New developments for fugacity expansion in lattice QCD

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(Lattice) QCD at finite chemical potential - 1

- Nature: Big bang, neutron stars.
- Experiment: Heavy-ion collisions @ RHIC, LHC.
- Temperature driven phase transition of QCD is well understood (ab initio information from lattice methods).
- At finite baryon chemical potential: Lattice QCD has sign problem.
- The Boltzmann factor becomes negative/complex and we cannot use usual Monte Carlo techniques.

(Lattice) QCD at finite chemical potential - 2

- Different approaches: Reweighting, Taylor expansion, dual variables.
- Here: **Fugacity expansion**.
- Fugacity expansion has different properties than Taylor expansion (Laurent- v.s. Taylor-series and finite sum for finite V).
- Interesting physics in the expansion coefficients (canonical partition sums).

Fugacity expansion

The **grand canonical determinant** with **real chemical potential** μ can be written as the **fugacity series** (exact for $q_{\text{cut}} = 6\,V$)

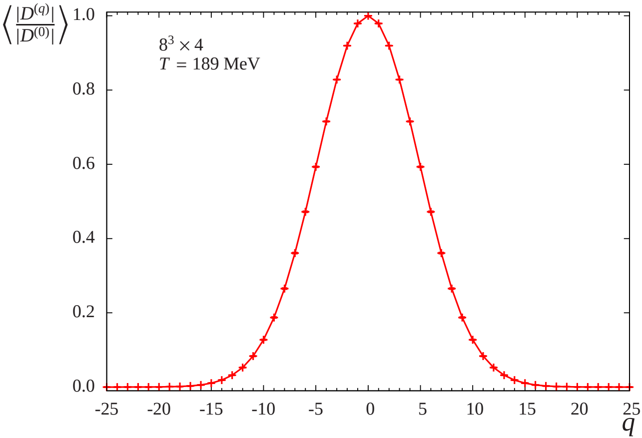
$$\det[D(\mu)] = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu\beta q}\, D^{(q)} \,,$$

$D^{(q)}$: **Canonical determinants** with net quark number q,

$$D^{(q)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-iq\phi} \det[D(\mu\beta = i\phi)] \,.$$

Fourier integral is done numerically $\Rightarrow q_{\text{cut}} \ll 6\,V$.

Behavior of the canonical determinants



$$\det[D(\mu)] = \sum_{q=-q_{\mathrm{cut}}}^{q_{\mathrm{cut}}} e^{\mu\beta q} D^{(q)}$$

Observables related to quark number

Moments of $D^{(q)}$:

$$M^n = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu\beta q}\, q^n \, \frac{D^{(q)}}{det[D(\mu=0)]} \, .$$

Quark number density:

$$\frac{n_q}{T^3} = 2\, \frac{\beta^3}{V} \, \frac{\langle M^0 M^1 \rangle_0}{\langle (M^0)^2 \rangle_0} \, ,$$

Quark number susceptibility:

$$\frac{\chi_q}{T^2} = 2\, \frac{\beta^3}{V} \left[\frac{\langle (M^1)^2 \rangle_0 + \langle M^0 M^2 \rangle_0}{\langle (M^0)^2 \rangle_0} - 2 \left(\frac{\langle M^0 M^1 \rangle_0}{\langle (M^0)^2 \rangle_0} \right)^2 \right] \, ,$$

$\langle \ldots \rangle_0$: expectation value evaluated on configurations with $\mu = 0$.

Lattice parameters

- Two flavor degenerate Wilson fermions & Wilson gauge action
- Lattices $N_s^3 \times N_t$: $8^3 \times 4$, $10^3 \times 4$, $12^3 \times 4$, $12^3 \times 6$ $\quad (\beta = N_t = 1/T)$
- Inverse coupling: $5.00 \leq \frac{6}{g^2} \leq 5.65$
- Lattice spacing: $0.343\,\text{fm} \geq a \geq 0.260\,\text{fm}$ (+ finer currently running)
- Temperature: $96\,\text{MeV} \leq T \leq 211\,\text{MeV}$
- $\kappa = 0.158, 0.160, 0.162$
- Pion mass: $m_\pi = 950\,\text{MeV}$ (+ lighter currently running)
- Configurations generated using the MILC code
  (http://www.physics.utah.edu/ detar/milc/)

Technical details

Important to have $D^{(q)}$ at high precision.

