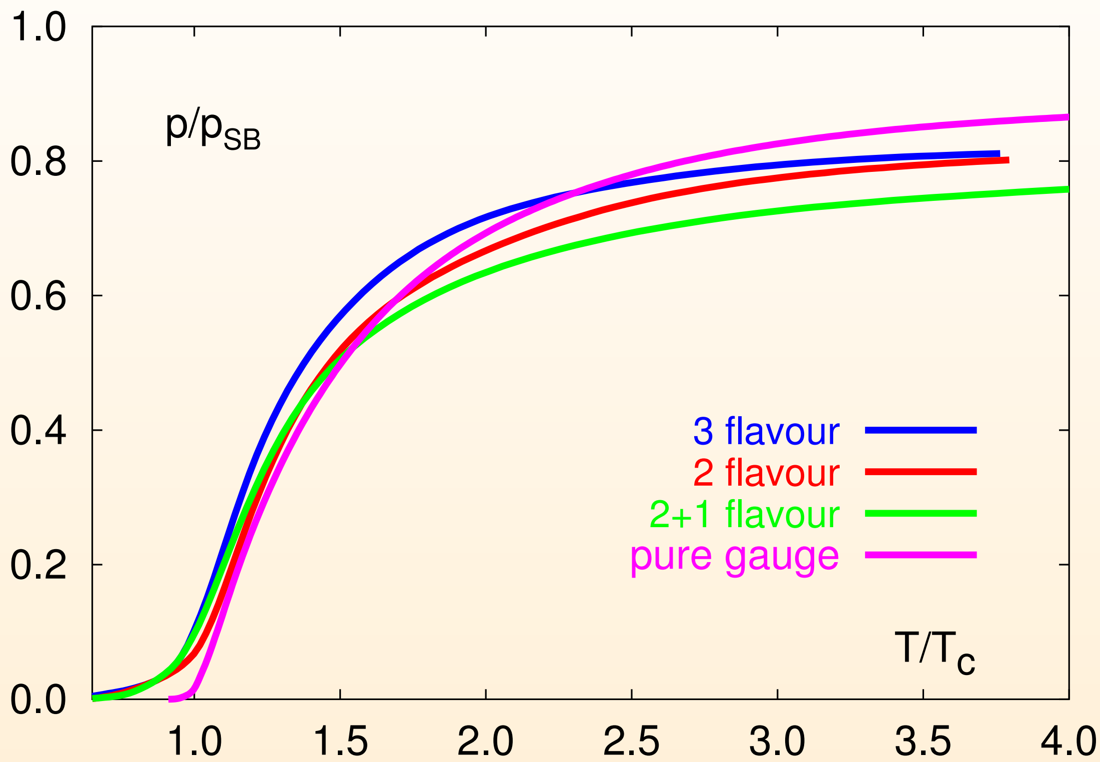
$p(T)$ in heavy ion collisions:

- expansion rate (after thermalization) given by

$$\partial_\mu T^{\mu\nu} = 0 \quad , \quad T^{\mu\nu} = [p(T) + e(T)]\, u^\mu u^\nu - p(T) g^{\mu\nu}$$

- with flow velocity $u^\mu(t,x)$
  - ▷ *hydrodynamic expansion: hadronization at $T \sim 100 - 150$ MeV $\Rightarrow$ observed hadron spectrum depends (indirectly) on $p(T)$*

$p(T)$ via (large) computer ($\mu_B = 0$)



[lattice data from Karsch et.al.]

at $T \to \infty$, expect ideal gas: $p_{\text{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$

confirms simplicity: 3 dofs (π) $\to$ 52 ($3 \times 3 \times 2 \times 2$ qu + 8×2 gl)

Energy scales in hot QCD

Interactions make QCD a multiscale system

At asymptotically high T, $g \ll 1 \Rightarrow$ clean separation of 3 scales

expansion parameter:

$$g^2\, n_b(|k|) = \frac{g^2}{e^{|k|/T} - 1} \overset{|k| \lesssim T}{\approx} \frac{g^2 T}{|k|}$$

- $|k| \sim \pi T$ aka ''hard'': fully perturbative at high T
  thermal fluctuations; effective mass of non-static field modes
- $|k| \sim gT$ aka ''soft'': dynamically generated; barely perturbative at high T
  inverse screening length of static color-electric fluctuations; thermal/Debye mass
- $|k| \sim g^2 T$ aka ''ultrasoft'': dynamically generated; non-perturbative at high T
  inverse screening length of static color-magnetic fluctuations; ''magnetic mass''
- no smaller momentum scales / larger length scales due to confinement

treatment of a multiscale system: effective field theory !

$p(T)$ via weak-coupling expansion

need to explain 20% deviation from ideal gas at $T \sim 4T_c$

- structure of pert series is non-trivial !
- $$p(T) \equiv \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}[A_\mu^a, \psi, \bar\psi] \exp\left(-\frac{1}{\hbar}\int_0^{\hbar/T} d\tau \int d^{3-2\epsilon}x\, \mathcal{L}_{\text{QCD}}\right)$$
  $$= c_0 + c_2 g^2 + c_3 g^3 + (c_4' \ln g + c_4) g^4 + c_5 g^5 + (c_6' \ln g + c_6) g^6 + \mathcal{O}(g^7)$$

[c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold/Zhai 94, c_5 Zhai/Kastening 95, Braaten/Nieto 96, c_6' KLRS 03]

- root cause of nonanalytic (in α_s) behavior well understood: above-mentioned dynamically generated scales
- clean separation best understood in effective field theory setup [this talk]
- other re-organizations possible, e.g. 2PI skeleton-expansion [see talk by JP Blaizot]

Effective theory prediction for $p(T)$

$$\frac{p_{\mathsf{QCD}}(T)}{p_{\mathsf{SB}}} = \frac{p_{\mathsf{E}}(T)}{p_{\mathsf{SB}}} + \frac{p_{\mathsf{M}}(T)}{p_{\mathsf{SB}}} + \frac{p_{\mathsf{G}}(T)}{p_{\mathsf{SB}}} \quad , \quad p_{\mathsf{SB}} = \left(16 + \frac{21}{2} N_f\right) \frac{\pi^2 T^4}{90}$$

$$= 1 + g^2 + g^4 + g^6 + \ldots \quad \Leftarrow \text{4d QCD}$$

$$+ g^3 + g^4 + g^5 + g^6 + \ldots \quad \Leftarrow \text{3d adj H}$$

$$+ \frac{1}{p_{\mathsf{SB}}} \frac{T}{V} \int \mathcal{D}[A_k^a] \exp\Big(-S_{\mathsf{M}}\Big) \quad \Leftarrow \text{3d YM}$$

- this could be coined the *physical leading-order (!) approximation*
- collect contributions to $p(T)$ from all physical scales
  - ▷ *weak coupling, effective field theory setup*
  - ▷ *faithfully adding up all Feynman diagrams*
  - ▷ *get long-distance input from clean lattice observable:*

$$p_{\mathsf{G}}(T) \equiv \frac{T}{V} \ln \int \mathcal{D}[A_k^a] \exp\Big(-S_{\mathsf{M}}\Big) = T \# \, g_{\mathsf{M}}^6$$

only one non-perturbative (but computable!) coeff needed

- how does this work in detail?

Effective theory setup: QCD $\to$ EQCD

high T: QCD dynamics contained in 3d EQCD

integrate out $|p| \gtrsim 2\pi T$: ψ, $A_\mu(n \neq 0)$

$$p_{\mathsf{QCD}}(T) \equiv p_{\mathsf{E}}(T) + \frac{T}{V} \ln \int \mathcal{D}[A_k^a, A_0^a] \exp\Big(-\int d^{3-2\epsilon}x\, \mathcal{L}_{\mathsf{E}}\Big)$$

$$\mathcal{L}_{\mathsf{E}} = \frac{1}{2} Tr\, F_{kl}^2 + Tr\,[D_k, A_0]^2 + m_{\mathsf{E}}^2 Tr\, A_0^2 + \lambda_{\mathsf{E}}^{(1)} (Tr\, A_0^2)^2 + \lambda_{\mathsf{E}}^{(2)} Tr\, A_0^4 + \ldots$$

five matching coefficients [E. Braaten, A. Nieto, 95; KLRS 02; M. Laine, YS, 05]

$$p_{\mathsf{E}} = T^4 \left[\# + \# g^2 + \# g^4 + \# g^6 + \ldots\right],\ m_{\mathsf{E}}^2 = T^2 \left[\# g^2 + \# g^4 + \ldots\right],$$

$$g_{\mathsf{E}}^2 = T\left[g^2 + \# g^4 + \# g^6 + \ldots\right],\ \lambda_{\mathsf{E}}^{(1),(2)} = T\left[\# g^4 + \ldots\right].$$

higher order operators do not (yet) contribute [S. Chapman, 94; Kajantie et al, 97, 02]

$$\frac{\delta p_{\mathsf{QCD}}(T)}{T} \sim \delta \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{D_k D_l}{(2\pi T)^2} \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{(gT)^2}{(2\pi T)^2} (gT)^3 \sim g^7 T^3$$

Effective theory setup: QCD → EQCD → MQCD

the IR of 3d EQCD is contained in 3d MQCD

integrate out $|p| \gtrsim gT$: A_0

$$p_{\mathsf{QCD}}(T) \equiv p_{\mathsf{E}}(T) + p_{\mathsf{M}}(T) + \frac{T}{V} \ln \int \mathcal{D}[A_k^a] \exp\Big(- \int d^{3-2\epsilon}x \, \mathcal{L}_{\mathsf{M}}\Big)$$

$$\mathcal{L}_{\mathsf{M}} = \frac{1}{2} Tr\, F_{kl}^2 + \dots$$

two matching coefficients [KLRS 03; P. Giovannangeli 04, M. Laine/YS 05]

$$p_{\mathsf{M}} = T m_{\mathsf{E}}^3 \left[\# + \# \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \# \frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \# \frac{g_{\mathsf{E}}^6}{m_{\mathsf{E}}^3} + \dots\right], \; g_{\mathsf{M}}^2 = g_{\mathsf{E}}^2 \left[1 + \# \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \# \frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \dots\right].$$

higher order operators do not (yet) contribute

$$\frac{\delta p_{\mathsf{QCD}}(T)}{T} \sim \delta \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{D_k D_l}{m_{\mathsf{E}}^3} \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{(g^2 T)^2}{m_{\mathsf{E}}^3} (g^2 T)^3 \sim g^9 T^3$$

Shopping list for c_6

$\ldots + g^6$

- 4-loop sum-integrals needed, const term
- DOABLE?! manpower OR brainpower? [YS/AV ??]

matching coeffs

- 2-loop ϵ-terms for m_{E}^2, g_{E}^2 DONE. ML/YS 05: IBP, reduction, master sum-ints

$\ldots + g^6$

- 4-loop integrals needed DONE. KLRS 03: reduction, master ints, HPL

match $\overline{\text{MS}}$/LAT

- 4-loop const in LAT reg via NSPT DONE. LMRST 06: LAT pert

$\ldots + g^6$

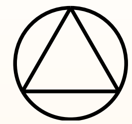
- measure $<$ Plaquette $>$ in 3d SU(N) DONE. HKLRS 05: LAT Monte Carlo

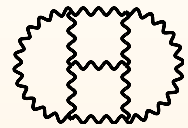
Shopping list for c_6

4-loop sum-integrals?

