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Thermal lattice QCD as a spin model

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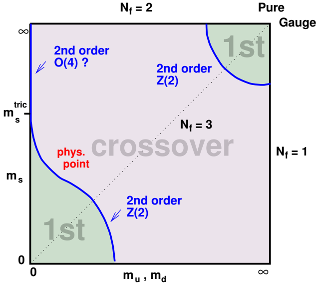
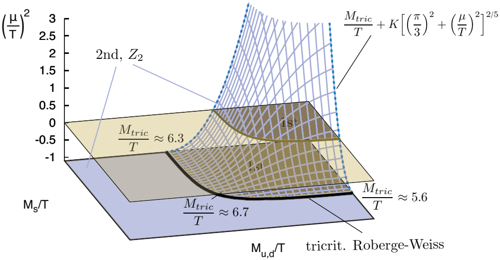
Phase diagram in the quark mass plane





Outline

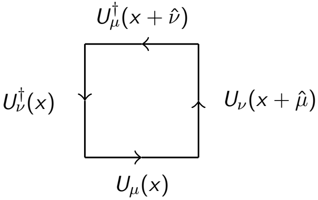
- Define effective theory by integrating out spatial degrees of freedom
- Effective theory can be simulated very fast by different algorithms
- No solution to the sign problem, but a huge reduction of its severity
- Disadvantage: Expansion starts from the unphysical strong coupling and infinite quark mass region

Starting point: QCD with Wilson's Action

- Partition function

$$Z = \int [dU_0][dU_k] \exp\left[\frac{\beta}{3}\sum_p \mathrm{Re}\,\mathrm{Tr}\, U_p\right] \qquad \beta = \frac{6}{g^2}$$

- Plaquettes consist of 4 links: $U_\mu(x) = \exp\left[iagA_\mu(x)\right]$



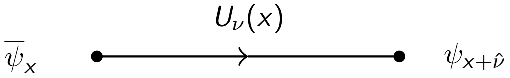
- Expansion in $\beta \triangleq$ Strong coupling expansion

Starting point: QCD with Wilson's Action

- Quark part after Grassmann integration (per flavor and omitting spin and color indices):

$$e^{S_q} = \det\left[\delta_{xy} - \kappa \sum_{\pm\nu}(1+\gamma_\nu)U_\nu(x)\,\delta_{x,y-\hat{\nu}}\right] \qquad \kappa = \frac{1}{8+2am}$$

- Expansion in $\kappa \hat{=}$ Hopping parameter expansion



$\rightarrow$ Everything is expressed in link variables $\in SU(3)$

Series expansion

- Physical observables are functions of (β, κ). Here we expand around $(\beta = 0, \kappa = 0)$, i.e. infinite coupling and quark mass
- Graphical expansion: Build graphs of plaquette and link variables
- Plaquettes contribute a factor of β
- Quark hops contribute a factor of κ
- Integrate over all link variables

Thermal lattice QCD

- Finite temperature:
  - Compactified time direction
  - Periodic boundary conditions for bosons
  - Antiperiodic boundary conditions for fermions
  - Order parameter: Polyakov loop

$$\mathrm{Tr}\,W(\vec{x}_i) = \mathrm{Tr}\prod_{\tau=1}^{N_\tau} U_0(\tau,\vec{x}_i) = L(\vec{x}_i) = L_i$$

- Finite chemical potential:
  - Modified temporal hopping parameter

$$\begin{aligned} \kappa &\rightarrow \kappa e^{a\mu} && \text{positive direction} \\ \kappa &\rightarrow \kappa e^{-a\mu} && \text{negative direction} \end{aligned}$$

The effective action

- Integrating out spatial link variables: Defines effective action

$$Z = \int [dU_0][dU_k] e^{S_g + S_q} \equiv \int [dU_0] e^{S_{\mathrm{eff}}}$$

- Advantages:
  - S_{eff} depends only on Polyakov loops; (3+1)d theory can be reduced to effective 3d theory
  - Complex numbers instead of group elements
- Disadvantages:
  - Infinite number of effective interaction terms and couplings
  - Couplings only known to some order from strong coupling and hopping parameter expansion

Leading order effective theory

Quark part

- Neglect spatial plaquettes and spatial quark hops
  $\rightarrow$ The spatial integrations can be calculated exactly
- The quark part has no spatial link dependence at all

$$e^{S_q} = \prod_i \det\left[1 + h_1 W_i\right]^2 \left[1 + \bar{h}_1 W_i^\dagger\right]^2$$

