

Universality of Phase Diagrams in QCD and QCD-like Theories

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“Quarks, Gluons, and Hadronic Matter under Extreme
Condition,” March 22, 2013

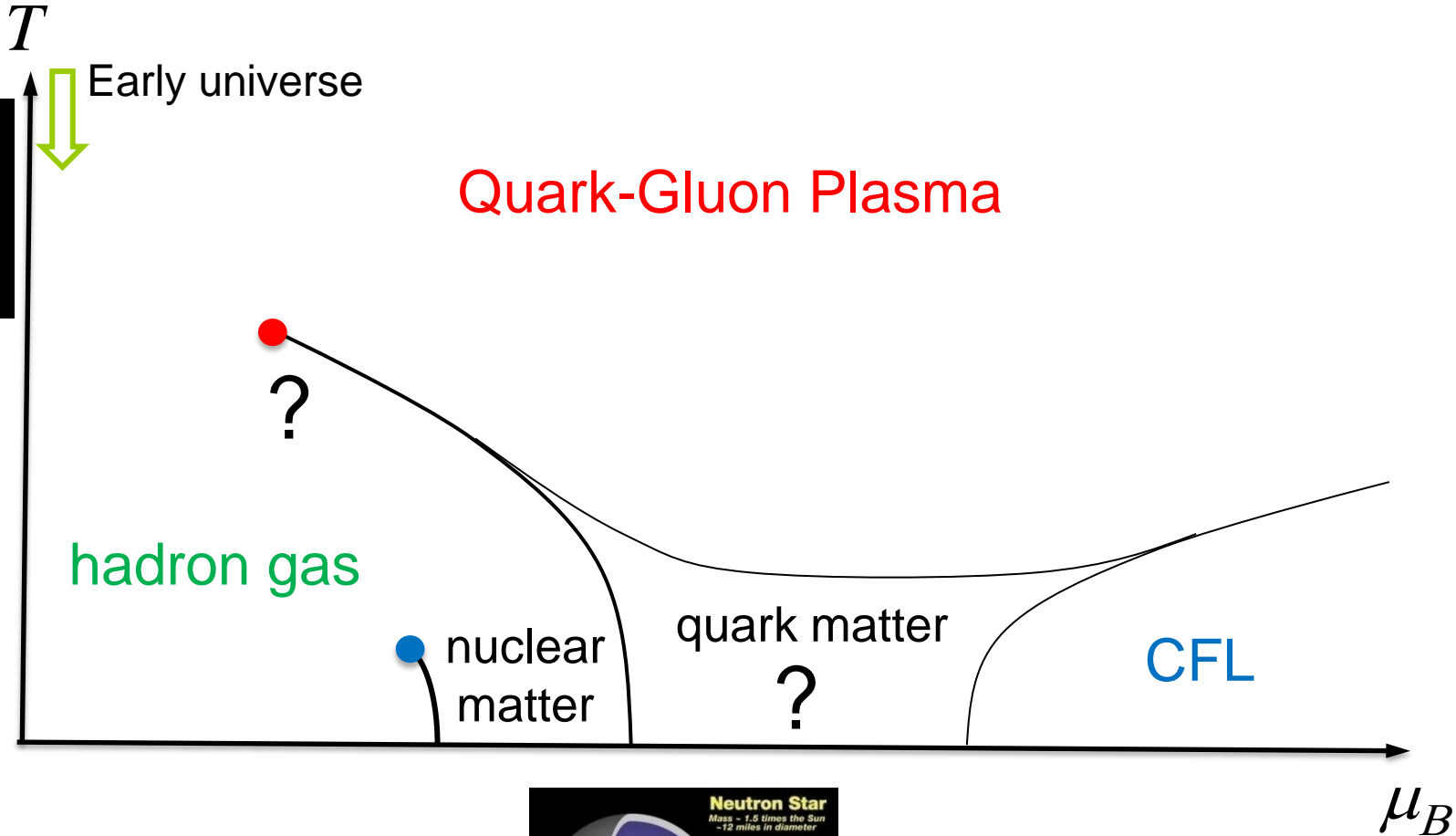
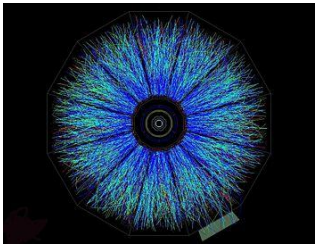
Outline

1. Universality of phases in QCD and QCD-like theories
2. A theorem on the QCD critical point
3. Summary and discussion

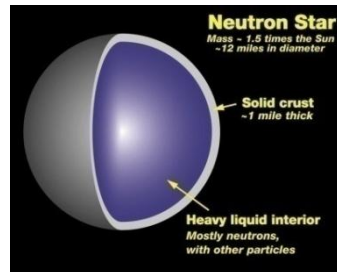
Refs.: **M. Hanada, NY, (JHEP 2012)**
Y. Hidaka, NY, (PRL 2012)
M. Hanada, Y. Matsuo, NY, (PRD 2012)