- a single one has already been computed
  - ▷ *painfully disentangled (sub-)divergences by hand*
  - ▷ *constant term only numerically*
  - ▷ *gave the g^6 term in scalar ϕ^4* [GLSTV 2008]
- in QCD, need $\mathcal{O}(10^8)$ of them
- ideas to profit from algorithmic $T = 0$ methods not fruitful (yet?)
  - ▷ *as used - and tested extensively - for the 3d part* [⇐ see below]
- find a smart duality to map the problem to sth simpler?

Algorithmic methods: reduction, IBP

can do 4-loop scalar theory on paper:  1 integral

for QCD, need a computer:  25M integrals ($2^9 6^6$)

powerful method: integration by parts (IBP) [Chetyrkin/Tkachov 81]

$\Rightarrow$ systematically use $0 = \int d^d k \, \partial_{k_\mu} f_\mu(k)$

many incarnations: Laporta, Baikov, Gröbner

key idea: lexicographic ordering among all loop integrals [Laporta 00]

arrive at rep in terms of irreducible ($\equiv$ master) integrals

$$\sum_i \frac{\text{poly}_i(d,\xi)}{\text{poly}_i(d)} \text{Master}_i(d)$$

Algorithmic methods: integration

Evaluating Masters

- numerical integration; cave: precision (MC?)
- explicit integration; can be an ''art''
- difference equations
  - *solve directly*
  - *solve numerically*
  - *Laplace transform*
- differential equations

Mathematical structure

- interested in the coefficients of an ϵ expansion
- in many cases, these are from a generic class of functions/numbers
- e.g. harmonic polylogarithms HPL(x) [Remiddi/Vermaseren 00]
- e.g. harmonic sums $S(N)$ [Vermaseren 98]

Algorithmic methods: harmonic sums

find interesting new numbers

$$
\begin{aligned}
S_{a,\bar{m}}(N) &\equiv \sum_{i=1}^{N} \frac{[\mathsf{sgn}(a)]^i}{i^{|a|}} S_{\bar{m}}(i) && , \quad S(i \geq 0) = 1 \quad , \quad S(i < 0) = 0 \\
\ln 2 &= -S_{-1}(\infty) \\
\zeta_{n\geq 2} &= S_n(\infty) && = \zeta(n) \\
a_{n\geq 4} &= -S_{-1,\bar{1}_{n-1}}(\infty) && = Li_n\,(1/2) \\
s_6 &= S_{-5,-1}(\infty) && \approx +0.98744142640329971377165000804182021413602714\,89 \\
s_{7a} &= S_{-5,1,1}(\infty) && \approx -0.95296007575629860340865215892596050767328040\,17 \\
s_{7b} &= S_{5,-1,-1}(\infty) && \approx +1.02912126296432453422440408804384184300201671\,26 \\
s_{8a} &= S_{5,3}(\infty) && \approx +1.04178502918279188338999002080231238008156211\,01 \\
s_{8b} &= S_{-7,-1}(\infty) && \approx +0.99644774839783766598087290122422927214404887\,82 \\
s_{8c} &= S_{-5,-1,-1,-1}(\infty) && \approx +0.98396667382173367092073025030655946912191093\,09 \\
s_{8d} &= S_{-5,-1,1,1}(\infty) && \approx +0.99996261346268344769671661371698277760950413\,87
\end{aligned}
$$

"the language that Feynman diagrams speak"? [J. Vermaseren]

Parametric behavior of some observables

pressure, energy density, ..

- $\frac{p}{T^4} \sim 1 + g^2 + g^3 + g^4 \ln g + g^4 + g^5 + g^6(\ln g + [\text{np}]) + \dots$

correlation lengths $\xi = m_\mathsf{E}^{-1}$ für $Tr\, F_{0i} F_{jk}$, Tr Pol etc.

- $m_\mathsf{E} \sim gT + g^2 T(\ln g + [\text{np}]) + \dots$

correlation lengths $\xi = m_\mathsf{G}^{-1}$ für $Tr\, F_{ij}^2$

- $m_\mathsf{G} \sim [\text{np}] \times g^2 T + \dots$

spatial string tension

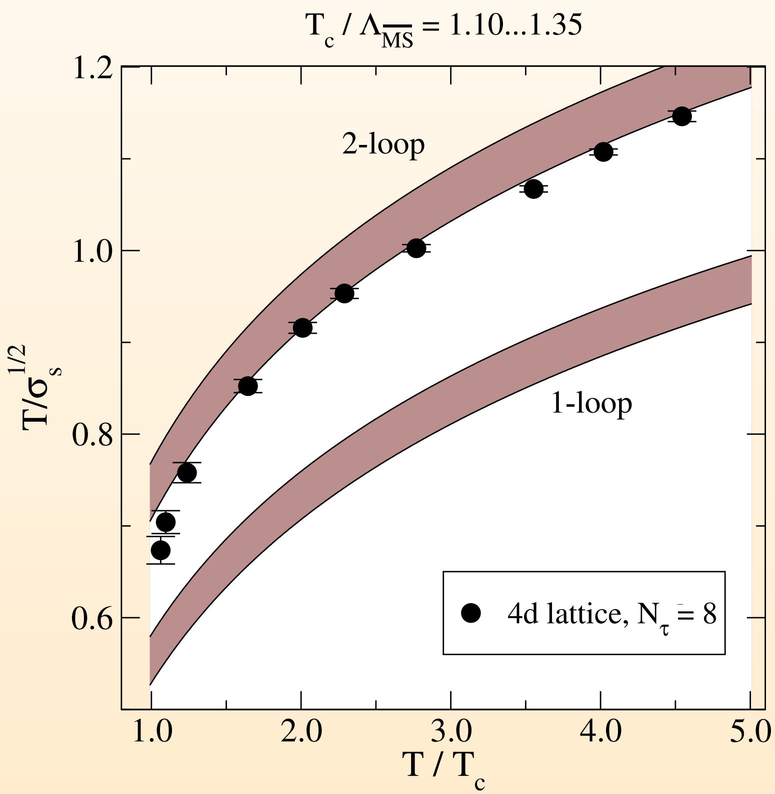
- $\sqrt{\sigma_s} \sim [\text{np}] \times g^2 T + \dots$

$\Rightarrow$ use these quantities e.g. as precision test of eff. th. setup

Spatial string tension: $W_s(R_1, R_2) = \exp(-\sigma_s\, R_1 R_2)$ at large R_1, R_2

SU(3), 4d lat: $\frac{\sqrt{\sigma_s}}{T} = \text{fct}\left(\frac{T}{T_c}\right)$; $T_c \approx 1.2\Lambda_{\overline{\text{MS}}}$

SU(3), 3d MQCD: $\frac{\sqrt{\sigma_s}}{T} = \#\, \frac{g_{\text{M}}^2}{g_{\text{E}}^2} \frac{g_{\text{E}}^2}{T} = \text{fct}\left(\frac{T}{\Lambda_{\overline{\text{MS}}}}\right)$; $\# = 0.553(1)$ [Teper, Lucini 02]



[4d lattice data from Boyd et al, 96] (cave: no cont. extrapolation)

parameter-free comparison; support for hard/soft+ultrasoft picture

Outlook for $p(T)$: $g^6 \to g^7 \to g^8$

$$\frac{p_{\mathsf{G}}}{p_{\mathsf{SB}}} = \#_{(6)}\left(\frac{g_{\mathsf{M}}^2}{T}\right)^3 + [\delta\mathcal{L}_{\mathsf{M}}]_{(9)}$$

$$g_{\mathsf{M}}^2 = g_{\mathsf{E}}^2\left[1 + \#_{(7)}\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \left(\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\right)^2\left(\#_{(8)} + \#_{(10)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right) + \dots_{(9)}\right]$$

$$\frac{p_{\mathsf{M}}}{p_{\mathsf{SB}}} = \frac{m_{\mathsf{E}}^3}{T^3}\left[\#_{(3)} + \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\left(\#_{(4)} + \#_{(6)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right) + \left(\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\right)^2\left(\#_{(5)} + \#_{(7)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2} + \#_{(9)}\left(\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right)^2\right)\right.$$

$$+ \left(\frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}}\right)^3\left(\#_{(6)} + \#_{(8)}\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2} + \#_{(10)}\left(\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right)^2 + \#_{(12)}\left(\frac{\lambda_{\mathsf{E}}}{g_{\mathsf{E}}^2}\right)^3\right)$$

$$\left. + [\text{3d 5loop 0pt}]_{(7)} + [\delta\mathcal{L}_{\mathsf{E}}]_{(7)} + [\text{3d 6loop 0pt}]_{(8)} + \dots_{(9)}\right]$$

$$m_{\mathsf{E}}^2 = T^2\left[\#_{(3)}g^2 + \#_{(5)}g^4 + [\text{4d 3loop 2pt}]_{(7)} + \dots_{(9)}\right]$$

$$\lambda_{\mathsf{E}} = T\left[\#_{(6)}g^4 + \#_{(8)}g^6 + \dots_{(10)}\right]$$

$$g_{\mathsf{E}}^2 = T\left[g^2 + \#_{(6)}g^4 + \#_{(8)}g^6 + \dots_{(10)}\right]$$

$$\frac{p_{\mathsf{E}}}{p_{\mathsf{SB}}} = \#_{(0)} + \#_{(2)}g^2 + \#_{(4)}g^4 + \#_{(6)}g^6 + [\text{4d 5loop 0pt}]_{(8)} + \dots_{(10)}$$

