Universality of Phase Diagrams in QCD and QCD-like Theories

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Outline

1. Universality of phases in QCD and QCD-like theories

2. A theorem on the QCD critical point

3. Summary and discussion

Refs.:  M. Hanada, NY, (JHEP 2012)  
Y. Hidaka, NY, (PRL 2012)  
M. Hanada, Y. Matsuo, NY, (PRD 2012)
A conjectured QCD phase diagram

RHIC/LHC

$T$

Early universe

Quark-Gluon Plasma

hadron gas

nuclear matter

quark matter

CFL

Neutron star

Neutron Star

Solid crust

Heavy liquid interior

+1 km thick

with other particles
Phase diagrams of QCD-like theories

QCD at $\mu_i \neq 0$
Son-Stephanov (2001)

SO($N_c$) gauge theory at $\mu_B \neq 0$

Sp($N_c$) gauge theory at $\mu_B \neq 0$
Holography vs. Orbifolding

Gauge/gravity duality
Maldacena (1998), ...

- strongly-coupled gauge theory
- weakly-coupled gravity theory
- AdS/QGP, AdS/QCD, AdS/CMT, ...

Orbifold equivalence
Kachru-Silverstein (1998), ...

- strongly-coupled gauge theory
- strongly-coupled gauge theory
- Cherman-Hanada-Robles-Llana;
  Cherman-Tiburzi;
  Hanada-NY; Hidaka-NY (2012)
**Holography vs. Orbifolding**

**Gauge/gravity duality**
- Maldacena (1998), ...
  - **strongly-coupled** QCD-like theory
  - **weakly-coupled** gravity theory
  - AdS/QGP, AdS/QCD, AdS/CMT, ...

**Orbifold equivalence**
- Kachru-Silverstein (1998), ...
  - QCD at $\mu_B \neq 0$
    - w/ sign problem
  - Example
    - gauge theory
      - w/o sign problem
  - Cherman-Hanada-Robles-Llana; Cherman-Tiburzi; Hanada-NY; Hidaka-NY (2012)
  - Monte Carlo
  - Exact results
Orbifold equivalence

[How to use]

1. Identify a discrete global symmetry $S$ of a theory $P$ (parent).

2. Eliminate all the d.o.f.s of parent not invariant under $S$ (projection).

3. This gives a new theory $D$ (daughter).

4. A class of observables are identical between $P$ and $D$ at large $N_c$ or MFA. Valid as long as $S$ is not broken spontaneously.

**Orbifold equivalence**

[How to use]

1. Identify a discrete global symmetry $S$ of a theory $P$ (parent).
   \[ J_c = -i\sigma_2 \times 1_{N_c} \in \text{SO}(2N_c), \quad \omega = e^{i\pi} \in U(1)_B \]

2. Eliminate all the d.o.f.s of parent not invariant under $S$ (projection).
   \[ A_\mu^{SO} = J_c A_\mu^{SO} J_c^{-1}, \quad \psi^{SO} = \omega J_c \psi^{SO} \]

3. This gives a new theory $D$ (daughter).
   \[ D = \text{SU}(N_c) \text{ QCD}. \]

4. A class of observables are identical between $P$ and $D$ at large $N_c$ or MFA.
   Valid as long as $S$ is not broken spontaneously.

Universality of phase diagrams

\[ \text{sign-free} \]

\[ \text{SO}(N_c) \text{ or } \text{Sp}(N_c) \text{ YM } + \mu_B \]

\[ \text{SU}(N_c) \text{ QCD } + \mu_B \]

\[ \text{SU}(N_c) \text{ QCD } + \mu_I \]

Hanada-NY (JHEP, 2012)
Universal sign-free nature of phase diagrams

- SO($N_c$) or Sp($N_c$) YM + $\mu_B$
  - outside diquark cond.

- SU($N_c$) QCD + $\mu_B$
  - outside pion cond.

- SU($N_c$) QCD + $\mu_I$
  - everywhere

Hanada-NY (JHEP, 2012)
Sign problem and phase quenching in finite-density QCD: models, holography, and lattice

Masanori Hanada\textsuperscript{1,2}, Yoshinori Matsuo\textsuperscript{1}, and Naoki Yamamoto\textsuperscript{3,4}

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Intuitive derivation

- Consider $\langle \bar{q}q \rangle$ in QCD at finite $\mu_I$ or $\mu_B$ with $N_f = 2$ ($u$ & $d$).

$$\langle \bar{q}q \rangle(\mu_B) = $$
Consider $\langle \bar{q}q \rangle$ in QCD at finite $\mu_I$ or $\mu_B$ with $N_f = 2$ ($u$ & $d$).
Intuitive derivation

Consider $\langle \bar{q}q \rangle$ in QCD at finite $\mu_I$ or $\mu_B$ with $N_f = 2$ ($u$ & $d$).

$$\langle \bar{q}q \rangle(\mu_B) = f(\mu_u) + f(\mu_d)$$
**Intuitive derivation**

- Consider $\langle \bar{q}q \rangle$ in QCD at finite $\mu_I$ or $\mu_B$ with $N_f = 2$ ($u$ & $d$).