A conjectured QCD phase diagram

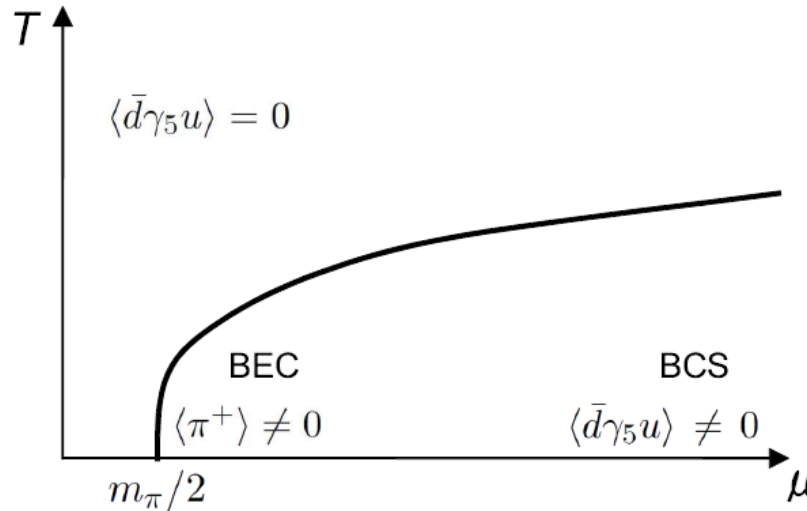
RHIC/LHC



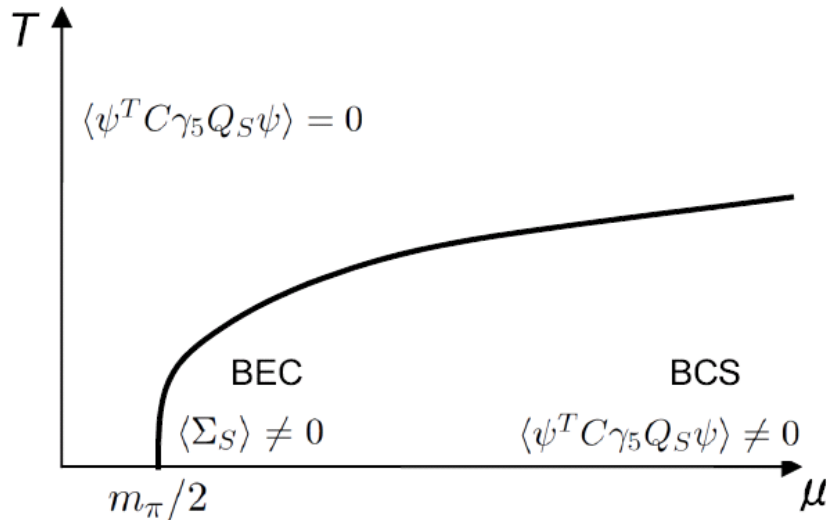
Neutron star



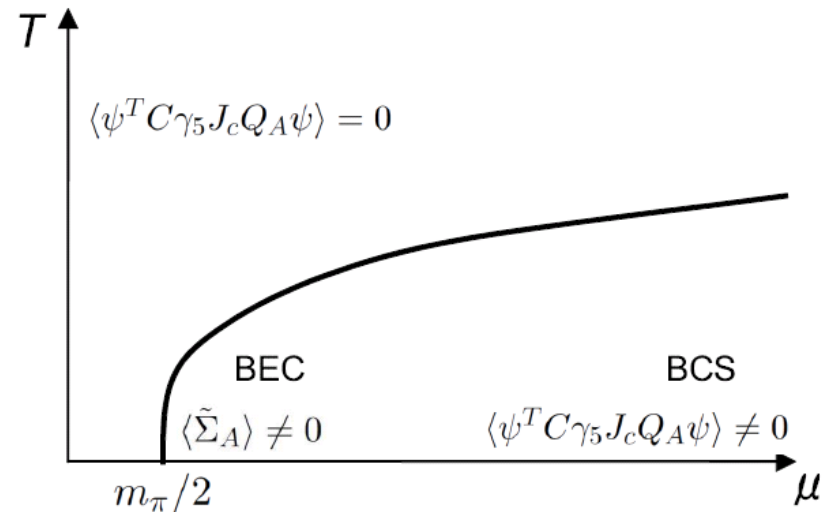
Phase diagrams of QCD-like theories



QCD at $\mu \neq 0$
 Son-Stephanov (2001)



SO(N_c) gauge theory at $\mu_B \neq 0$

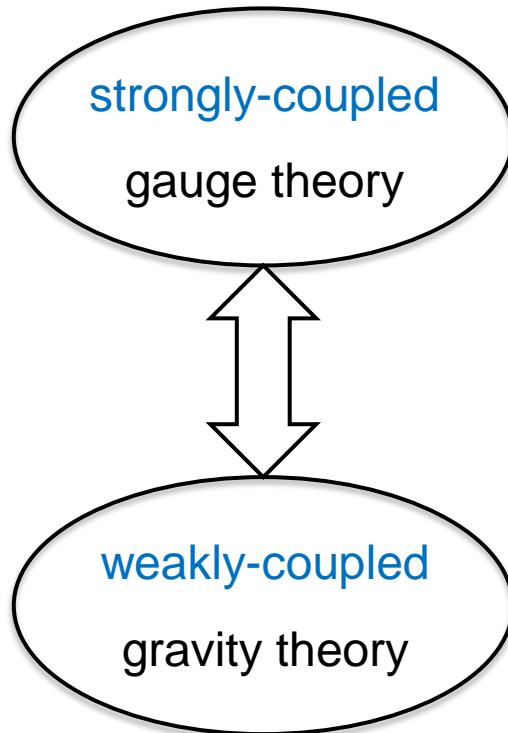


Sp(N_c) gauge theory at $\mu_B \neq 0$

Holography vs. Orbifolding

Gauge/gravity duality

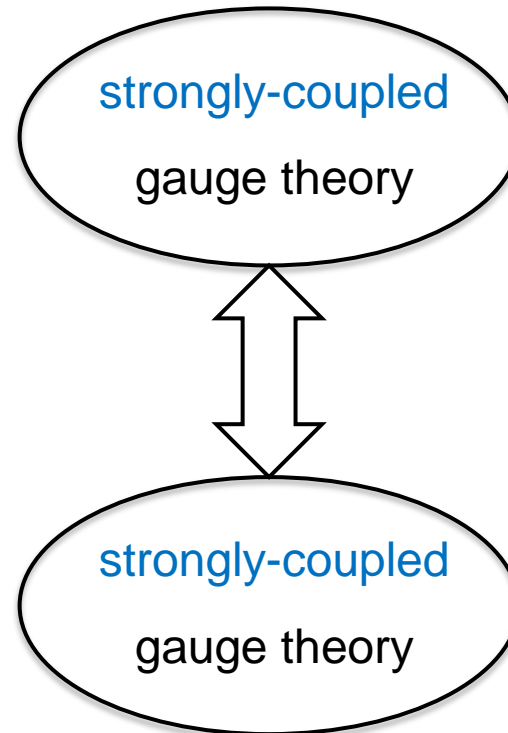
Maldacena (1998), ...



AdS/QGP,
AdS/QCD,
AdS/CMT, ...

Orbifold equivalence

Kachru-Silverstein (1998), ...

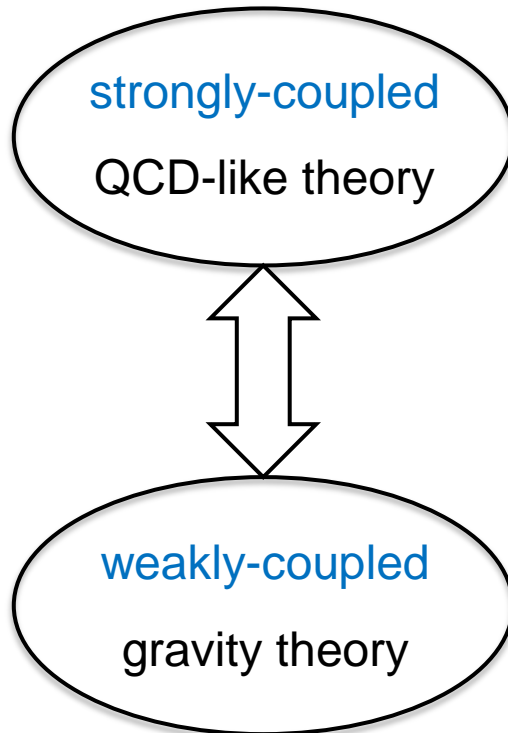


Cherman-Hanada-Robles-Llana;
Cherman-Tiburzi;
Hanada-NY; Hidaka-NY (2012)

Holography vs. Orbifolding

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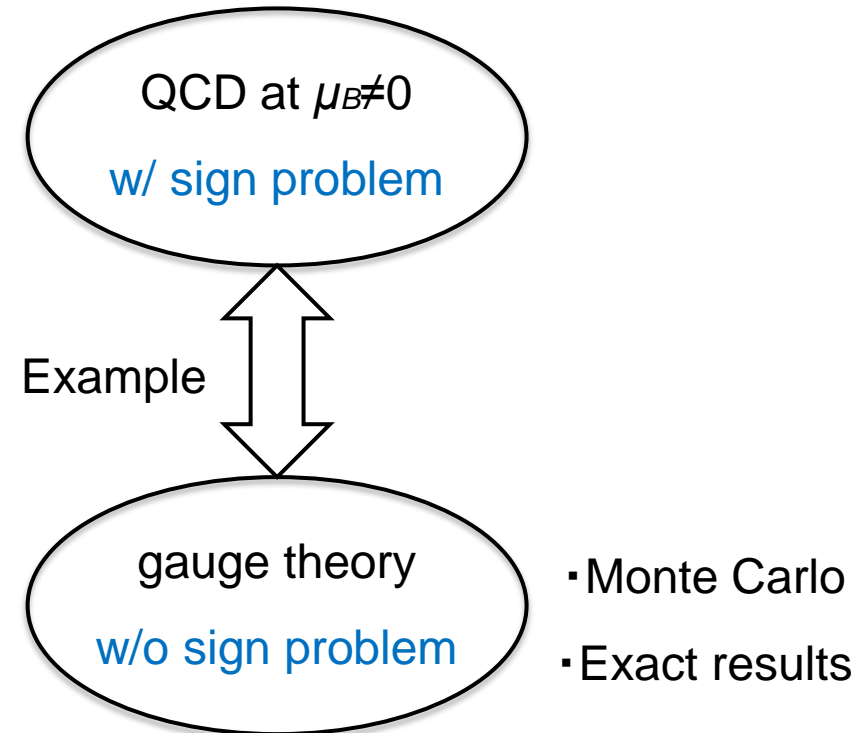
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Cherman-Hanada-Robles-Llana;
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Orbifold equivalence

[How to use]

1. Identify a discrete global symmetry S of a theory P (parent).
2. Eliminate all the d.o.f.s of parent not invariant under S (projection).
3. This gives a new theory D (daughter).
4. A class of observables are identical between P and D at large N_c or MFA.
Valid as long as S is not broken spontaneously.