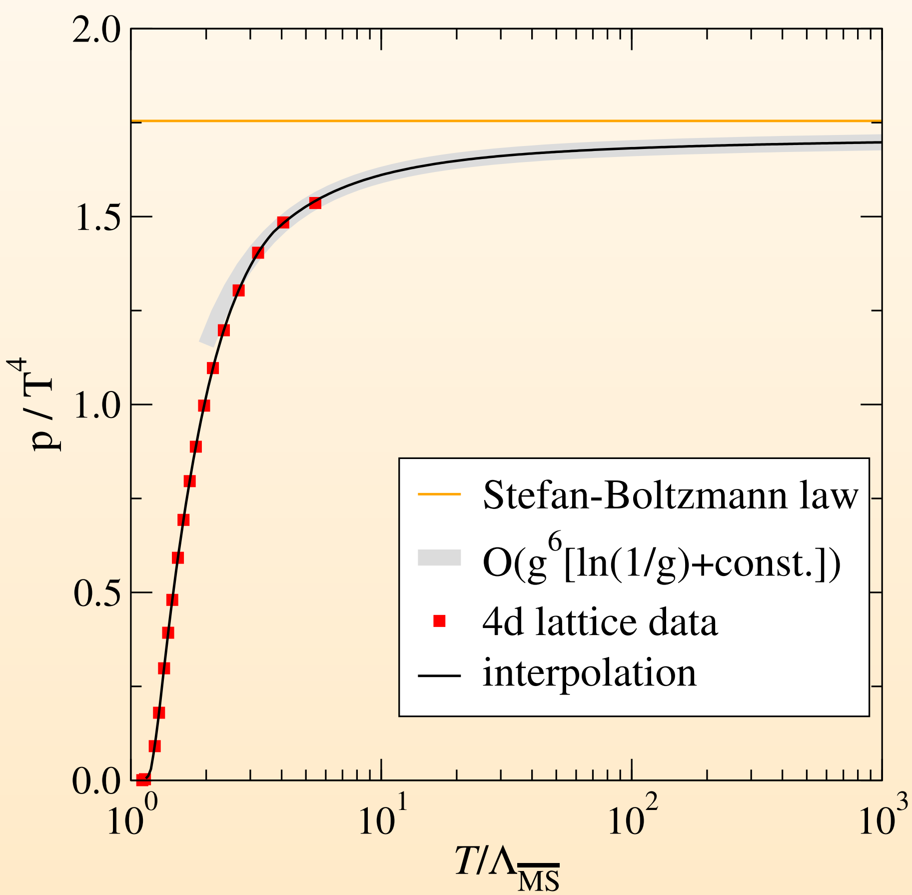
notation: $\#_{(n)}$ enters p_{QCD} at g^n

[cave: no $\frac{1}{\epsilon} + 1 + \epsilon$, no IR/UV, and no logs shown above]

Matching $p(T)$ **at** $N_f = 0$

in the meantime ...

- want to show results / tackle simpler problems / phenomenology
- strive for best possible description of pure-glue sector

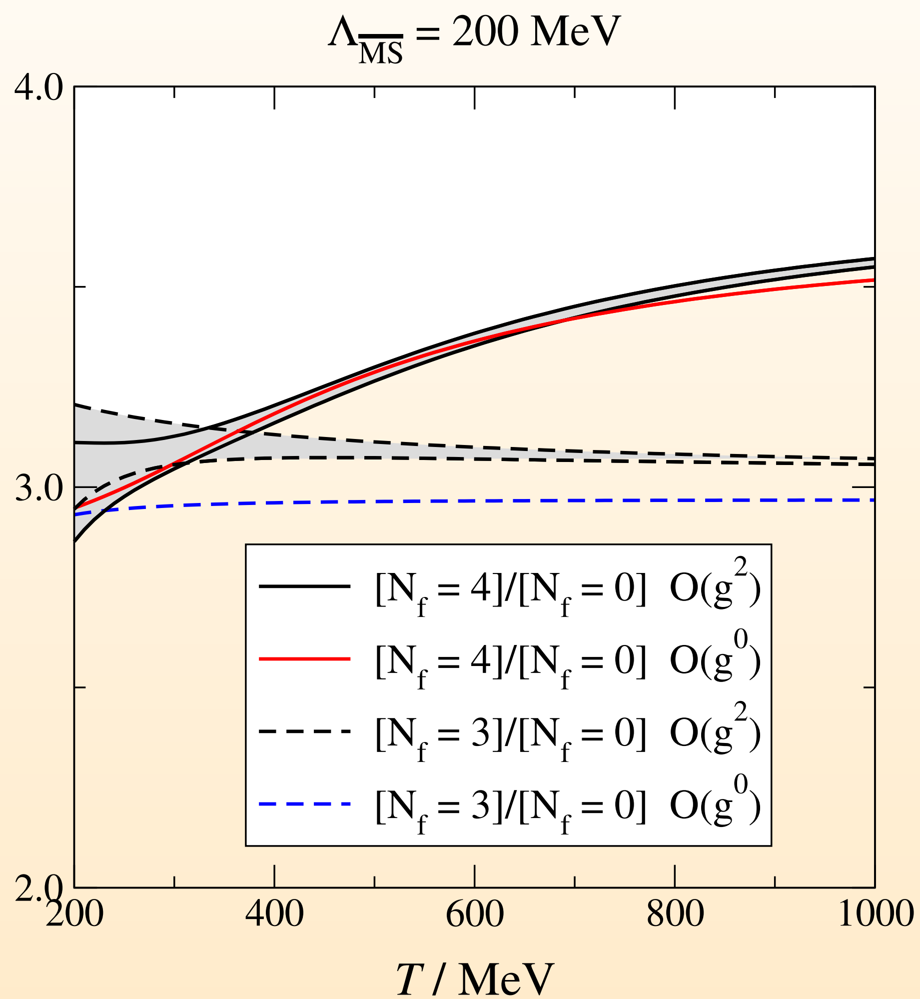


- fix unknown perturbative $\mathcal{O}(g^6)$ coeff
- use available lattice data
  here: at 3-5T_c [Boyd et al 1996]
- translate via $T_c/\Lambda_{\overline{\text{MS}}} \approx 1.20$
- match at intermediate $T \sim 3 - 5T_c$
- (at high T? [Endrödi et al LAT07; Borsanyi])

Quark mass dependence

analyze quark mass dependence to NLO

- strategy: ''unquenching''
  start from $N_f = 0$, i.e. $m_q = \infty$
  lower N_f quark masses to $m_{q,phys}$
  p at any T increases
- estimate this ''correction factor''
- aproach is systematic
  LO: $c_0(N_f)/c_0(0)$
  NLO: $[c_0 + g^2 c_2](N_f) \;/\; [c_0 + g^2 c_2](0)$
- computed $c_{0,2}(T, N_c, N_f, m_i, \mu_i)$
- good convergence LO→NLO
  - ▷ *$N_f = 3$: 5% effect*
  - ▷ *$N_f = 4$: even better*



charm quark contributes already at low $T \sim 350 MeV$

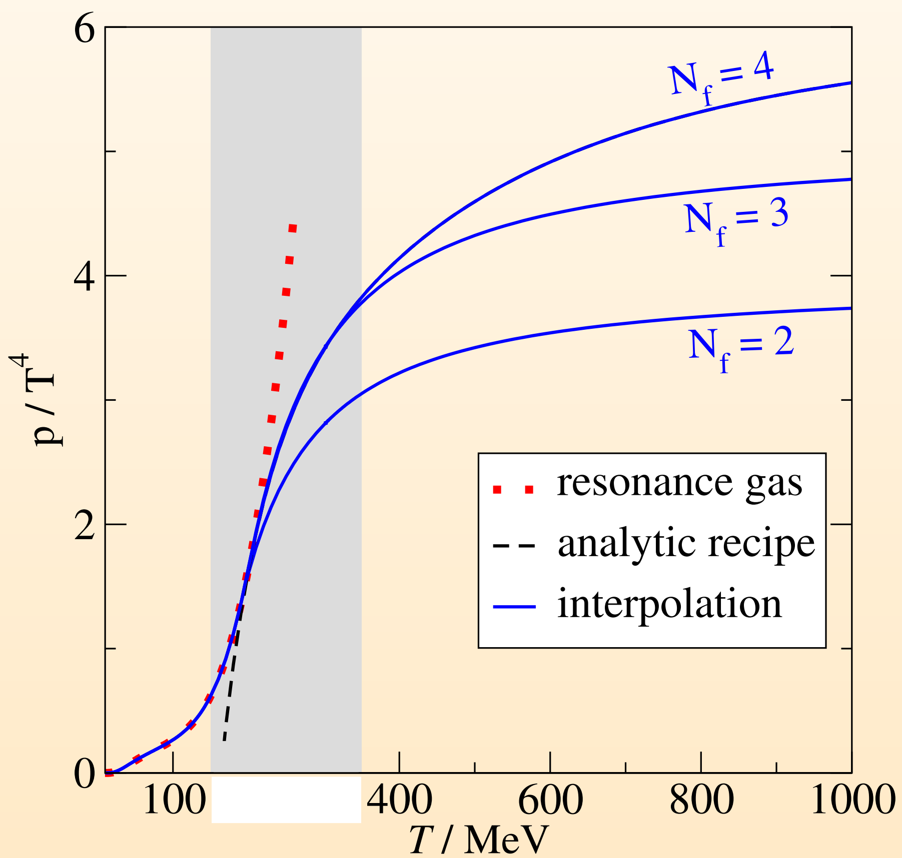
Setting the scale

now ready to estimate thermodynamic quantities

multiply best $N_f = 0$ result with correction factor

$$g^2(\bar\mu) = \frac{24\pi^2}{(11C_A - 2N_f)\ln(\bar\mu/\Lambda_{\overline{\mathrm{MS}}})}\ , \quad m_i(\bar\mu) = m_i(\bar\mu_{\mathrm{ref}})\left[\frac{\ln(\bar\mu_{\mathrm{ref}}/\Lambda_{\overline{\mathrm{MS}}})}{\ln(\bar\mu/\Lambda_{\overline{\mathrm{MS}}})}\right]^{\frac{9C_F}{11C_A-2N_f}}$$

