New developments for fugacity expansion in lattice QCD

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Nature: Big bang, neutron stars.

Experiment: Heavy-ion collisions @ RHIC, LHC.

Temperature driven phase transition of QCD is well understood (ab initio information from lattice methods).

At finite baryon chemical potential: Lattice QCD has sign problem.

The Boltzmann factor becomes negative/complex and we cannot use usual Monte Carlo techniques.
Different approaches: Reweighting, Taylor expansion, dual variables.

Here: **Fugacity expansion**.

Fugacity expansion has different properties than Taylor expansion (Laurent- v.s. Taylor-series and finite sum for finite $V$).

Interesting physics in the expansion coefficients (canonical partition sums).
The grand canonical determinant with real chemical potential $\mu$ can be written as the fugacity series (exact for $q_{\text{cut}} = 6 V$)

$$\det[D(\mu)] = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu \beta q} D(q),$$

$D(q)$: Canonical determinants with net quark number $q$,

$$D(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \ e^{-i q \phi} \det[D(\mu \beta = i \phi)] .$$

Fourier integral is done numerically $\Rightarrow q_{\text{cut}} \ll 6 V$. 

**Fugacity expansion**
Behavior of the canonical determinants

\[
\det[D(\mu)] = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu \beta q} D^{(q)}
\]

\(8^3 \times 4\)

\(T = 189\ \text{MeV}\)
Observables related to quark number

Moments of $D^{(q)}$:

$$M^n = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu \beta q} q^n \frac{D^{(q)}}{\det[D(\mu = 0)]}.$$ 

Quark number density:

$$\frac{n_q}{T^3} = 2 \frac{\beta^3}{V} \frac{\langle M^0 M^1 \rangle_0}{\langle (M^0)^2 \rangle_0},$$

Quark number susceptibility:

$$\frac{\chi_q}{T^2} = 2 \frac{\beta^3}{V} \left[ \frac{\langle (M^1)^2 \rangle_0 + \langle M^0 M^2 \rangle_0}{\langle (M^0)^2 \rangle_0} - 2 \left( \frac{\langle M^0 M^1 \rangle_0}{\langle (M^0)^2 \rangle_0} \right)^2 \right],$$

$\langle \ldots \rangle_0$: expectation value evaluated on configurations with $\mu = 0$. 
Lattice parameters

- Two flavor degenerate Wilson fermions & Wilson gauge action
- Lattices $N_s^3 \times N_t$: $8^3 \times 4$, $10^3 \times 4$, $12^3 \times 4$, $12^3 \times 6$ ($\beta = N_t = 1/T$)
- Inverse coupling: $5.00 \leq \frac{6}{g^2} \leq 5.65$
- Lattice spacing: $0.343 \text{ fm} \geq a \geq 0.260 \text{ fm}$ (+ finer currently running)
- Temperature: $96 \text{ MeV} \leq T \leq 211 \text{ MeV}$
- $\kappa = 0.158, 0.160, 0.162$
- Pion mass: $m_\pi = 950 \text{ MeV}$ (+ lighter currently running)
- Configurations generated using the MILC code (http://www.physics.utah.edu/ detar/milc/)
Technical details

Important to have $D^{(q)}$ at high precision.

\[ \Downarrow \]

Calculation of $\det[D(\mu \beta = i \phi)]$ for many values of $\phi$.

\[ \Downarrow \]

Expensive!

Dimensional reduction$^1$:

Use a domain decomposition of the Dirac operator $D(x, y)$ to obtain

$$\det[D(\mu)] = A_0 \, W,$$

with a $\mu$-independent factor $A_0$ and

$$W = \det[K_0 - e^{\mu \beta} K - e^{-\mu \beta} K^\dagger].$$

$K_0, K$ live on a single time slice, but are dense matrices.

Quark number density (l.h.s) and susceptibility (r.h.s) as a function of $\frac{6}{g^2}$ (proportional to temperature).
Quark number susceptibility @ $\mu = 0$

\[ \frac{\chi_q}{T^2} \]

\[ 8^3 \times 4 \quad 10^3 \times 4 \quad 12^3 \times 4 \]

Allton et.al.†

Extrapolation to finite $\mu$.