Refs: Bershadsky-Johansen (1998); Kovtun-Ünsal-Yaffe (2003, 2005, 2006);

Generalization w/ fermions at finite μ/T , Hanada-NY (JHEP, 2012).

Orbifold equivalence

[How to use]

1. Identify a discrete global symmetry S of a theory P (parent).

$$J_c = -i\sigma_2 \times \mathbf{1}_{N_c} \in \text{SO}(2N_c), \quad \omega = e^{i\pi} \in \text{U}(1)_B$$

2. Eliminate all the d.o.f.s of parent not invariant under S (projection).

$$A_\mu^{\text{SO}} = J_c A_\mu^{\text{SO}} J_c^{-1}, \quad \psi^{\text{SO}} = \omega J_c \psi^{\text{SO}}$$

3. This gives a new theory D (daughter).

$$D = \text{SU}(N_c) \text{ QCD.}$$

4. A class of observables are identical between P and D at large N_c or MFA.

Valid as long as S is not broken spontaneously.

Refs: Bershadsky-Johansen (1998); Kovtun-Ünsal-Yaffe (2003, 2005, 2006);

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Universality of phase diagrams

sign-free

$\text{SO}(N_c)$ or $\text{Sp}(N_c)$ YM + μ_B

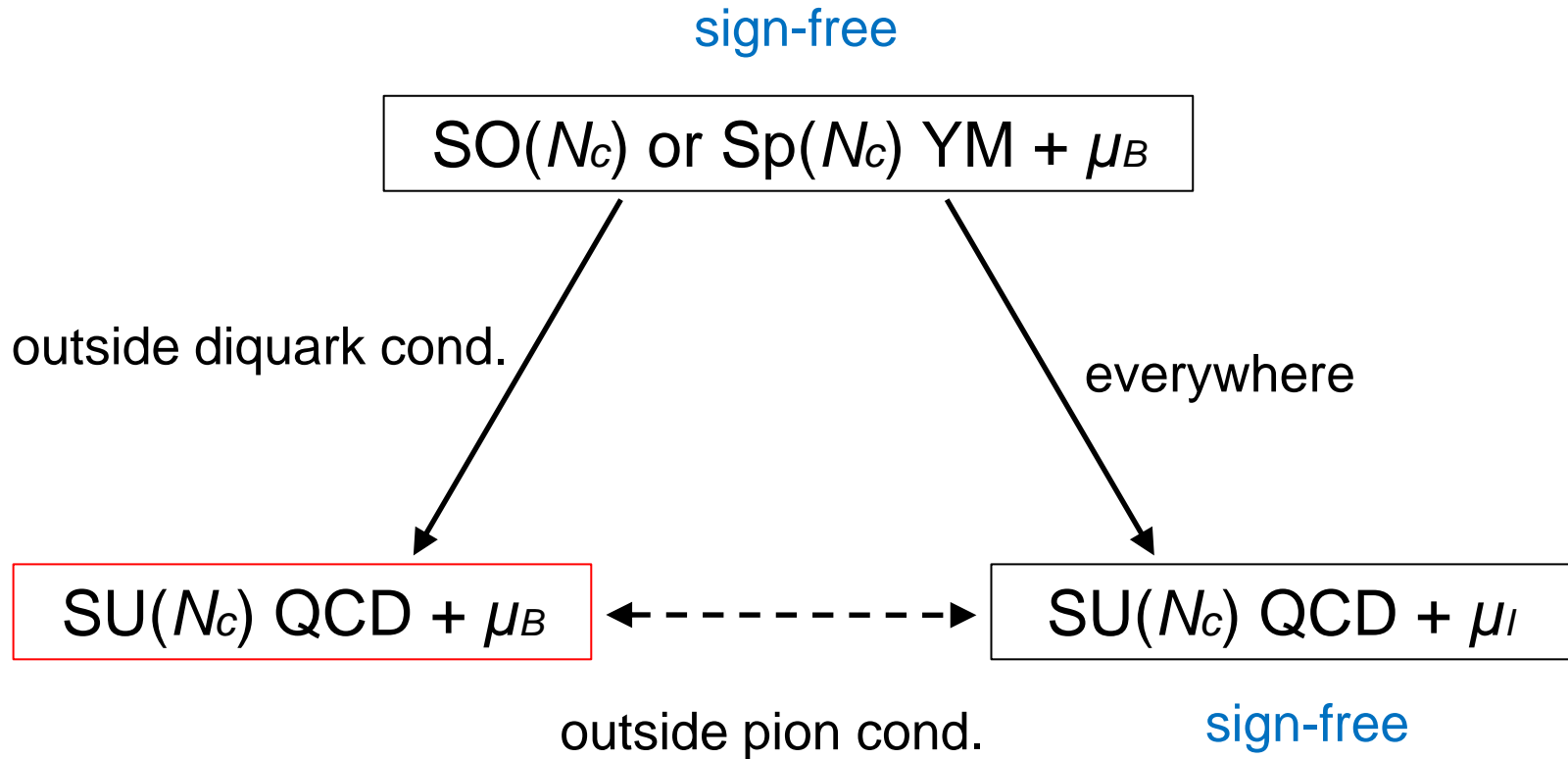
$\text{SU}(N_c)$ QCD + μ_B

$\text{SU}(N_c)$ QCD + μ_I

sign-free

Hanada-NY (JHEP, 2012)

Universality of phase diagrams



Sign problem and phase quenching in finite-density QCD: models, holography, and lattice

Masanori Hanada^{1,2}, Yoshinori Matsuo¹, and Naoki Yamamoto^{3,4}

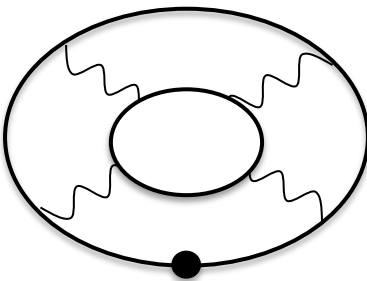
arXiv:1205.1030 (PRD, 2012)

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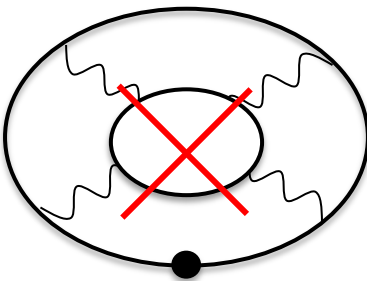
Intuitive derivation

- Consider $\langle \bar{q}q \rangle$ in QCD at finite μ_I or μ_B with $N_f = 2$ (u & d).

$$\langle \bar{q}q \rangle(\mu_B) = \text{Diagram}$$
The diagram is a torus (a donut shape) representing a fermion loop. It consists of an outer circle and an inner circle. The region between the two circles is filled with wavy lines, representing gluon exchanges. At the bottom of the outer circle, there is a solid black dot, which represents a quark insertion point.

Intuitive derivation

- Consider $\langle \bar{q}q \rangle$ in QCD at finite μ_I or μ_B with $N_f = 2$ (u & d).

$$\langle \bar{q}q \rangle(\mu_B) = \text{Diagram}$$
The diagram shows a fermion loop (represented by a solid line) with a gluon loop (represented by a wavy line) inside it. The gluon loop is crossed out with a red 'X', indicating it is neglected in the derivation. The fermion loop has a black dot at the bottom, representing the quark condensate operator $\bar{q}q$.