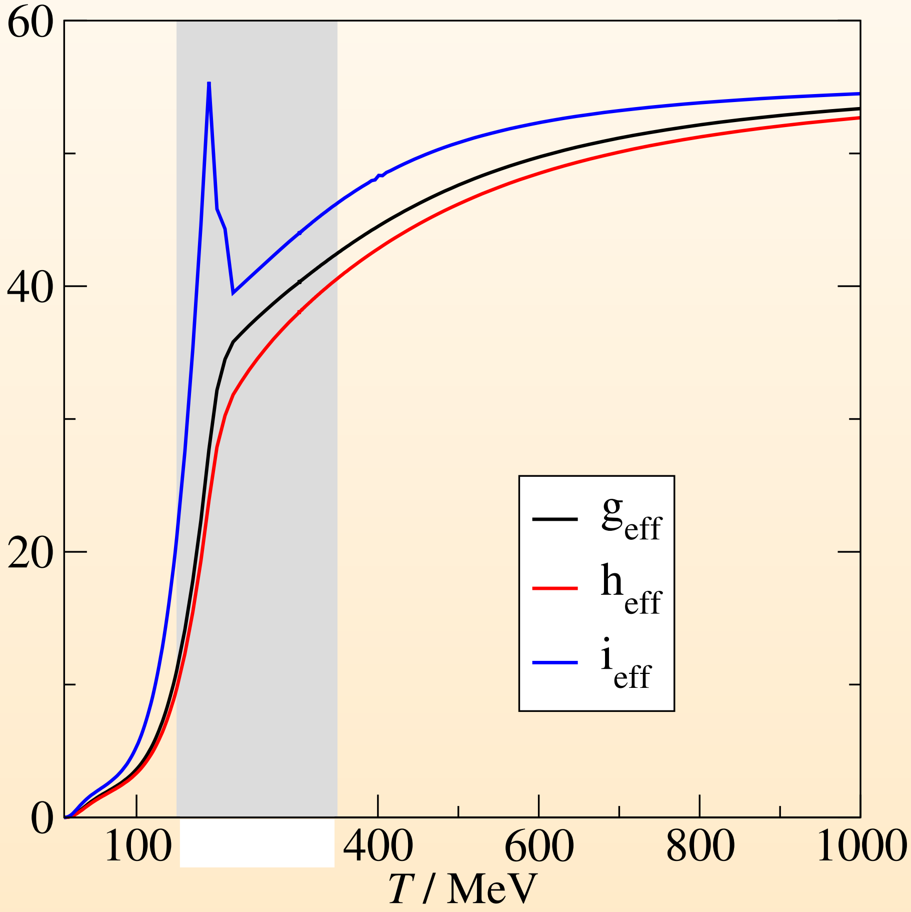
- need to fix $\Lambda_{\overline{\mathrm{MS}}}$ in physical units!
- strategy: matching
  take p of hadronic resonances
  match p and p' to our recipe
- obtain $\Lambda_{\overline{\mathrm{MS}}}^{(eff)} \approx 175...180 MeV$
- shaded: lattice simulations needed!



Thermodynamic quantities

now use the recipe $\boxed{p(N_f{=}0)\times \text{corr.fct}}$ to obtain

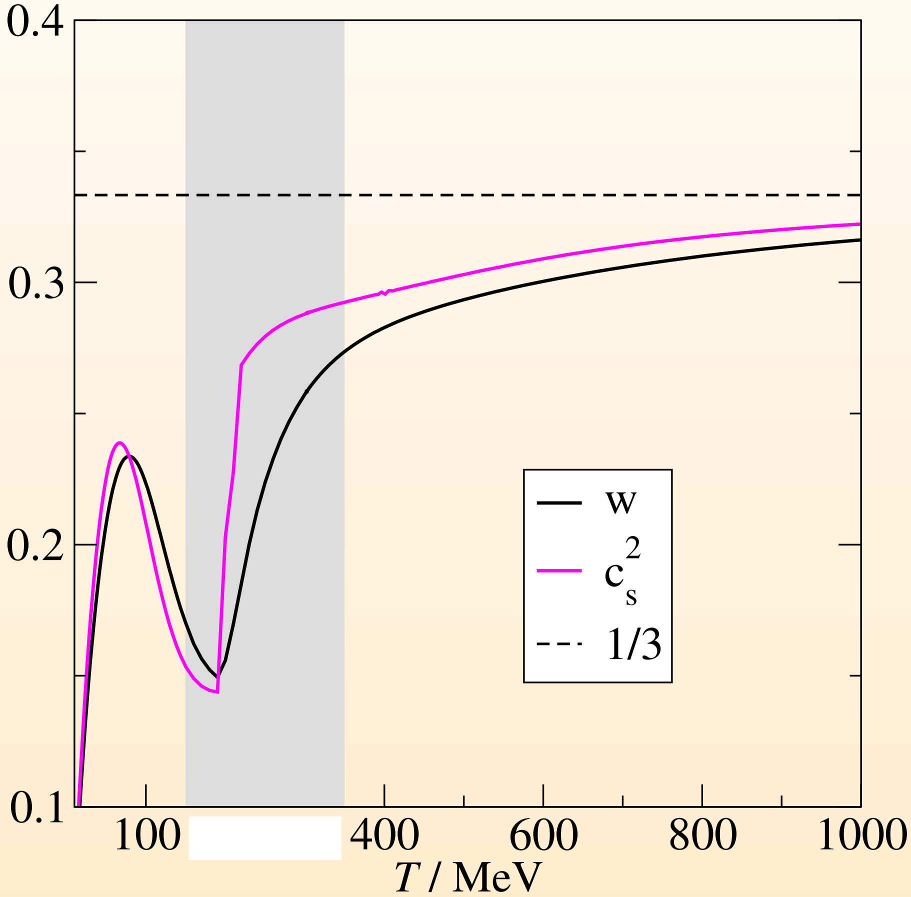
$$s(T) = p'(T)\ ,\ \ e(T) = Ts(T) - p(T)\ ,\ \ c(T) = e'(T) = Tp''(T)$$



- use eff numbers of bosonic dof's
  $g_{\text{eff}}(T) \equiv e(T)/\left[\frac{\pi^2 T^4}{30}\right]$
  $h_{\text{eff}}(T) \equiv s(T)/\left[\frac{2\pi^2 T^3}{45}\right]$
  $i_{\text{eff}}(T) \equiv c(T)/\left[\frac{2\pi^2 T^3}{15}\right]$
- observe significant structure
- at 2nd order phase transition
  $i(T) \sim (T - T_c)^{-\gamma}$

Thermodynamic quantities

consider dimensionless ratios



- equation of state

$$w(T) \equiv \frac{p(T)}{e(T)} = \frac{p(T)}{Tp'(T) - p(T)}$$

- sound speed (squared)

$$c_s^2(T) \equiv \frac{p'(T)}{e'(T)} = \frac{p'(T)}{Tp''(T)} = \frac{s(T)}{c(T)}$$

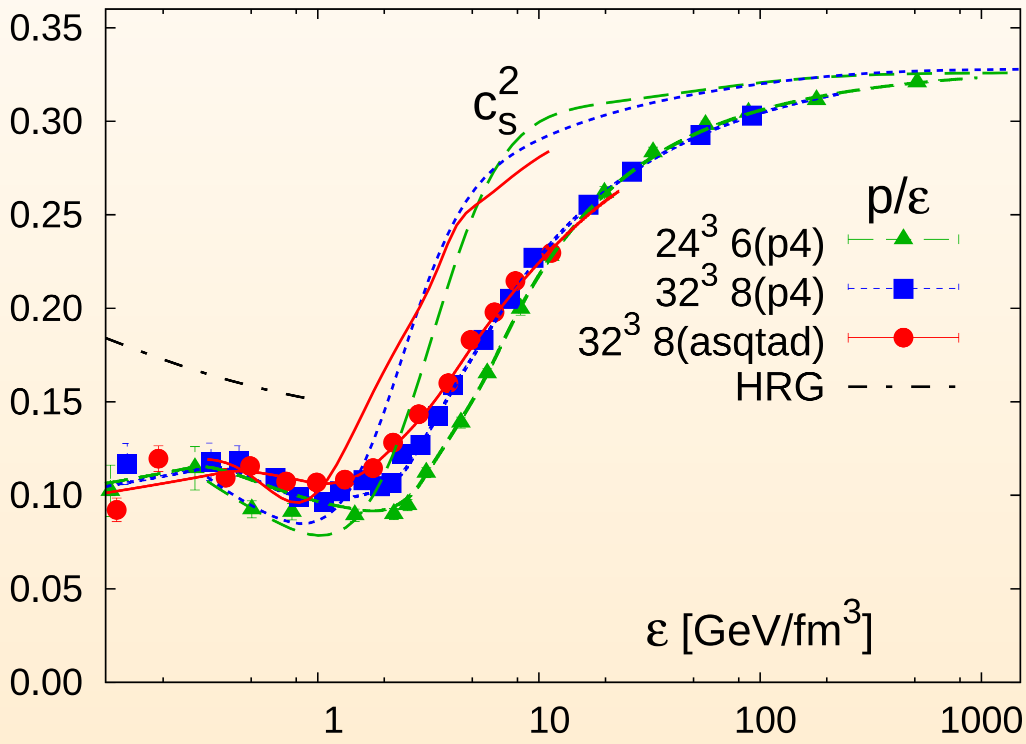
- $\left(\frac{1}{3} - w(T)\right) \propto$ "trace anomaly" (or "interaction measure")

peak around 70MeV not (yet) visible in lattice simulations

Thermodynamic quantities

most recent lattice data

[Bazavov et al, 2009]



- HotQCD 2009
- $N_f = 2 + 1$
  m_s physical
  light quarks not
- $N_\tau = 8$
  two (staggered) actions

Summary

- thermodynamic quantities of QCD are relevant for cosmology and heavy ion collisons
- these quantities can be determined numerically at $T \sim 200$ MeV, and analytically at $T \gg 200$ MeV; multi-loop sports, eff. theories convenient
- effective field theory opens up tremendous opportunities: analytic treatment of fermions, universality, superrenormalizabilty
- spatial string tension
  - ▷ *successful test of effective theory setup*
- QCD pressure
  - ▷ *not even known at "physical leading order"*
  - ▷ *problem reduced to one (hard) perturbative computation*
  - ▷ *shows friendly functional behavior with fitted unknown coefficient*
- quark mass dependence in EoS
  - ▷ *shows good convergence*
  - ▷ *charm quark contributes already at fairly low T*
  - ▷ *need reliable lattice simulations in transition region*

Summing up IR contributions beyond 4-loop

Can one do without the EQCD $\rightarrow$ MQCD matching?