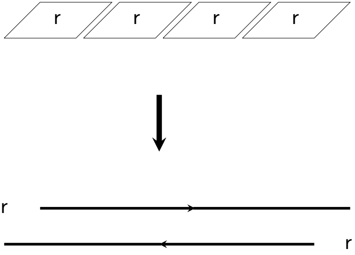
- Effective coupling: $h_1(\kappa, \mu) = (2\kappa e^{\mu})^{N_\tau} = \bar{h}_1(\kappa, -\mu)$
- Further simplification (in case of $SU(3)$):

$$\det\left[1 + h_1 W\right] = 1 + h_1 L + h_1^2 L^* + h_1^3$$

Leading order effective theory

Gauge part

- Chain of N_τ plaquettes in the same representation



- Nearest neighbor Polyakov Loop interaction

Leading order effective theory: Remarks

$$Z = \int [dW] \prod_i \det \left[1 + h_1 \, W_i\right]^2 \left[1 + \bar{h}_1 \, W_i^\dagger\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re}\, L_i L_j^*\right]$$

- Simulation yields critical h_1^c and λ_1^c $\quad\rightarrow\quad$ β^c and κ^c
- Can be done for each N_τ, need h_1^c and λ_1^c only once
- The well-known $SU(3)$ spin model is the first order approximation to this
- Spatial plaquettes and quark hops contribute higher orders to the leading couplings and introduce new interaction terms

Solving the effective theory: Pure gauge theory

- Solve effective partition function for λ_1^c

$$Z = \int [dL] e^{V(L)} \prod_{<ij>} \left[1 + \lambda_1 \left(L_i L_j^* + L_i^* L_j\right)\right]$$

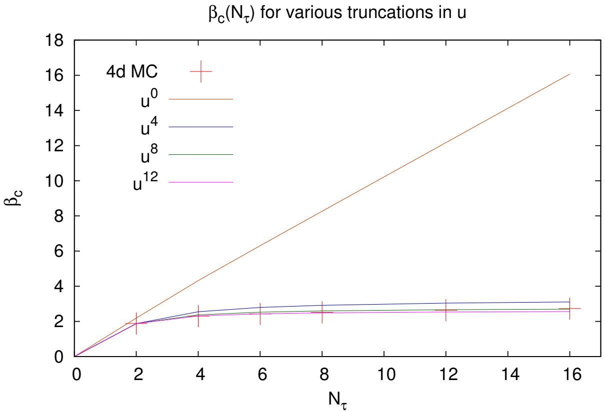
- We get $\lambda_1^c = 0.18805(2)$
- Use this value to convert to $\beta_c(N_\tau)$

$$\lambda_1(\beta, N_\tau) = \left(\frac{\beta}{2N_c^2}\right)^{N_\tau} \exp\left[N_\tau \left(P(\beta)\right)\right]$$

- Crucial point: Knowledge of λ_1 as a function of β and N_τ
- Polynomial $P(\beta)$ known up to $\mathcal{O}(\beta^{12})$

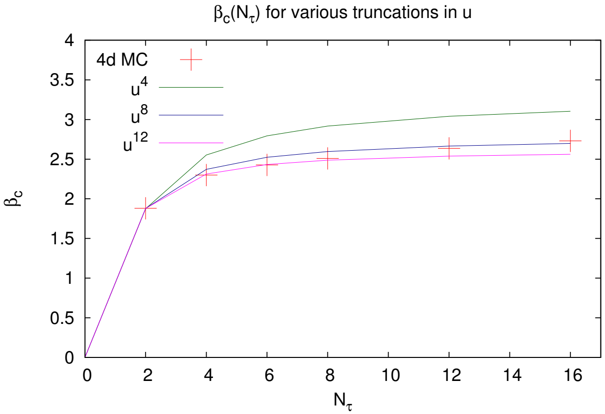
Deconfinement transition

Evolution of $\beta^c(N_\tau)$ for different truncations and $SU(2)$



Deconfinement transition

Evolution of $\beta^c(N_\tau)$ for different truncations and $SU(2)$



Comparison with full simulations

$SU(2)$

N_τ	3d Eff. Th.	4d YM
2	2.1929(13)	2.1768(30)
4	2.3102(08)	2.2991(02)
6	2.4297(05)	2.4265(30)
8	2.4836(03)	2.5104(02)
12	2.5341(02)	2.6355(10)
16	2.5582(02)	2.7310(20)

4d Monte Carlo results taken from [Fingberg et al. (1992), Bogolubsky et al. (2004) and Velytsky (2007)]