Intuitive derivation

- Consider $\langle \bar{q}q \rangle$ in QCD at finite μ_I or μ_B with $N_f = 2$ (u & d).

$$\langle \bar{q}q \rangle(\mu_B) = \text{Diagram} = f(\mu_u) + f(\mu_d)$$

Intuitive derivation

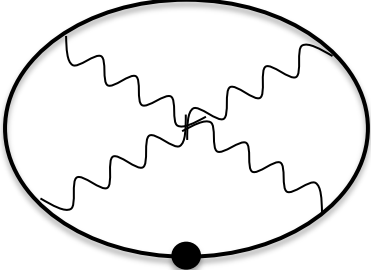
- Consider $\langle \bar{q}q \rangle$ in QCD at finite μ_I or μ_B with $N_f = 2$ (u & d).

$$\langle \bar{q}q \rangle(\mu_B) = \text{Diagram} = f(\mu_u) + f(\mu_d) = f(\mu_u) + f(-\mu_d) = \langle \bar{q}q \rangle(\mu_I)$$

see also T. D. Cohen (2004)

Intuitive derivation

- Consider $\langle \bar{q}q \rangle$ in QCD at finite μ_I or μ_B with $N_f = 2$ (u & d).

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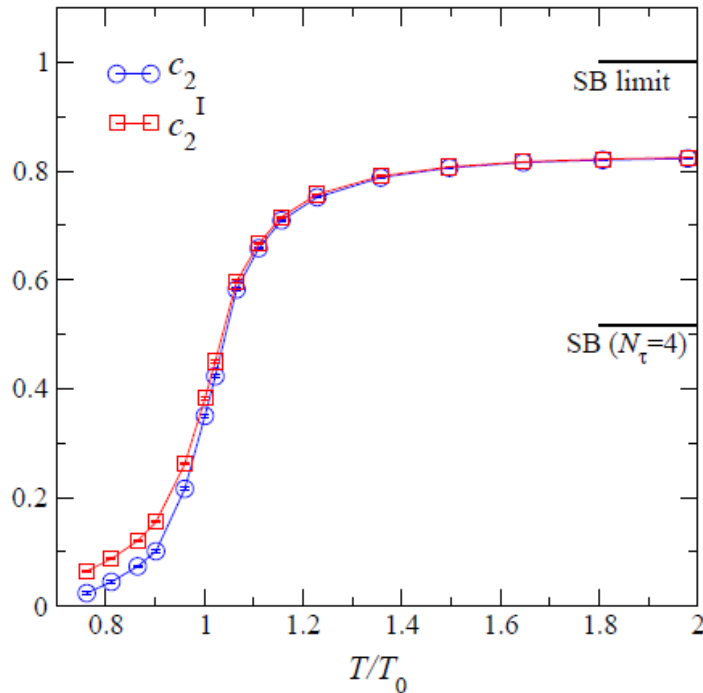
see also T. D. Cohen (2004)

- For gluonic observables, equivalence holds up to $O(N_c^{-2})$. D. Toublan (2005)
- Not applicable when u & d are mixed \rightarrow MFA outside π condensation
- Orbifold equivalence: more systematic for a larger class of theories.

Lattice results

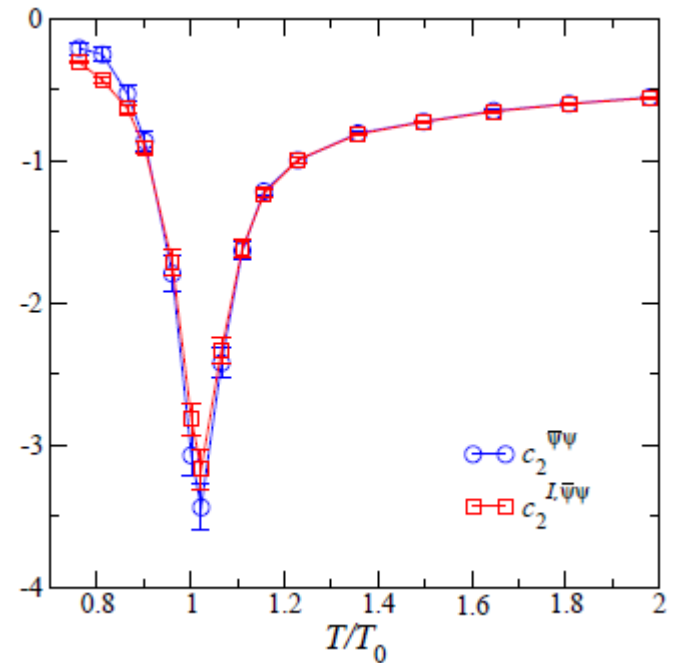
Pressure

$$\frac{p}{T^4} \equiv \Omega(T, \mu_q, \mu_q) = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n$$



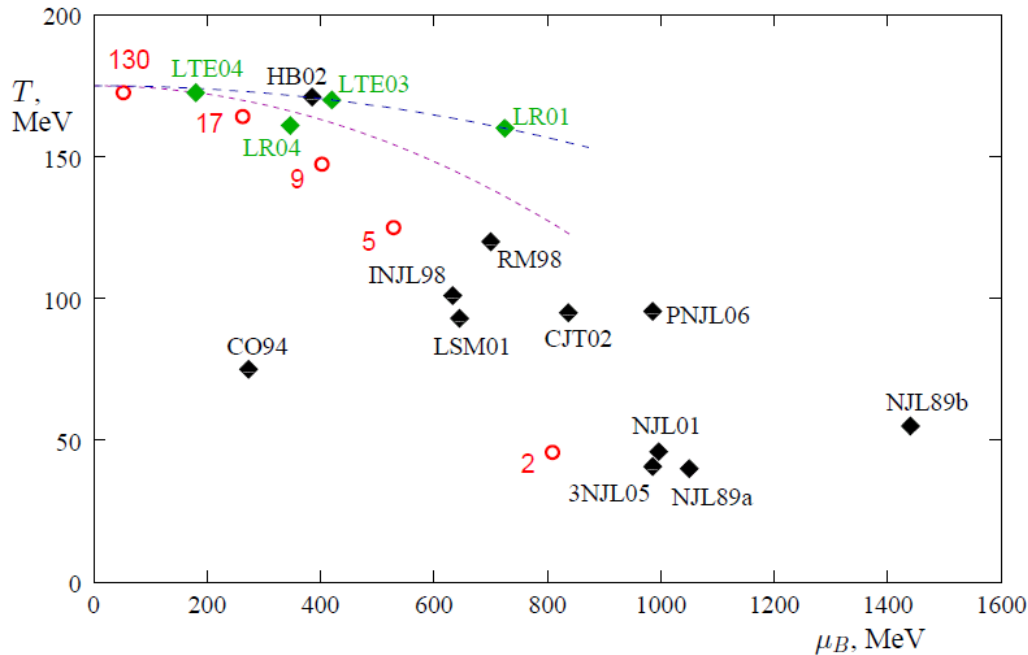
Chiral condensate

$$\frac{\langle \bar{\psi}\psi \rangle}{T^3} = \left(\frac{N_\tau}{N_\sigma}\right)^3 \frac{\partial \ln Z}{\partial m/T} = \sum_{n=0}^{\infty} c_n^{\bar{\psi}\psi}(T) \left(\frac{\mu_q}{T}\right)^n$$

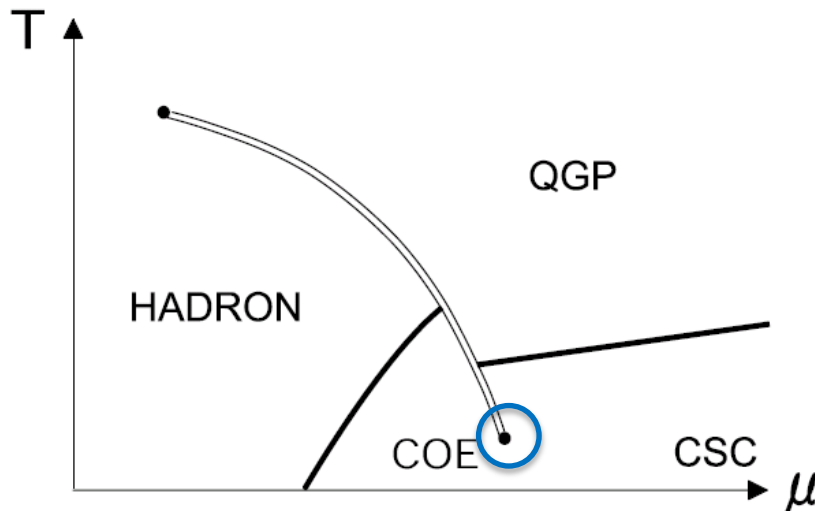


Allton *et al.* (Bielefeld-Swansea collaboration), 2005, PRD

Where are QCD critical point(s)?