- treat 3d EQCD on the lattice
- still much simpler than full QCD
- measure condensates on physical line
- 3d EQCD is superrenormalizable
  - ▷ *can perform LAT $\leftrightarrow$ $\overline{\text{MS}}$ matching exactly in perturbation theory*
  - ▷ *before cont. limit, all numbers are $f(am)$*
  - ▷ *action not (yet) completely $\mathcal{O}(a)$ improved*
  - ▷ *large discretization effects such as $a\ln(a)$ present*

Compute parameters needed for reliable continuum extrapolation

- need to compute (4-loop) diagrams in lattice regularization
  - ▷ *via Numerical Stochastic Perturbation Theory* *[Di Renzo et al 2008]*
- invested $\sim 4\cdot 10^{18}$ flops on APE (Parma), Ben (Trento)

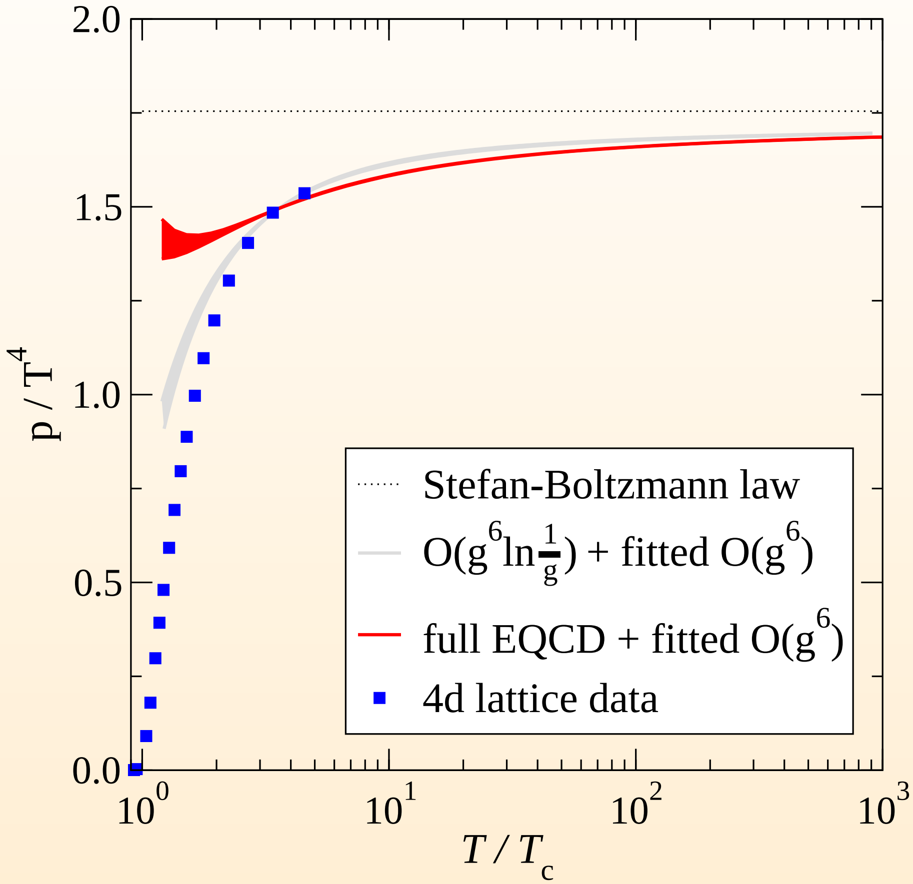
Summing up IR contributions beyond 4-loop

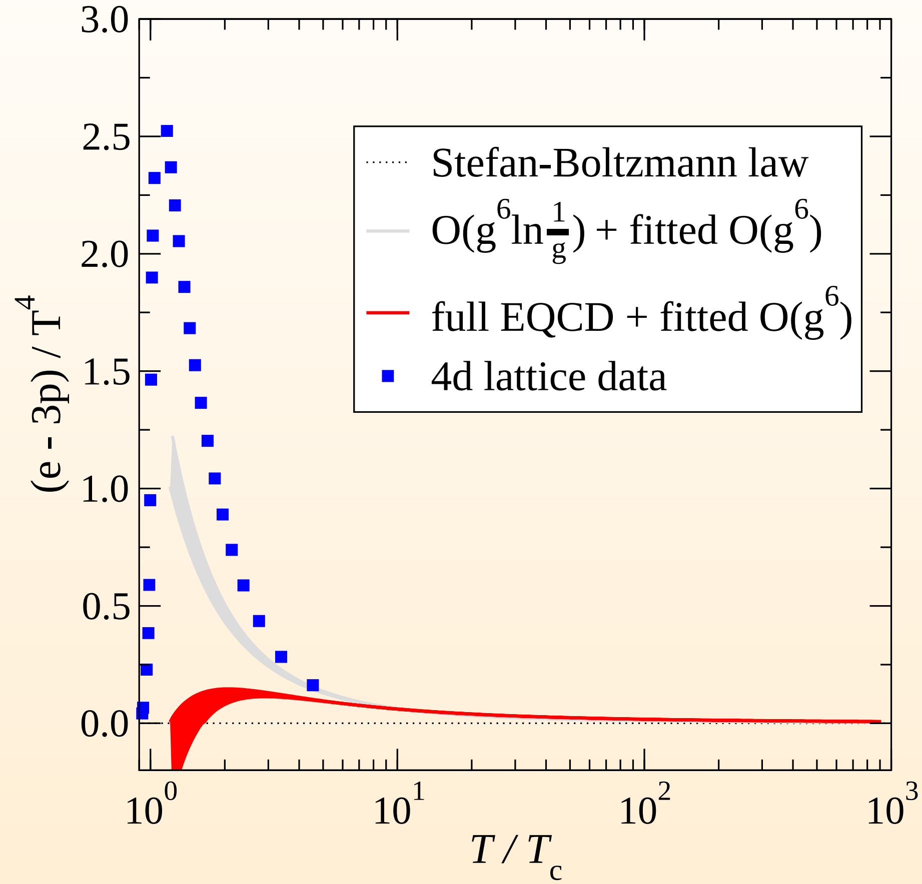
Measure condensates [HKLRS 2008]

- main idea: measure $\langle Tr A_0^2 \rangle \sim \partial_{m^2}\mathcal{F}$ and integrate
- know integration constant from perturbative analysis
  - ▷ *since dimensionless expansion parameter is* g_E^2/m_E
- sampled the function at 13 points
  used 186 lattices, with $(\beta = \frac{6}{a g_E^2}, N_s)$ from (24,48) up to (240,512)
- get a good perturbation-theory inspired fit and integrate
- invested $\sim 1.4 \cdot 10^{18}$ flops at CSC (Fin)

... do not yet fully understand the result ...

Summing up IR contributions beyond 4-loop





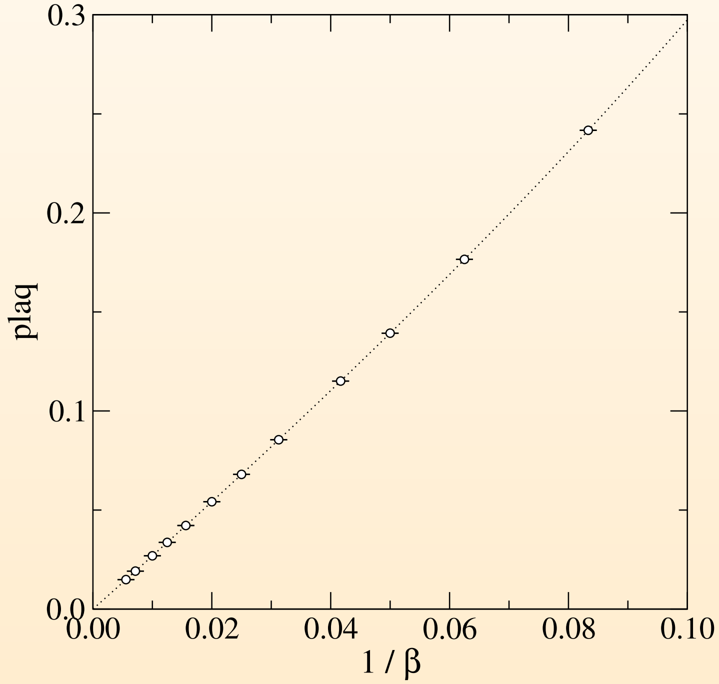
- need to include $\langle (TrA_0^2)^2 \rangle$?
- $\langle TrA_0^2 \rangle$ fluctuates too much at small m_{E} (or T) $\longrightarrow$ systematically overestimated?
  - ▷ *physical phase of EQCD is metastable* *[Kajantie et al 1997]*
  - ▷ *cure this by improved eff. theories?* *[Vuorinen/Yaffe; Pisarski; ...]*

Methods: Lattice simulation

3d, finite box $(aL)^3$. infinite-volume ($L \to \infty$) and continuum ($\frac{1}{\beta} \equiv \frac{g_{\mathsf{M}}^2 a}{2N_c} \to 0$) limits

$$\frac{1}{2g_{\mathsf{M}}^2}\left\langle Tr\,[F_{kl}^2]\right\rangle_{\overline{\mathsf{MS}}} \equiv g_{\mathsf{M}}^2 \frac{\partial}{\partial g_{\mathsf{M}}^2} p_{\mathsf{G},\overline{\mathsf{MS}}} = 3g_{\mathsf{M}}^6 \frac{d_A C_A^3}{(4\pi)^4}\left[\alpha_{\mathsf{G}}\left(\ln\frac{\bar\mu}{2C_A g_{\mathsf{M}}^2} - \frac{1}{3}\right) + B_{\mathsf{G}} + \mathcal{O}(\epsilon)\right]$$

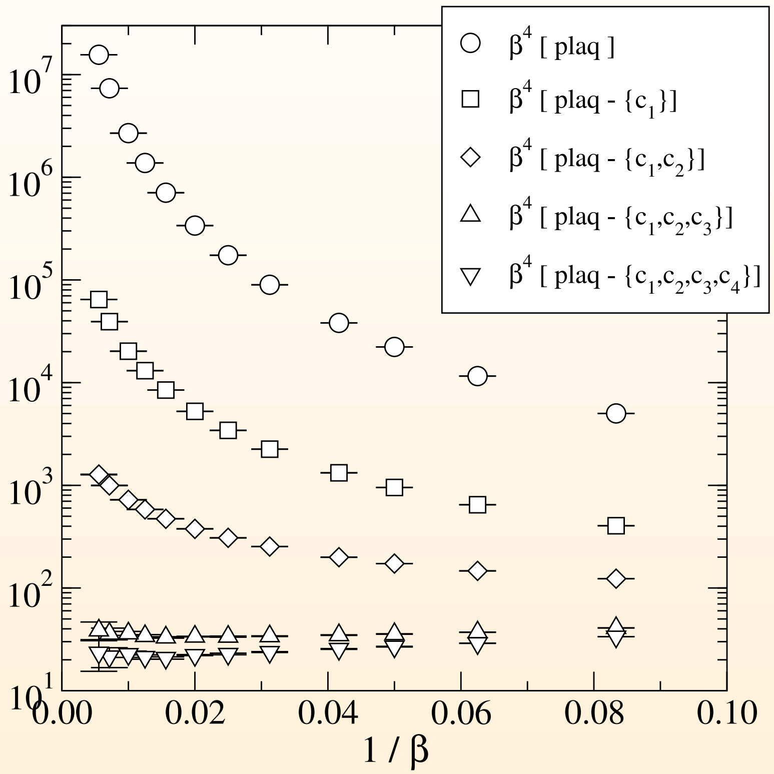
$$8\frac{d_A C_A^6}{(4\pi)^4} B_{\mathsf{G}} = \lim_{\beta\to\infty} \beta^4 \left\{\left\langle 1 - \frac{1}{C_A} Tr\,[P_{12}]\right\rangle_a - \left[\frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4}\left(\ln\beta + c_4'\right)\right]\right\}$$