Comparison with full simulations

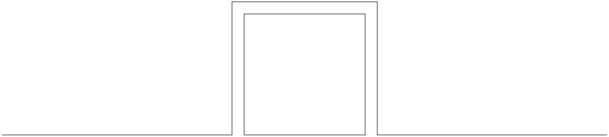
$SU(3)$

N_τ	3d Eff. Th	4d YM
2	5.1839(2)	5.10(5)
4	6.09871(7)	5.6925(2)
6	6.32625(4)	5.8941(5)
8	6.43045(3)	6.001(25)
12	6.52875(2)	6.268(12)
16	6.57588(1)	6.45(5)

4d Monte Carlo results taken from [Fingberg et al. (1992)]

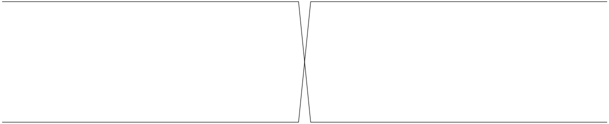
Fermionic corrections: Examples

- Corrections to the leading coupling: $\mathcal{O}(\kappa^{N_\tau+2}u)$



$\longrightarrow$ Deconfinement transition

- New interaction terms: $\mathcal{O}(\kappa^{2N_\tau+2})$



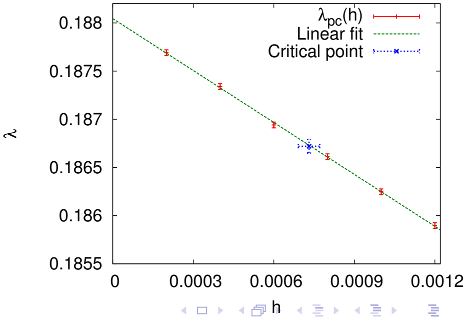
$\longrightarrow$ Cold, dense matter

Deconfinement transition: $\mu = 0$

- Use the leading order effective theory and $\bar{h}_1 = h_1$

$$Z = \int [dW] \prod_i \det \left[1 + h_1 W_i\right]^2 \left[1 + h_1 W_i^\dagger\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re}\, L_i L_j^*\right]$$

- With increasing h_1, the transition turns from first order to crossover at a second order endpoint
- Corrections of higher interaction terms negligible



Deconfinement transition: $\mu = 0$

Comparison with other approaches

- Comparison with 4d simulations
- Conversion to quark masses via $\kappa = \frac{1}{2}e^{-aM_q}$

N_f	M_c/T	$\kappa_c(N_\tau = 4)$	$\kappa_c(4)$, Ref. [1]	$\kappa_c(4)$, Ref. [2]
1	7.22(5)	0.0822(11)	0.0783(4)	~ 0.08
2	7.91(5)	0.0691(9)	0.0658(3)	–
3	8.32(5)	0.0625(9)	0.0595(3)	–

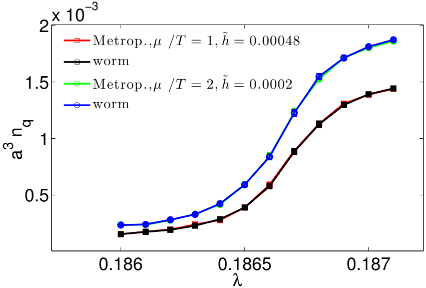
Table : Location of the critical point for $\mu = 0$ and $N_\tau = 4$. Existing literature: [1] Saito et al. (2011), [2] Alexandrou et al. (1998)

Deconfinement transition: $\mu \neq 0$

$$Z = \int [dW] \prod_i \left[1 + h_1 L_i\right]^2 \left[1 + \bar{h}_1 L_i^*\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re}\, L_i L_j^*\right]$$

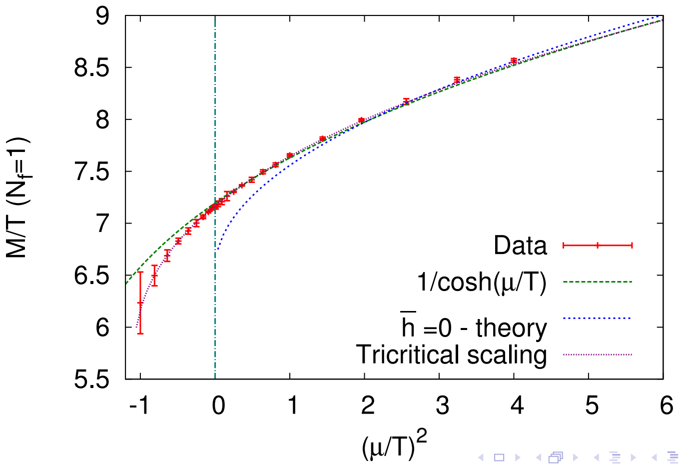
- Metropolis algorithm: Mild sign problem
- Worm algorithm: No sign problem