Summarized in
M. Stephanov, hep-lat/0701002



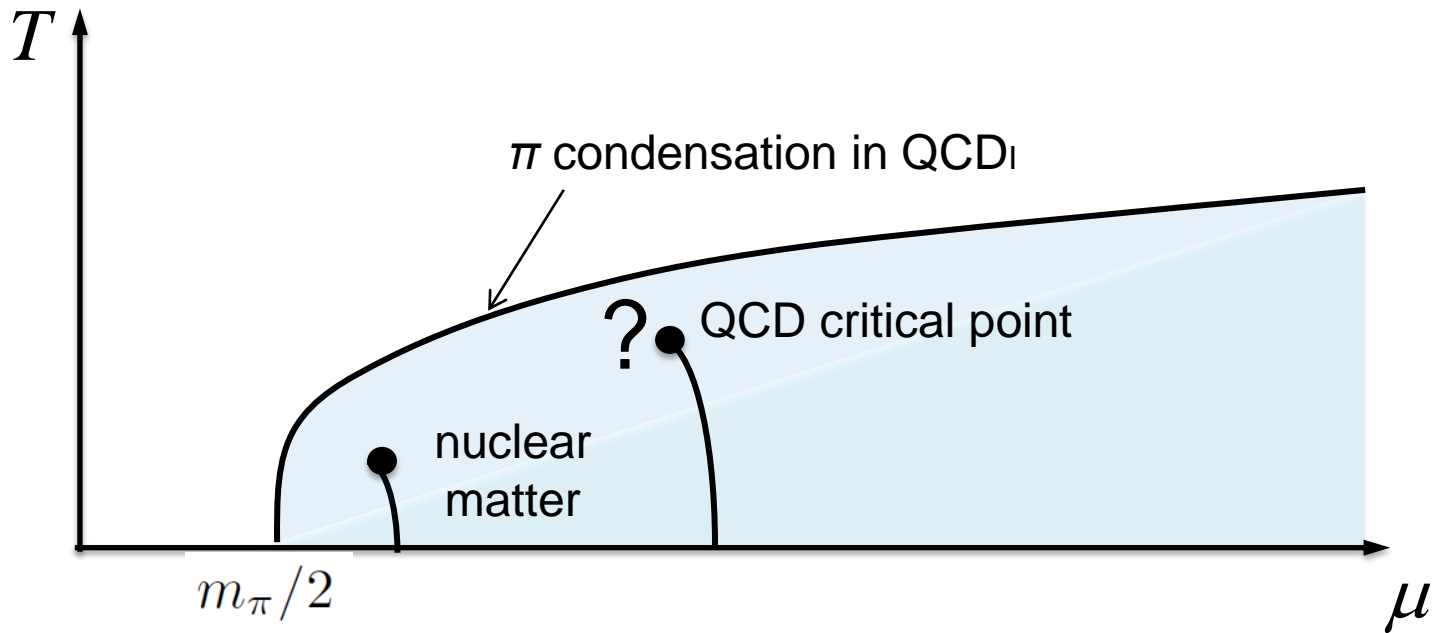
Baym-Hatsuda-Tachibana-NY
(PRL, 2006)

A theorem

- π^\pm are the lightest in QCD at finite μ_I Son-Stephanov (2001)
 - ∴ QCD inequalities: π^\pm propagator \geq any meson propagator, which follows from $\tau_1 \gamma_5 D \gamma_5 \tau_1 = D^\dagger$ and Cauchy-Schwarz inequality.
- $m_\sigma \geq m_\pi > 0$ for $m_q > 0$ outside π condensation phase.
- QCD critical point ($\xi = \infty$ or $m_\sigma = 0$) is prohibited there.
 - Via the orbifold equivalence, it also holds in QCD at finite μ_B .
 - Hidaka-NY (PRL, 2012)
- The only assumption: suppression of disconn. diagrams (OZI rule)

A theorem

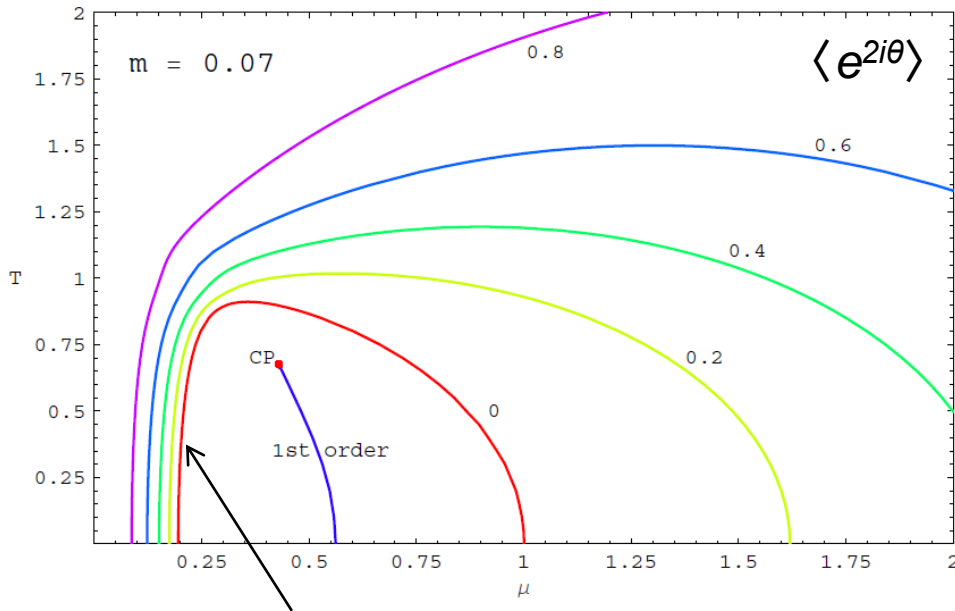
QCD critical point in QCD_B can **only be inside** π condensation of QCD_I where **reweighting method breaks down**.