$\Downarrow$

Calculation of $\det[D(\mu\beta = i\phi)]$ for many values of ϕ.

$\Downarrow$

Expensive!

Dimensional reduction[1]:
Use a domain decomposition of the Dirac operator $D(x, y)$ to obtain

$$\det[D(\mu)] = A_0 \, W \; ,$$

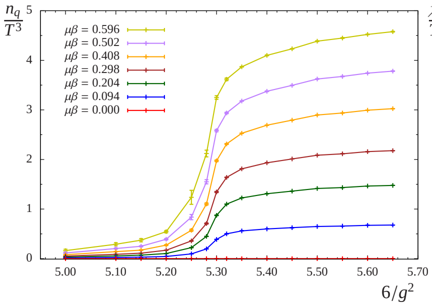
with a μ-independent factor A_0 and

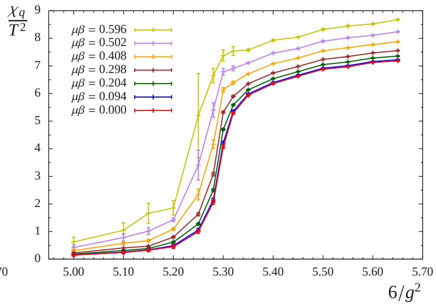
$$W = \det[K_0 - e^{\mu\beta} K - e^{-\mu\beta} K^{\dagger}] \; .$$

K_0, K live on a single time slice, but are dense matrices.

[1] J. Danzer, C. Gattringer, Phys. Rev. D 78, 114506 (2008).

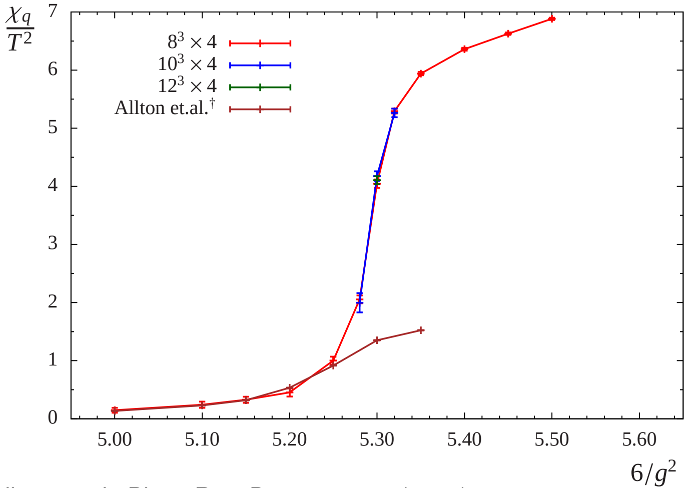
Quark number density & susceptibility





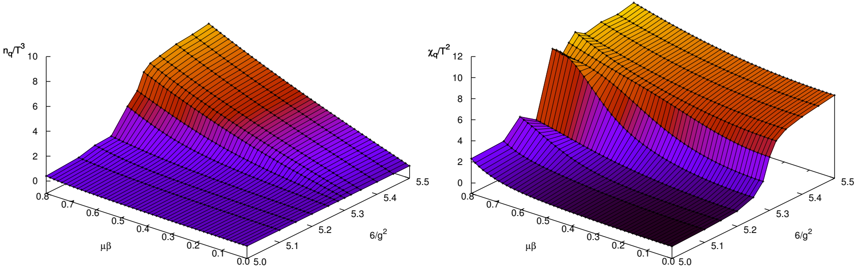
Quark number density (l.h.s) and susceptibility (r.h.s) as a function of $\frac{6}{g^2}$ (proportional to temperature).