Comparison of the two algorithms: Quark number density for $\frac{\mu}{T} = 1; 2$



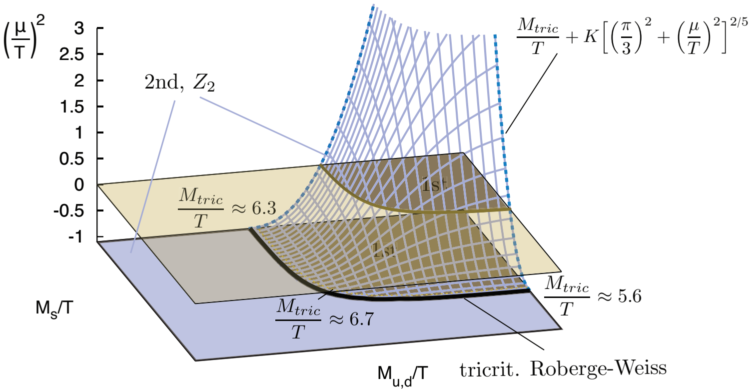
Deconfinement transistion: $\mu \neq 0$

Critical $\frac{M}{T}$ for all chemical potentials



Deconfinement transition: $\mu \neq 0$

3d columbia plot



Cold and dense matter

- $T \simeq 0$ is at finite a realized by large N_τ

$$\lambda_1(\beta = 5.7, N_\tau = 115) \sim 10^{-27}$$

- $\Rightarrow$ Effective gauge part can be neglected
- Not to be confused with strong coupling limit: λ_1 is small, not β
- Effective theory then reads:

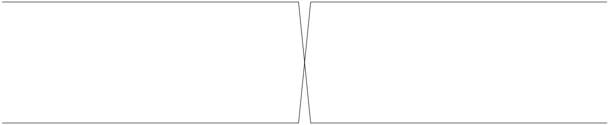
$$Z = \int [dW] \prod_i \det \left[1 + h_1 W_i\right]^2 \left[1 + \bar{h}_1 W_i^\dagger\right]^2$$

- No interactions, single-site problem: Can be solved analytically

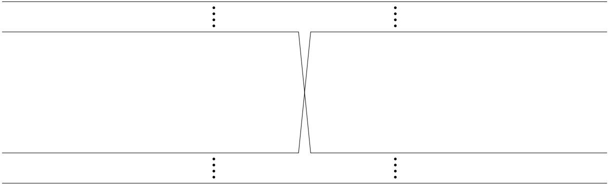
Cold and dense matter

Interactions

- Leading interaction term:



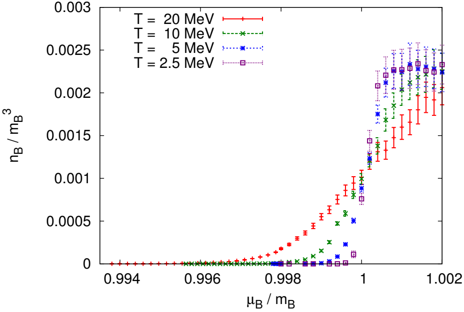
- This graph alone spoils baryon saturation at large densities: Need to resum all winding numbers



Cold and dense matter

Results:

- Transition to nuclear matter:



- Not yet clear, if this happens at $T = 0$ or $T > 0$ (as in nature)
- Binding energy exponentially suppressed with pion mass ($\sim$ 20GeV in our truncation)

Recent developments:

- Measure effective couplings nonperturbatively
- Ansatz for full effective action after spatial link integration:

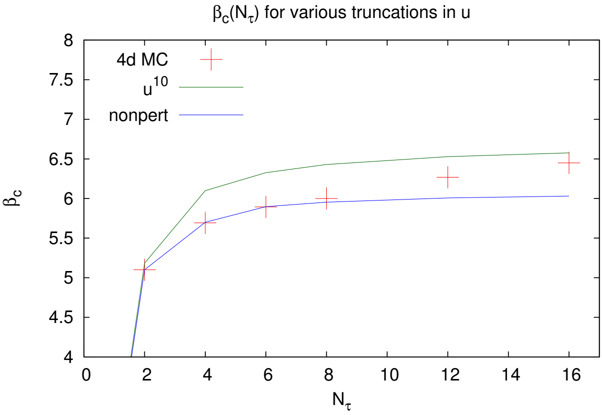
$$
\begin{aligned}
e^{S_{\mathrm{eff}}} &= \sum_{\{r_i\}} c\Big(\{r_i\}\Big) \prod_i \chi_{r_i}(W_i) \\
&= c(\{0\}) \left[1 + \sum_{\{r_i\}} \lambda\Big(\{r_i\}\Big) \prod_i \chi_{r_i}(W_i)\right]
\end{aligned}
$$