Model results ($N_f=2$)

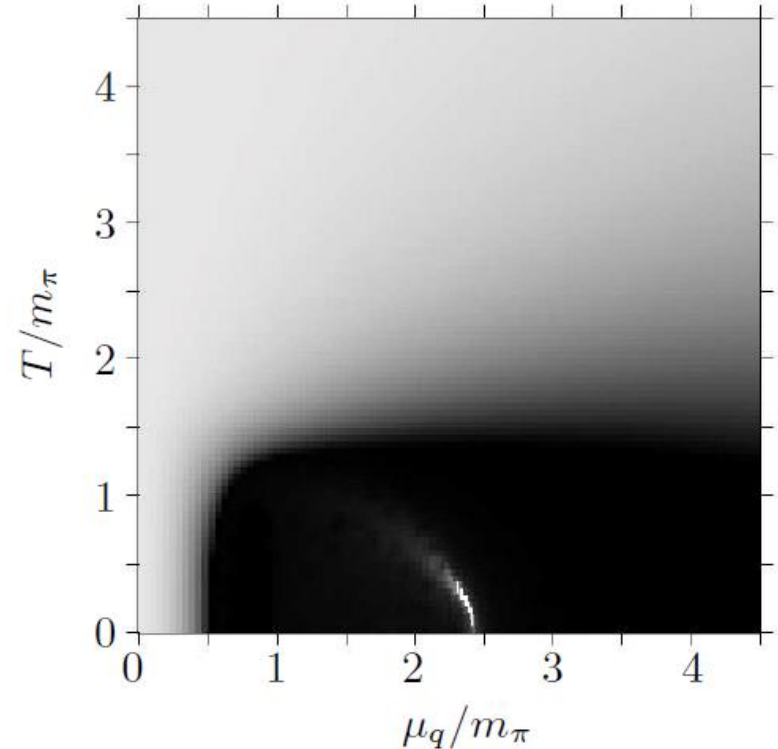
Random matrix model: [Han-Stephanov \(2008\)](#)

NJL model: [Andersen-Kyllingstad-Splittorff \(2009\)](#)



onset of pion condensation phase

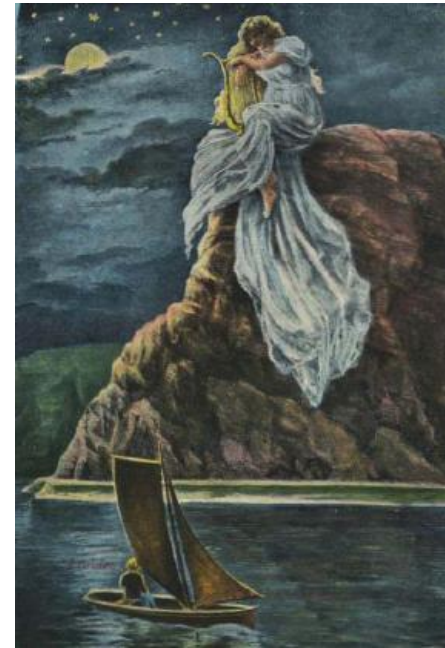
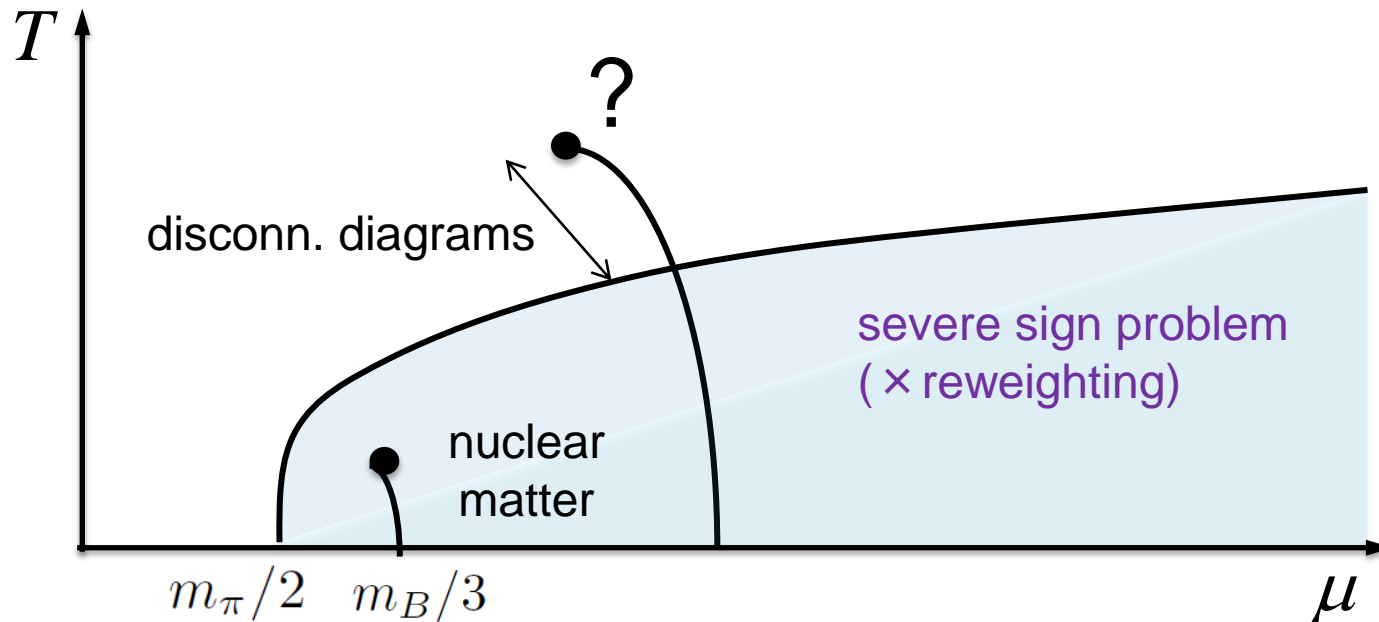
$$e^{2i\theta} = \frac{\det(D + \mu_q \gamma_0 + m)}{\det(D + \mu_q \gamma_0 + m)^*}$$



➤ Similar result in PNJL model: [Sakai-Sasaki-Kouno-Yahiro \(2010\)](#)

Summary & Outlook

- Universality of phase diagrams in QCD and QCD-like theories
- Importance of disconn. diagrams beyond MFA → DS & Functional RG?
- Many more applications of the orbifold equivalence.
- Challenge for theory: physics inside π condensation.



Back up slides

Discussion

- At MFA, chiral transition cannot be 1st outside π condensation.
→ only beyond-MFA effects can make it 1st.

Discussion

- At MFA, chiral transition cannot be 1st outside π condensation.
→ only beyond-MFA effects can make it 1st.



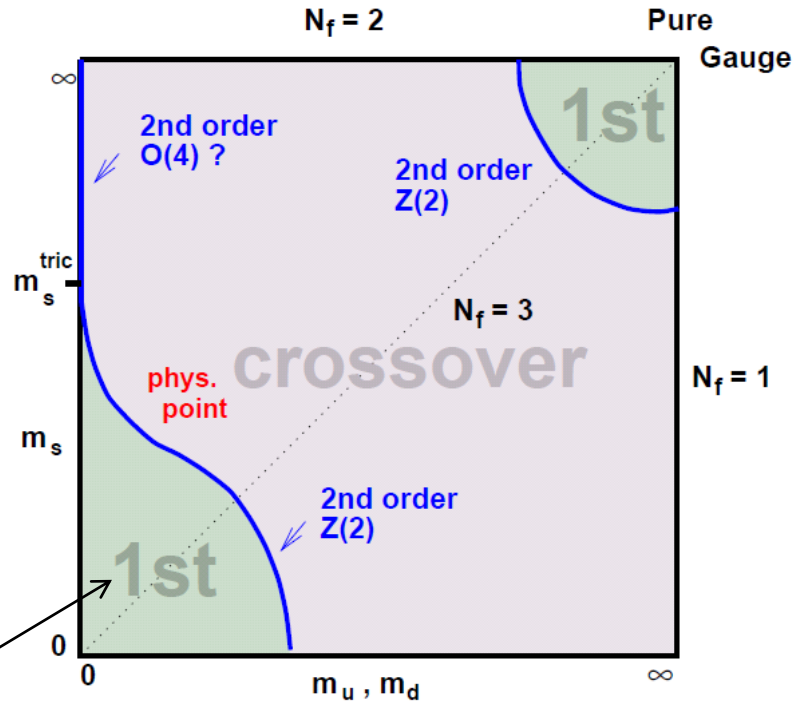
competition

quark mass weakens the chiral transition (1st \rightarrow 2nd \rightarrow crossover)

- At $\mu_B = 0$, chiral transition is crossover; beyond-MFA effects $<$ quark mass.
- QCD critical point is accessible on the lattice (outside π condensation) if beyond-MFA effects are enhanced at $\mu_B \neq 0$ for some reason.