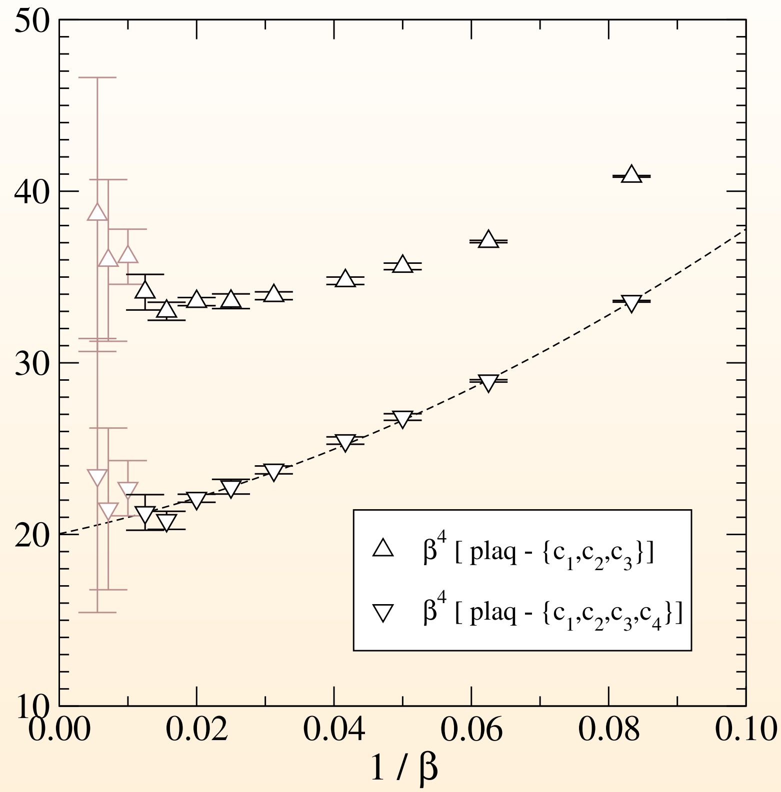


statistical errors are (much) smaller than the symbol sizes

Fit: $c_1/\beta + c_2/\beta^2 + c_3/\beta^3 + c_4 \ln\beta/\beta^4 + c_4'/\beta^4 + c_5/\beta^5 + c_6/\beta^6$

Methods: Lattice simulation



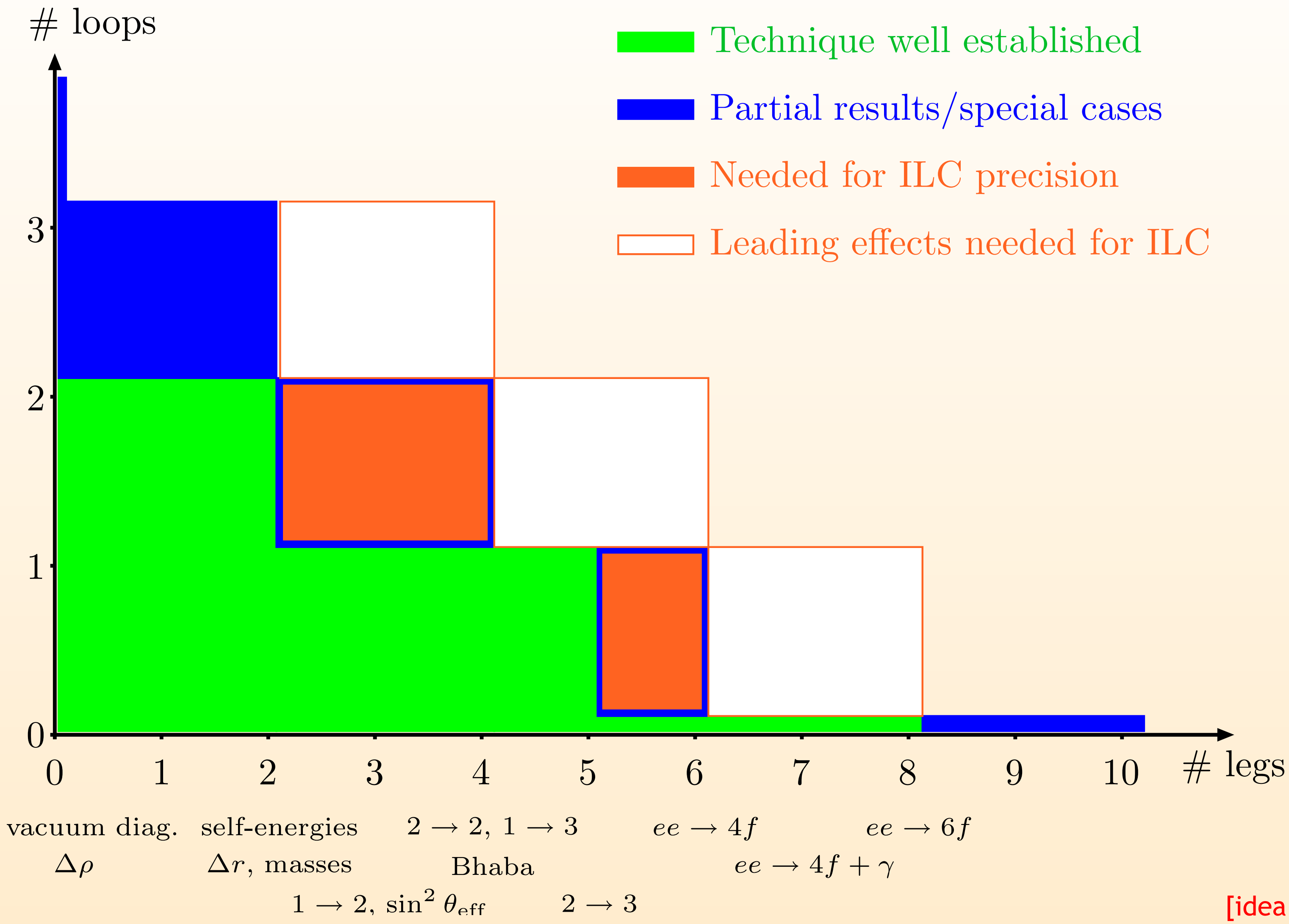


significance loss due to the UV subtractions

continuum limit of infinite-volume extrapolated data

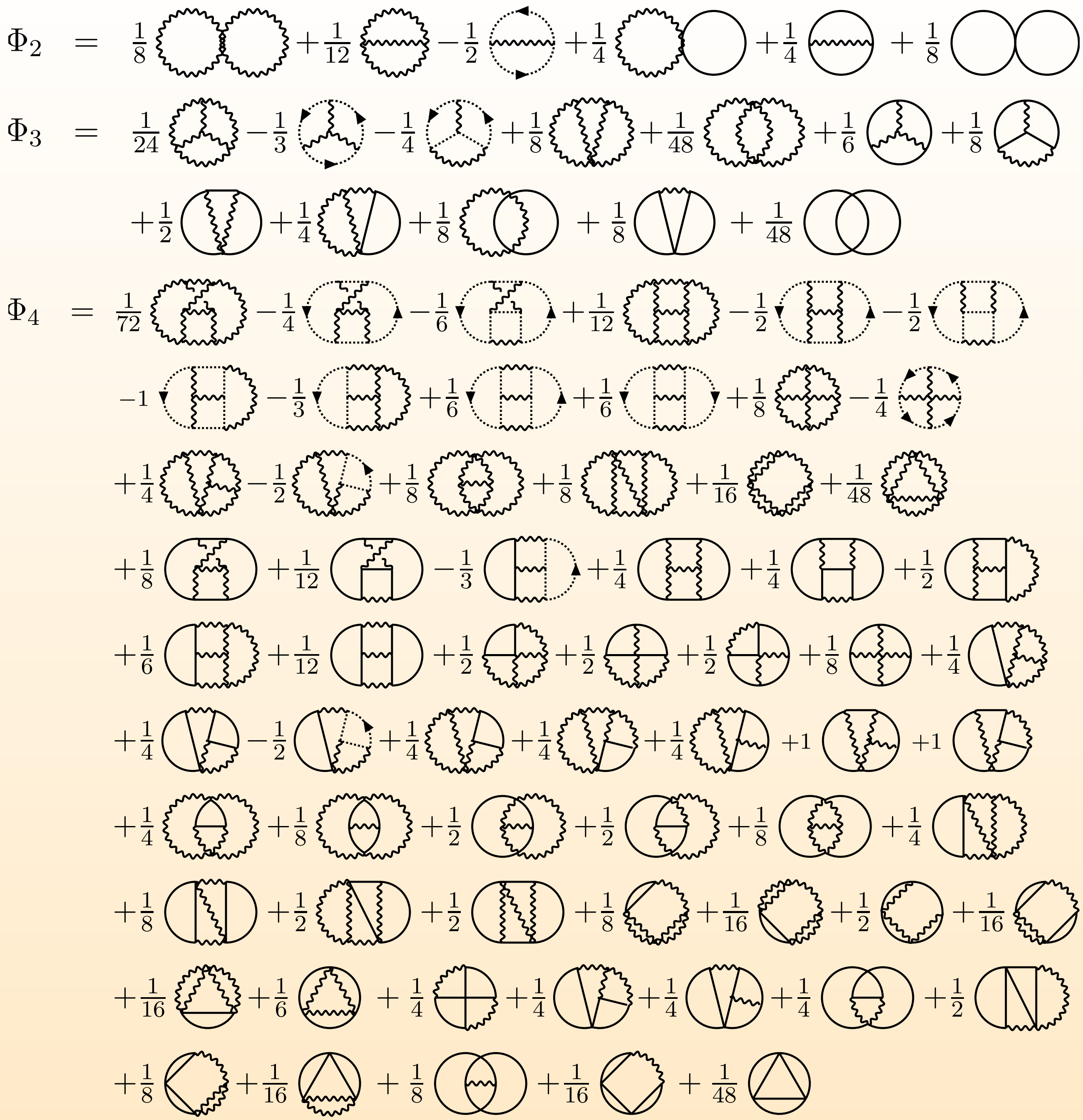
$$B_G + \left(\frac{43}{12} - \frac{157}{768}\pi^2\right) c'_4 \quad = \quad 10.7 \pm 0.4 \qquad (N_c = 3)$$