Quark number susceptibility @ $\mu = 0$



† C.R. Allton, et.al., Phys. Rev. D 71, 054508 (2005).

Extrapolation to finite μ



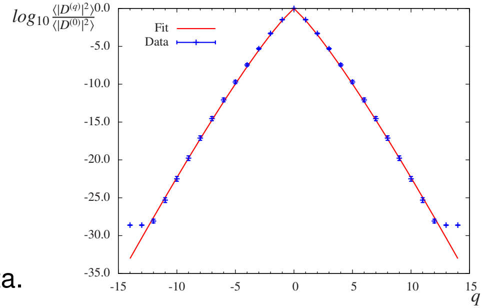
Quark number density (l.h.s) and susceptibility (r.h.s) as a function of $\frac{6}{g^2}$ and $\mu\beta$.

Canonical determinants $D^{(q)}$ at $\mu = 0$

Hadron resonance gas $\rightarrow$
Skellam distribution[†]

(preliminary results)

$$\frac{\langle |D^{(q)}|^2 \rangle}{\langle |D^{(0)}|^2 \rangle} \approx \frac{I_q(Z)}{I_0(Z)} .$$



$I_q \cdots$ modified Bessel function
$Z \cdots 2\sqrt{q\bar{q}}$

Fit in Z to our $12^3 \times 6$, $\frac{6}{g^2} = 5.00$ data.

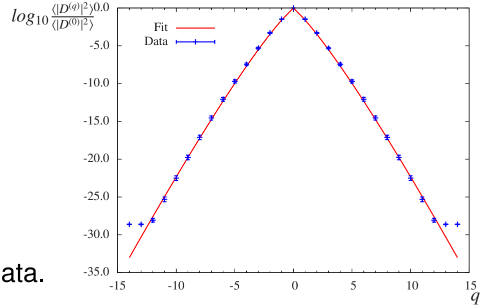
[†]P. Braun-Munzinger, et.al., Phys. Rev. C 84, 064911 (2011).

$D^{(q)}$ from FRG see: K. Morita, et.al., 1301.2873 [hep-ph].

$D^{(q)}$ from ChPT†

(preliminary results)

$$\frac{\langle |D^{(q)}|^2 \rangle}{\langle |D^{(0)}|^2 \rangle} = \frac{I_q\left(\frac{V}{\beta^3}\frac{(m_\pi \beta)^{3/2}}{\sqrt{2}\pi^{3/2}} e^{-m_\pi \beta}\right)}{I_0\left(\frac{V}{\beta^3}\frac{(m_\pi \beta)^{3/2}}{\sqrt{2}\pi^{3/2}} e^{-m_\pi \beta}\right)} .$$



$I_q \cdots$ modified Bessel function

Fit in $m_\pi \beta$ to our $12^3 \times 6$, $\frac{6}{g^2} = 5.00$ data.

† K. Splittorff, J. Verbaarshot, 1-loop ChPT, private communication.
(Valid for $m_\pi L \gg 1$, $m_\pi / m_N \ll 1$)

Summary:

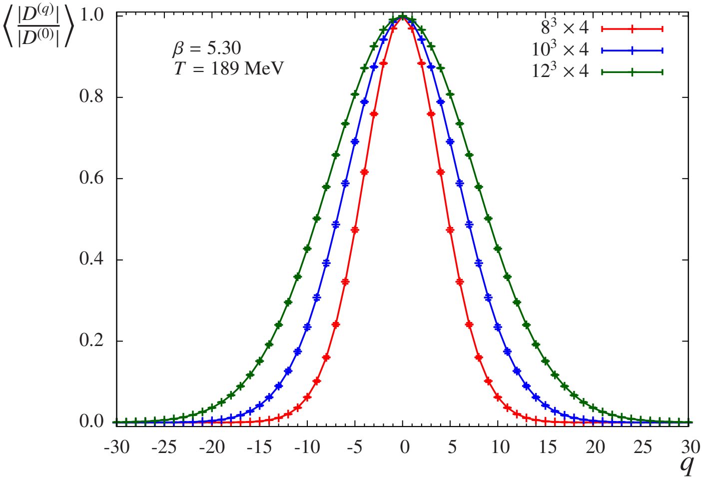
- Fugacity expansion: Laurent series in $e^{\mu\beta}$.
- Finite lattice $\Rightarrow$ finite series.
- Used to continue from $\mu = 0$ to small chemical potential.
- Calculation numerically challenging $\Rightarrow$ dimensional reduction.
- Observables related to n_q $\Rightarrow$ moments of $D^{(q)}$.

Outlook:

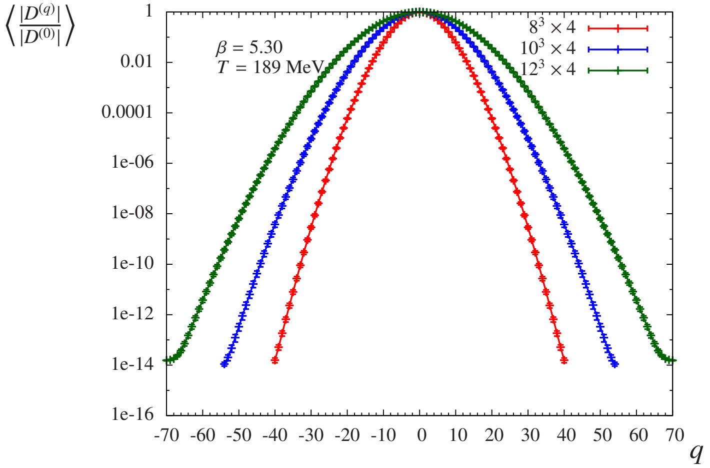
- Larger lattices and calculation at smaller pion masses.
- Calculation for other fermions (Clover improved Wilson, staggered).
- Comparison with other approaches in toy models.

Backup slides

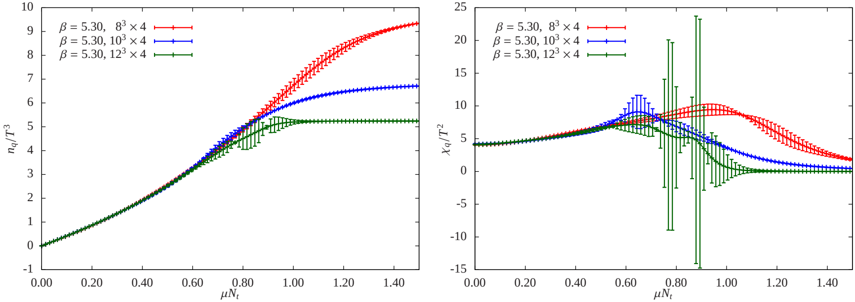
Volume dependence of the $D^{(q)}$ - 1



Volume dependence of the $D^{(q)}$ - 2



Expansion quality & lattice volume



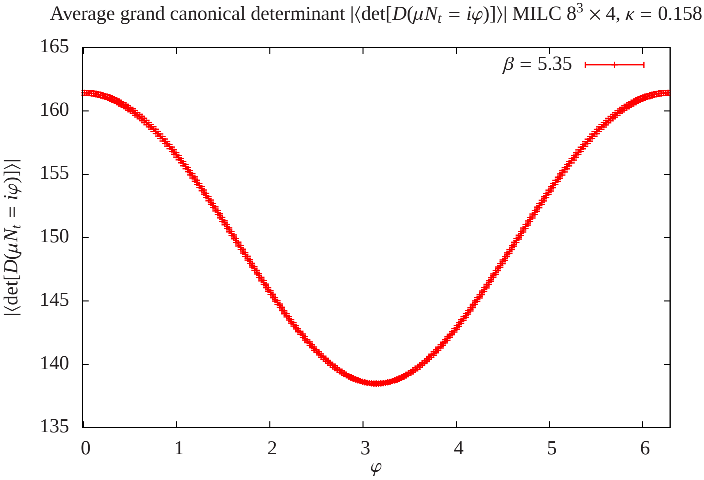
Quark number density (l.h.s) and susceptibility (r.h.s) as a function of μ.