A counter example?

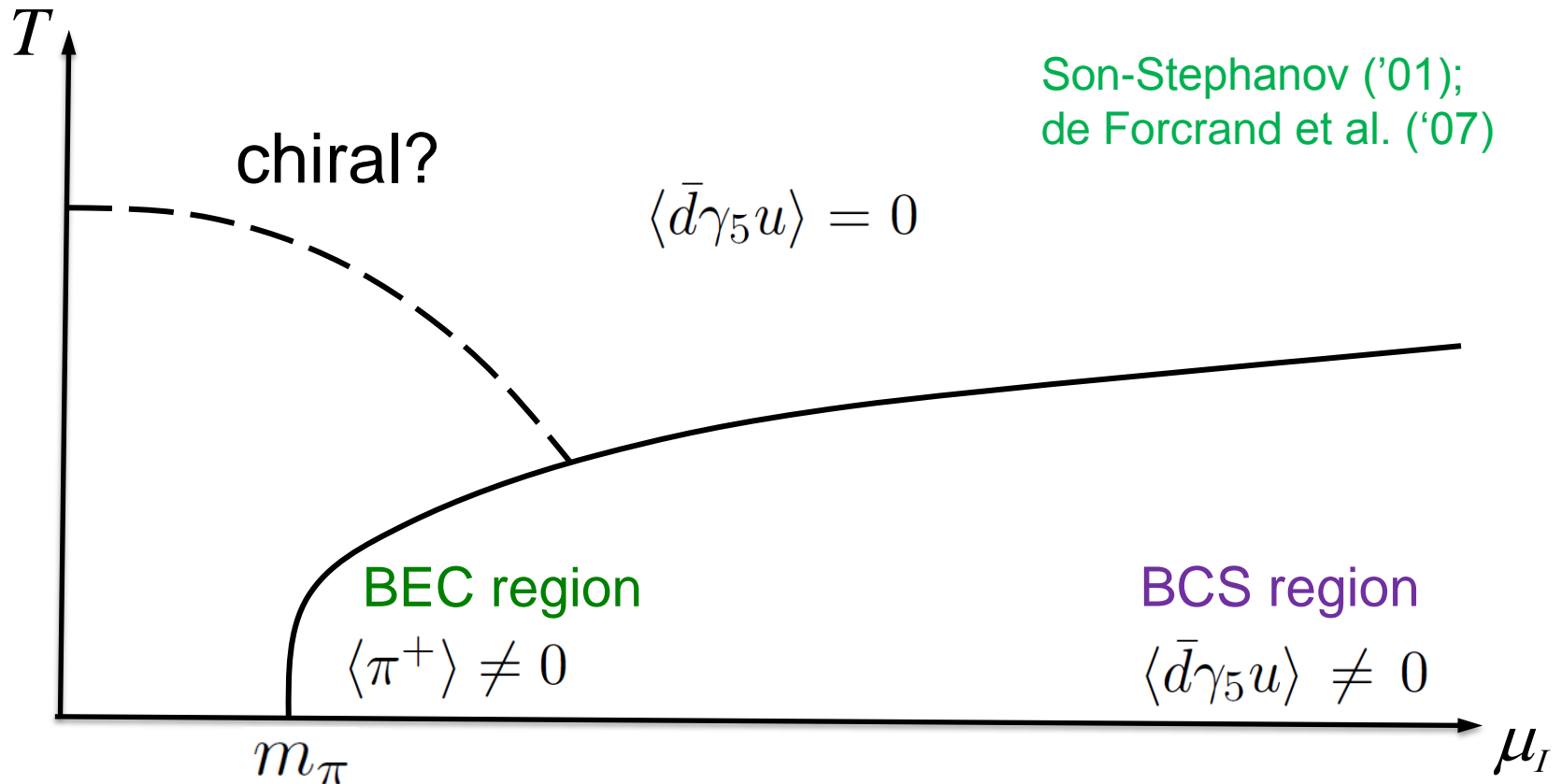
➤ Columbia plot:



axial anomaly (disconnected diagrams)

- One can make the critical point **on the T -axis** (outside the pion condensation) by fine tuning $m_q \sim$ anomaly.
- Ignoring disconn. diagrams: anomaly $<$ m_q

QCD at finite isospin density

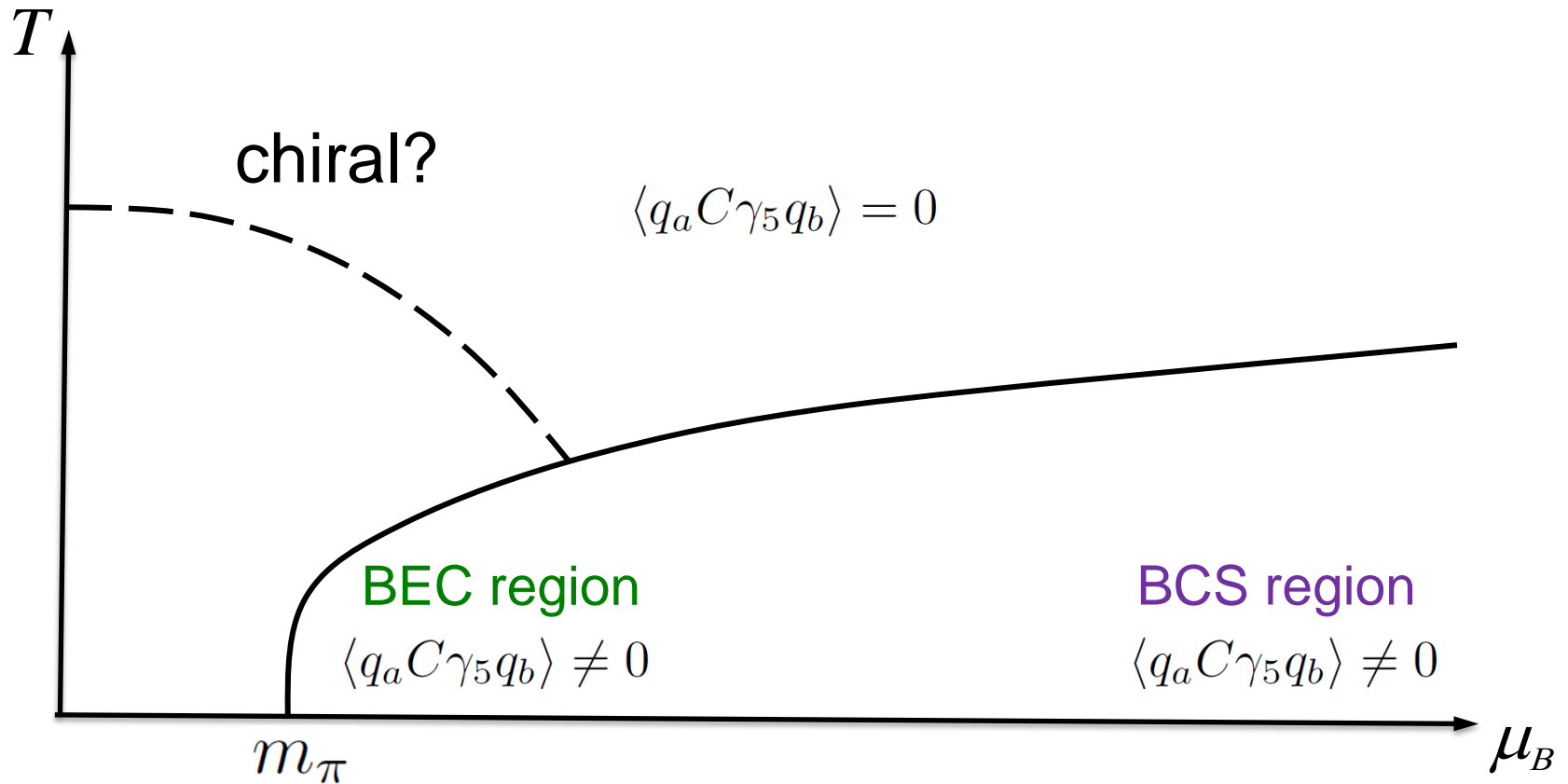


Finite $\mu_I=2\mu$: chemical potentials μ & $-\mu$ for u & d quarks

→ Dirac eigenvalues for u & d are complex conjugate.