Algorithmic methods: Loop-and-Leg sports

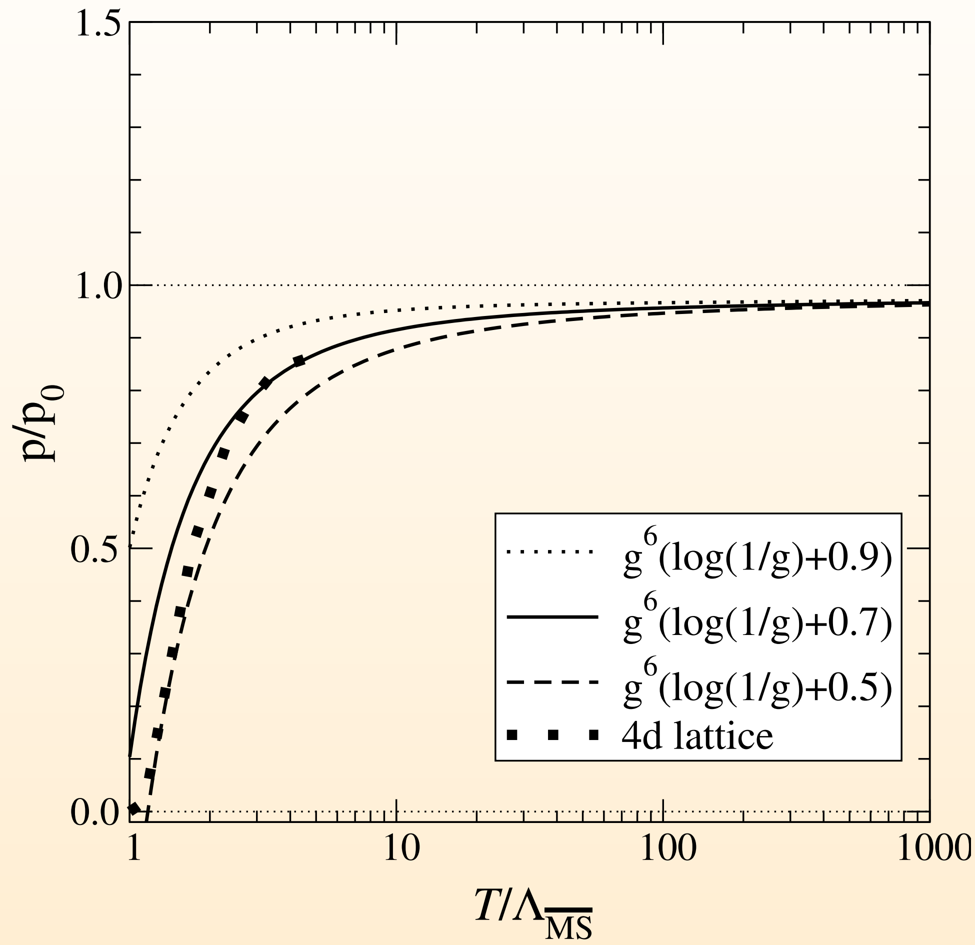


[idea: S. Dittmaier]

for pressure: 4 loops, 0 legs


$\Phi_2 =$
$\Phi_3 =$
$\Phi_4 =$

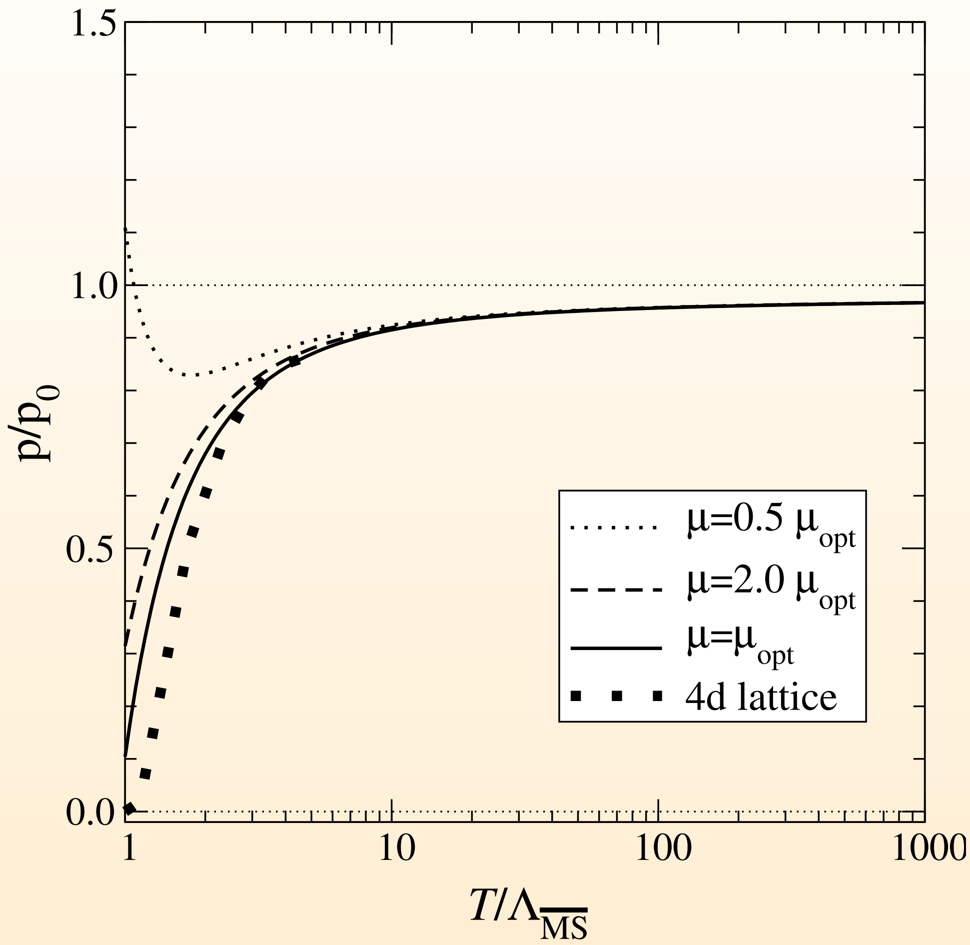
Thermal pressure $p(T)$: 4d vs 3d $(N_f = 0)$



depecence on g^6 constant

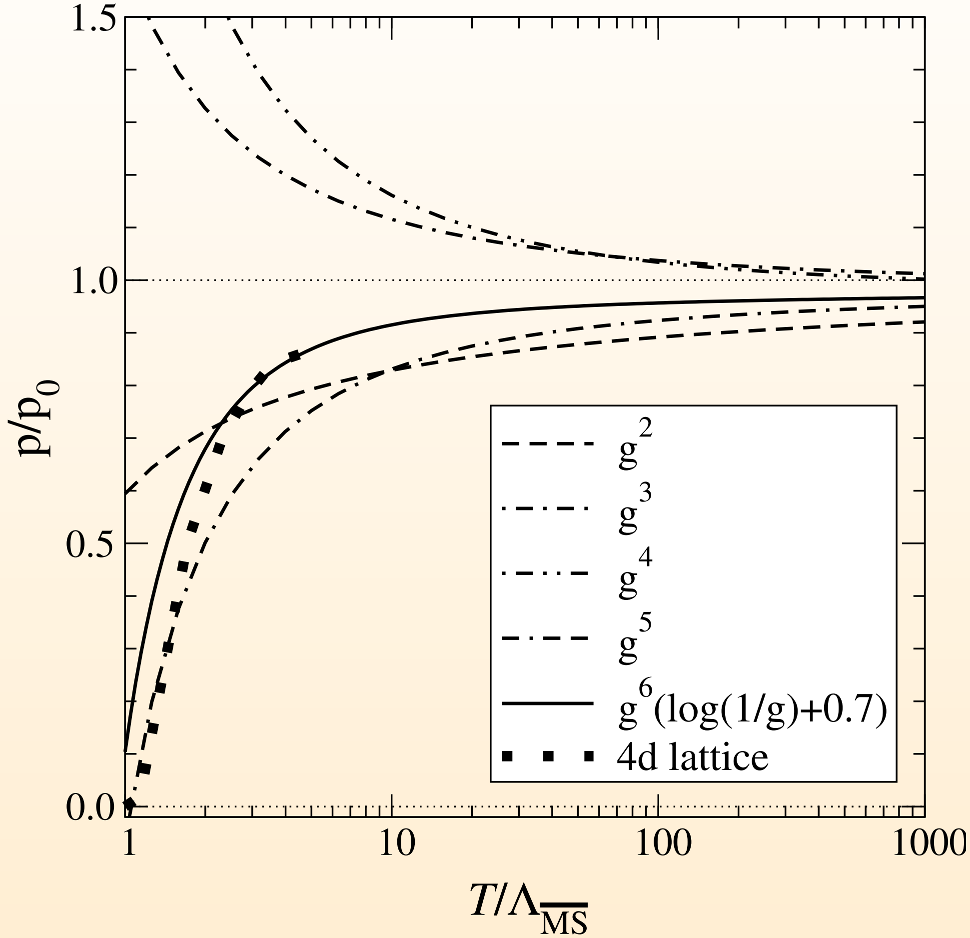
this non-perturbative contribution is unknown, but computable!

Thermal pressure $p(T)$: 4d vs 3d $(N_f = 0)$



scale dependence

Thermal pressure $p(T)$: 4d vs 3d $(N_f = 0)$



g^6 constant is a guess.

non-perturbative contrib not known, but computable!