Numerical values of the quark number susceptibility @ $\mu = 0$

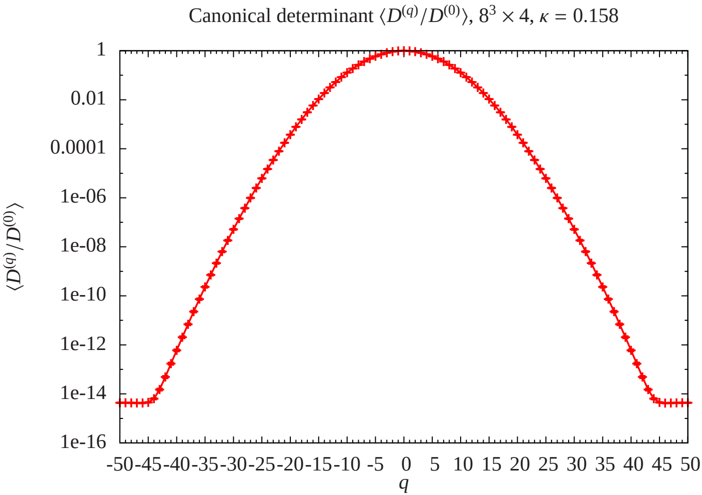
T[MeV]	β	χ_q (this work)	χ_q[2]	χ_q[3]
144	5.00	0.14(4)	0.25(8)	0.1360(44)
153	5.10	0.24(5)	0.14(9)	0.2312(52)
157	5.15	0.32(5)	0.41(8)	0.3190(58)
164	5.20	0.45(5)	0.52(7)	0.5340(62)
173	5.25	1.00(7)	0.90(8)	0.9170(66)
189	5.30	4.07(7)	3.41(8)	1.3514(36)
211	5.35	5.93(3)	5.85(4)	1.5240(30)

[2] J. Danzer, C. Gattringer, Phys. Rev. D 86, 014502 (2012).
[3] C. R. Allton et al., Phys. Rev. D 71, 054508 (2005).

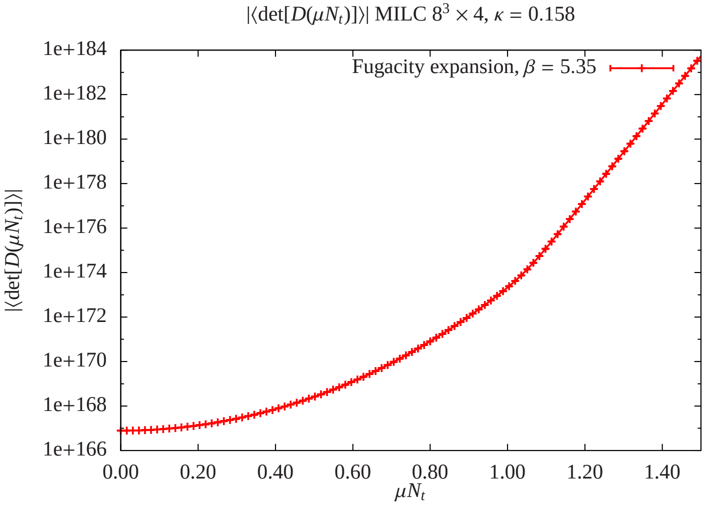
More $8^3 \times 4$ results - 1



More $8^3 \times 4$ results - 2



More $8^3 \times 4$ results - 3



More $8^3 \times 4$ results - 4