→ Positive fermion determinant (no sign problem)

SO & Sp gauge theories at $\mu_B > 0$



SO($2N_c$): real \rightarrow positive fermion determinant (no sign problem)

(similarly for pseudo-real Sp($2N_c$) with even N_f)

From $SO(2N_c)$ to $SU(N_c)$ at finite μ_I

Start with $SO(2N_c)$ gauge theory at $\mu_B > 0$.

1. Discrete symmetry: $J_c = -i\sigma_2 \otimes \mathbf{1}_{N_c} \in SO(2N_c)$
 $J_i = -i\sigma_2 \otimes \mathbf{1}_{N_f/2} \in SU(2)_{iso}$
2. Projection: $A_\mu^{SO} = J_c A_\mu^{SO} J_c^{-1}, \quad \psi^{SO} = J_c \psi^{SO} J_i^{-1}$
3. Daughter theory: $U(N_c) \approx SU(N_c)$ gauge theory at $\mu_I > 0$.

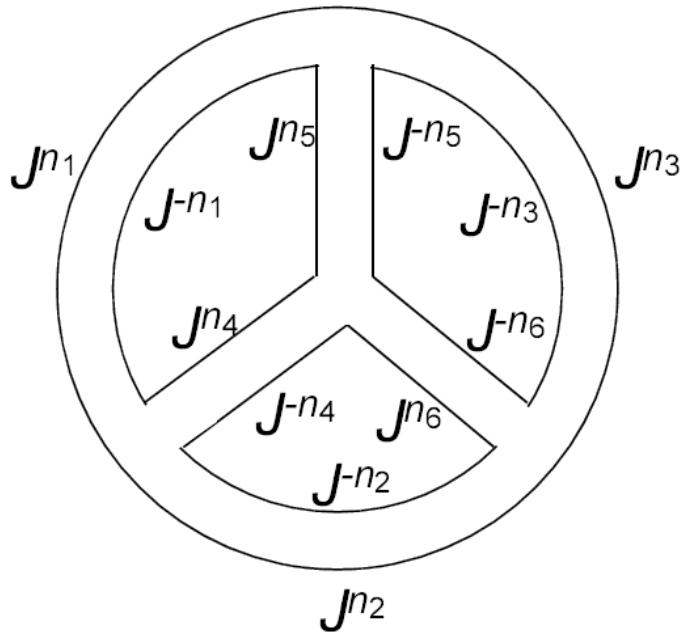
$$A_\mu^{\text{proj}} = \begin{pmatrix} (A_\mu^U)^C & 0 \\ 0 & A_\mu^U \end{pmatrix}, \quad \psi^{\text{proj}} = \begin{pmatrix} \psi_+^U \\ \psi_-^U \end{pmatrix}$$

4. Orbifold equivalence: $\langle \bar{\psi}\psi \rangle^{SO} = \langle \bar{\psi}\psi \rangle^{SU}$ etc.

[Caution]

- $Z_4 \in SU(2)_{iso}$ unbroken from $SO(2N_c)$ to QCD at $\mu_I > 0$ everywhere.

Perturbative proof



Bershadsky-Johansen ('98)

- Insert $\mathcal{P}(A_\mu^{\text{SO}}) = \frac{1}{2} (A_\mu^{\text{SO}} + J_c A_\mu^{\text{SO}} J_c^{-1})$ for each propagator.
- Take the same 't Hooft coupling.
- Difference comes from color factors.
- Condition: $\text{tr}(J_c^n) = 0$, when $J_c^n \neq \pm \mathbf{1}_{2N_c}$

$$\sum_{n_i=0,1} \left(\frac{1}{2}\right)^{N_P} \cdot \text{tr}(J^{-n_1} J^{n_4} J^{n_5}) \cdot \text{tr}(J^{-n_2} J^{-n_4} J^{n_6}) \cdot \text{tr}(J^{-n_3} J^{-n_5} J^{-n_6}) \cdot \text{tr}(J^{n_1} J^{n_2} J^{n_3})$$

$$= 2^{-6} \cdot 2^{6-3} \cdot 2^4 = 2$$

Generally, $2^{-N_P} \cdot 2^{N_P - (N_L - 1)} \cdot 2^{N_L} = 2$ for any planar diagrams.

What are (aren't) equivalent?

Not all the quantities are equivalent in the orbifold equivalence.

- Projection symmetry must be unbroken.
- Observables must keep the projection symmetry (**neutral**).
- Symmetry breaking patterns, quantum numbers of the condensates can be different, but their magnitudes are the same.
- Example: BCS gap (inside the BEC-BCS crossover)

$$\begin{aligned} \Delta_{\mu_B}^{\text{SU}} &\sim \mu \exp\left(-\frac{\pi^2}{g} \sqrt{\frac{6N_c}{N_c+1}}\right) \longrightarrow 0 \\ \Delta_{\mu_I}^{\text{SU}} &\sim \mu \exp\left(-\frac{\pi^2}{g} \sqrt{\frac{6N_c}{N_c^2-1}}\right) \\ \Delta_{\mu_B}^{\text{SO}} &\sim \mu \exp\left(-\frac{\pi^2}{g} \sqrt{\frac{12}{2N_c-1}}\right) \\ \Delta_{\mu_B}^{\text{Sp}} &\sim \mu \exp\left(-\frac{\pi^2}{g} \sqrt{\frac{12}{2N_c+1}}\right) \end{aligned} \begin{array}{l} \longrightarrow 0 \\ \text{'t Hooft limit (large } N_c, g^2 N_c \text{ fixed)} \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \sim \mu \exp\left(-\pi^2 \sqrt{\frac{6}{g^2 N_c}}\right)$$

QCD inequality

➤ Correlation function:

$$\begin{aligned}
 C_\Gamma(x, y) &\equiv \langle M_\Gamma(x) M_\Gamma^\dagger(y) \rangle_{\psi, A} & \bar{\Gamma} &\equiv \gamma_0 \Gamma^\dagger \gamma_0 \\
 &= -\langle \text{tr}[S_A(x, y) \Gamma S_A(y, x) \bar{\Gamma}] \rangle_A & S_A(x, y) &\equiv \langle x | \mathcal{D}^{-1} | y \rangle \\
 &= \langle \text{tr}[S_A(x, y) \Gamma i\gamma_5 S_A^\dagger(x, y) i\gamma_5 \bar{\Gamma}] \rangle_A \\
 &\leq \sqrt{\langle \text{tr}[S_A S_A^\dagger] \rangle_A} \sqrt{\langle \text{tr}[\Gamma \gamma_5 S_A^\dagger \gamma_5 \bar{\Gamma} (\Gamma \gamma_5 S_A^\dagger \gamma_5 \bar{\Gamma})^\dagger] \rangle_A} \\
 &= \langle \text{tr}[S_A(x, y) S_A^\dagger(x, y)] \rangle_A,
 \end{aligned}$$

Inequality saturated when $\Gamma = i\gamma_5$.

➤ Cauchy-Schwarz inequality: $\text{tr}(AB^\dagger) \leq \sqrt{\text{tr} AA^\dagger} \sqrt{\text{tr} BB^\dagger}$