Hadron resonance gas

from PDG, get `http://pdg.lbl.gov/2009/mcdata/mass_width_2008.csv`:

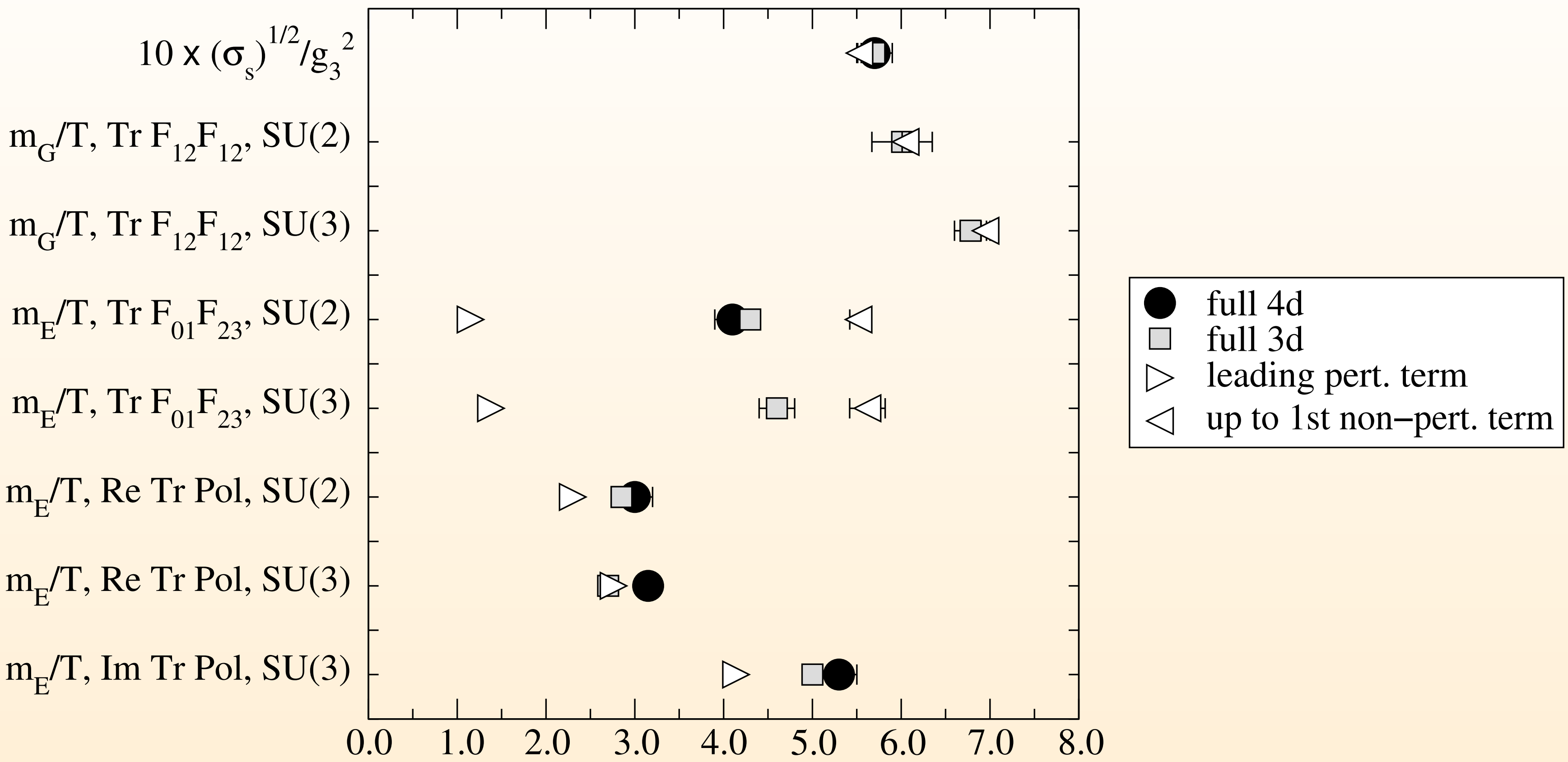
```
*MASS(MeV)     ,Err+    ,Err-    ,WIDTH(MeV)   ,I   ,G,J    ,P,C,A,Chrg,R,S,Name                ,Quarks
1.3957018E+02 ,3.5E-04,3.5E-04,2.5284E-14   ,1   ,-,0    ,-, ,B,   +, ,R,pi                  ,uD
1.349766E+02  ,6.0E-04,6.0E-04,7.8E-06      ,1   ,-,0    ,-,+, ,   0, ,R,pi                  ,(uU-dD)/sqrt(2)
5.4751E+02    ,1.8E-01,1.8E-01,1.30E-03     ,0   ,+,0    ,-,+, ,   0, ,R,eta                 ,x(uU+dD)+y(sS)
8.0E+02       ,4.0E+02,4.0E+02,8.0E+02      ,0   ,+,0    ,+,+, ,   0, ,R,f(0)(600)           ,Maybe non-qQ
7.755E+02     ,4.0E-01,4.0E-01,1.4940E+02   ,1   ,+,1    ,-, ,B,   +, ,R,rho(770)            ,uD
...
```

extract list of Mesons and Baryons, incl masses + deg.factors

take the $\sim$200 well established ones only $\Rightarrow$ $\sim$1000 resonances

$$
\begin{aligned}
\frac{p_{had}(T)}{T^4} &= \sum_{i\in Baryons} \frac{d_i}{2\pi^2}\int_0^\infty dp\, p^2 \ln\left(1+e^{-\sqrt{p^2+m_i^2/T^2}}\right) \\
&- \sum_{i\in Mesons} \frac{d_i}{2\pi^2}\int_0^\infty dp\, p^2 \ln\left(1-e^{-\sqrt{p^2+m_i^2/T^2}}\right)
\end{aligned}
$$

String tension and inverse correlation lengths



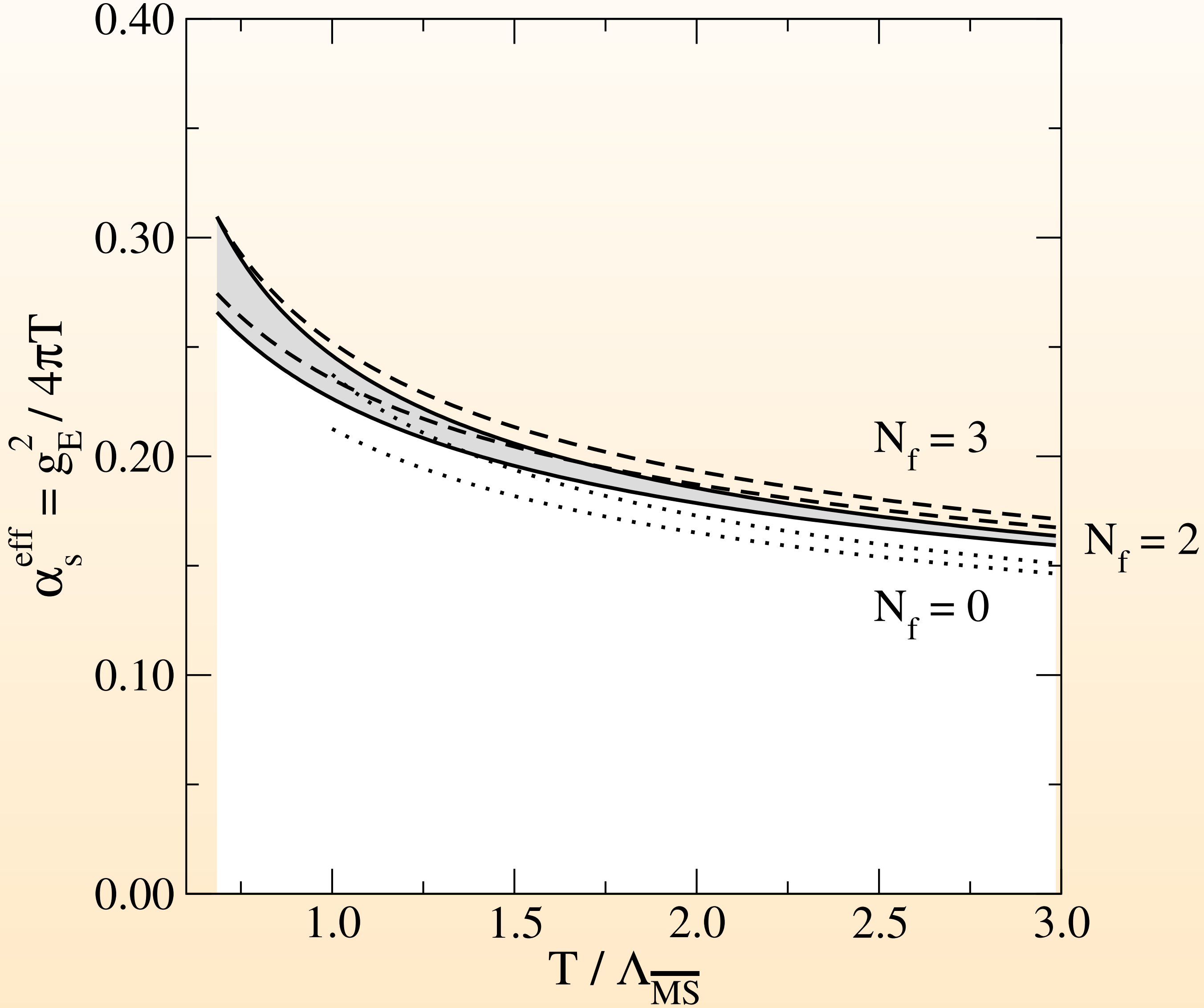
[lattice date from: Hart et al 00; Boyd et al 96; Kaczmarek at al 00; Teper 98; Laine et al 01; Datta et al 02]

''full 4d'': 4d lattice Monte Carlo

''full 3d'': 3d lattice, couplings(g^2,T)

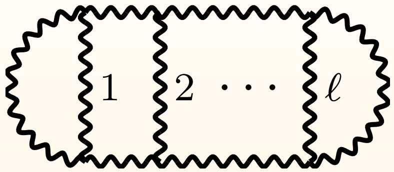
effective coupling constant g_E^2 numerically

[Laine, Schröder, hep-ph/0503061]



The IR problem

[Linde 1979; Gross/Pisarski/Yaffe 1981]



(ℓ+1) loops, 2ℓ vert, 3ℓ propags

$$\sim \left(T \sum_n \int d^3p \right)^{\ell+1} \frac{(gp)^{2\ell}}{[(2\pi n T)^2 + p^2 + \Pi(2\pi n T, p)]^{3\ell}}$$

IR power counting: n=0, define $\Pi(0, p \to 0) \equiv m^2$

$$\sim T^{\ell+1} g^{2\ell} m^{3(\ell+1)+2\ell-6\ell} = g^6 T^4 \left(\frac{g^2 T}{m} \right)^{\ell-3}$$

- $\Pi_L(0,p) \sim (gT)^2 \Leftarrow$ OK: get series in g
- $\Pi_T(0,p) \sim (g^2T)^2 \Leftarrow$ all orders important!