- Using character orthogonality λ_i are obtainable, e.g

$$
\begin{aligned}
c(\{0\}) &= \int [dW] e^{S_{\mathrm{eff}}} = \int [dW][dU_i] e^{S} = Z \\
\lambda_1 &= \frac{1}{Z} \int [dW] e^{S_{\mathrm{eff}}} \chi_f(W_i) \chi_{\bar{f}}(W_j) = \langle L_i L_j^* \rangle
\end{aligned}
$$

Recent developments:

- Neglect higher order interaction terms
- Goal: Reduce remaining uncertainty for $SU(3)$



Conclusions

- Constructed effective theory with much milder sign problem
- In good agreement with full simulations, where comparison is possible (heavy quarks)
- Gauge part seems to be under control especially with nonperturbatively extracted effective couplings
- Fermionic sector more complicated (as always) due to the 4d simulations involved
- Main advantage: Dependence of the couplings on chemical potential is trivial, determination at $\mu = 0$ suffices

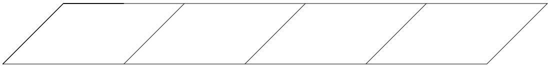
Thank you for your attention

Backup slides

Gauge corrections

Corrections to the leading coupling λ_1

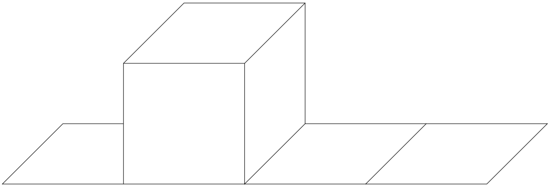
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim u^{N_\tau}$



Gauge corrections

Corrections to the leading coupling λ_1

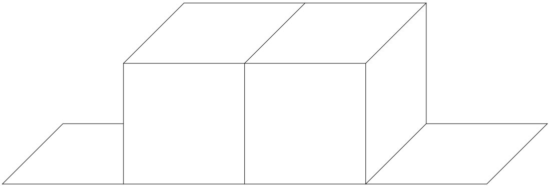
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau+4}$



Gauge corrections

Corrections to the leading coupling λ_1

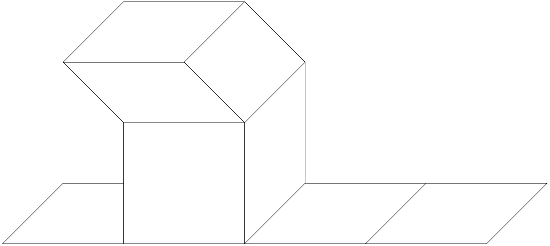
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau+6}$



Gauge corrections

Corrections to the leading coupling λ_1

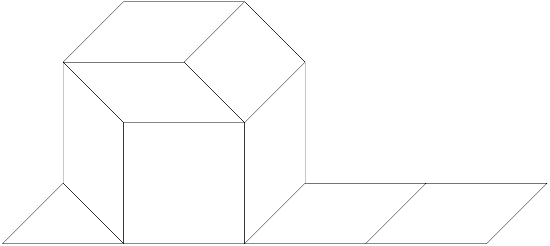
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau+8}$



Gauge corrections

Corrections to the leading coupling λ_1

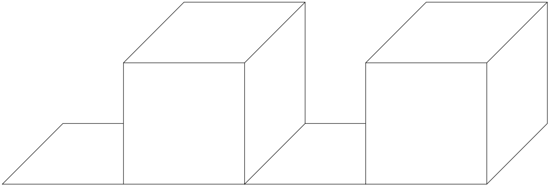
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau+10}$



Gauge corrections

Corrections to the leading coupling λ_1

- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim \frac{1}{2} N_\tau^2 u^{N_\tau+8}$



Gauge corrections

Corrections to the leading coupling λ_1

- Repetitions of these decorations exponentiate

$$\lambda_1(u, N_\tau) = u^{N_\tau} \exp\left[N_\tau\left(P_{N_\tau}(u)\right)\right]$$

- E.g. $SU(2)$ up to $\mathcal{O}(u^{12})$ and $N_\tau \geq 6$:

$$P(u) = 4u^4 - 4u^6 + \frac{140}{3}u^8 - \frac{37664}{405}u^{10} + \frac{863524}{1215}u^{12}$$

- $P(u, N_\tau < 6)$ also known to this order