Quark number density (l.h.s) and susceptibility (r.h.s) as a function of $\frac{6}{g^2}$ and $\mu \beta$. 
Canonical determinants $D(q)$ at $\mu = 0$

Hadron resonance gas $\rightarrow$ Skellam distribution$^\dagger$

$$\frac{\langle |D(q)|^2 \rangle}{\langle |D(0)|^2 \rangle} \approx \frac{l_q(Z)}{l_0(Z)} .$$

$l_q \cdots$ modified Bessel function
$Z \cdots 2\sqrt{q\bar{q}}$

Fit in $Z$ to our $12^3 \times 6$, $\frac{6}{g^2} = 5.00$ data.


$D(q)$ from FRG see: K. Morita, et.al., 1301.2873 [hep-ph].
$D(q)$ from ChPT$^\dagger$

$$\frac{\langle |D(q)|^2 \rangle}{\langle |D(0)|^2 \rangle} = \frac{l_q \left( \frac{V}{\beta^3} \frac{(m_\pi \beta)^{3/2}}{\sqrt{2\pi}^{3/2}} e^{-m_\pi \beta} \right)}{l_0 \left( \frac{V}{\beta^3} \frac{(m_\pi \beta)^{3/2}}{\sqrt{2\pi}^{3/2}} e^{-m_\pi \beta} \right)}. $$

$l_q \cdots$ modified Bessel function

Fit in $m_\pi \beta$ to our $12^3 \times 6$, $\frac{6}{g^2} = 5.00$ data.

$^\dagger$ K. Splittorff, J. Verbaarshot, 1-loop ChPT, private communication.

(Valid for $m_\pi L \gg 1$, $m_\pi / m_N \ll 1$)
Summary:

- Fugacity expansion: Laurent series in $e^{\mu \beta}$.
- Finite lattice $\Rightarrow$ finite series.
- Used to continue from $\mu = 0$ to small chemical potential.
- Calculation numerically challenging $\Rightarrow$ dimensional reduction.
- Observables related to $n_q \Rightarrow$ moments of $D^{(q)}$.

Outlook:

- Larger lattices and calculation at smaller pion masses.
- Calculation for other fermions (Clover improved Wilson, staggered).
- Comparison with other approaches in toy models.
Backup slides
Volume dependence of the $D(q) - 1$

$$\left\langle \left| \frac{D(q)}{D(0)} \right| \right\rangle$$

$\beta = 5.30$
$T = 189$ MeV

$8^3 \times 4$, $10^3 \times 4$, $12^3 \times 4$
Volume dependence of the $D^{(q)}$ - 2
Quark number density (l.h.s) and susceptibility (r.h.s) as a function of $\mu$. 
Numerical values of the quark number susceptibility $\chi_q$ at $\mu = 0$

<table>
<thead>
<tr>
<th>$T$ [MeV]</th>
<th>$\beta$</th>
<th>$\chi_q$ (this work)</th>
<th>$\chi_q^2$</th>
<th>$\chi_q^3$</th>
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<td>5.00</td>
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<td>0.14(9)</td>
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<tr>
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<td>5.85(4)</td>
<td>1.5240(30)</td>
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</tbody>
</table>

More $8^3 \times 4$ results - 1

Average grand canonical determinant $|\langle \det[D(\mu N_t = i\varphi)] \rangle|$ MILC $8^3 \times 4$, $\kappa = 0.158$

$\beta = 5.35$
More $8^3 \times 4$ results - 2

Canonical determinant $\langle D(q)/D(0) \rangle$, $8^3 \times 4$, $\kappa = 0.158$
More $8^3 \times 4$ results - 3

$|\langle \det[D(\mu N_t)] \rangle| \text{ MILC } 8^3 \times 4, \kappa = 0.158$

Fugacity expansion, $\beta = 5.35$
More $8^3 \times 4$ results - 4

$\text{arg}(\langle D(\mu N_t) \rangle)$ MILC $8^3 \times 4$, $\kappa = 0.158$

Fugacity expansion, $\beta = 5.35$