\[
\langle \bar{q}q \rangle(\mu_B) = f(\mu_u) + f(\mu_d) = f(\mu_u) + f(-\mu_d) = \langle \bar{q}q \rangle(\mu_I)
\]

see also T. D. Cohen (2004)
**Intuitive derivation**

- Consider $\langle \bar{q}q \rangle$ in QCD at finite $\mu_I$ or $\mu_B$ with $N_f = 2$ ($u$ & $d$).

$$\langle \bar{q}q \rangle(\mu_B) = f(\mu_u) + f(\mu_d) = f(\mu_u) + f(-\mu_d) = \langle \bar{q}q \rangle(\mu_I)$$

see also T. D. Cohen (2004)

- For gluonic observables, equivalence holds up to $O(N_c^{-2})$. D. Toublan (2005)

- Not applicable when $u$ & $d$ are mixed $\rightarrow$ MFA outside $\pi$ condensation

- Orbifold equivalence: more systematic for a larger class of theories.
Lattice results

Pressure

\[
\frac{p}{T^4} \equiv \Omega(T, \mu_q, \mu_q) = \sum_{n=0}^{\infty} c_n(T) \left( \frac{\mu_q}{T} \right)^n
\]

Chiral condensate

\[
\frac{\langle \bar{\psi} \psi \rangle}{T^3} = \frac{N_T}{N_\sigma} \frac{\partial \ln Z}{\partial m/T} = \sum_{n=0}^{\infty} c_n^{\bar{\psi} \psi}(T) \left( \frac{\mu_q}{T} \right)^n
\]

Allton et al. (Bielefeld-Swansea collaboration), 2005, PRD
Where are QCD critical point(s)?

Summarized in
M. Stephanov, hep-lat/0701002

Baym-Hatsuda-Tachibana-NY
(PRL, 2006)
A theorem

- $\pi^\pm$ are the lightest in QCD at finite $\mu_l$  
  Son-Stephanov (2001)
  \[ \therefore \] QCD inequalities: $\pi^\pm$ propagator $\geq$ any meson propagator,
  which follows from $\tau_1 \gamma_5 D \gamma_5 T_1 = D^{\dagger}$ and Cauchy-Schwarz inequality.

- $m_\sigma \geq m_\pi > 0$ for $m_\sigma > 0$ outside $\pi$ condensation phase.

- QCD critical point ($\xi = \infty$ or $m_\sigma = 0$) is prohibited there.
  \[ \rightarrow \] Via the orbifold equivalence, it also holds in QCD at finite $\mu_B$.
  Hidaka-NY (PRL, 2012)

- The only assumption: suppression of disconn. diagrams (OZI rule)
A theorem

QCD critical point in QCD$_B$ can only be inside $\pi$ condensation of QCD$_I$ where reweighting method breaks down.
Model results \((N_f=2)\)

Random matrix model: Han-Stephanov (2008)

NJL model: Andersen-Kyllingstad-Splittorff (2009)

\[ \langle e^{2i\theta} \rangle \]

onset of pion condensation phase

\[ e^{2i\theta} = \frac{\det(D + \mu_q \gamma_0 + m)}{\det(D + \mu_q \gamma_0 + m)^*} \]

Similar result in PNJL model: Sakai-Sasaki-Kouno-Yahiro (2010)
Summary & Outlook

- Universality of phase diagrams in QCD and QCD-like theories
- Importance of disconn. diagrams beyond MFA → DS & Functional RG?
- Many more applications of the orbifold equivalence.
- Challenge for theory: physics inside $\pi$ condensation.
Back up slides
**Discussion**

- At MFA, chiral transition cannot be 1st outside $\pi$ condensation.
  
  → *only beyond-MFA effects* can make it 1st.
Discussion

- At MFA, chiral transition cannot be 1st outside $\pi$ condensation.
  $\rightarrow$ only beyond-MFA effects can make it 1st.

  competition

  quark mass weakens the chiral transition (1st $\rightarrow$ 2nd $\rightarrow$ crossover)

- At $\mu_B = 0$, chiral transition is crossover; beyond-MFA effects $<$ quark mass.

- QCD critical point is accessible on the lattice (outside $\pi$ condensation) if beyond-MFA effects are enhanced at $\mu_B \neq 0$ for some reason.
A counter example?

- Columbia plot:

- One can make the critical point on the $T$-axis (outside the pion condensation) by fine tuning $m_q \sim \text{anomaly}$.

- Ignoring disconn. diagrams: \( \text{anomaly} < m_q \)
Finite $\mu_I=2\mu$: chemical potentials $\mu$ & -$\mu$ for $u$ & $d$ quarks
→ Dirac eigenvalues for $u$ & $d$ are complex conjugate.
→ Positive fermion determinant (no sign problem)
SO & Sp gauge theories at $\mu_B > 0$

SO(2$N_c$): real $\rightarrow$ positive fermion determinant (no sign problem)
(similarly for pseudo-real Sp(2$N_c$) with even $N_f$)
From $\text{SO}(2N_c)$ to $\text{SU}(N_c)$ at finite $\mu_I$

Start with $\text{SO}(2N_c)$ gauge theory at $\mu_B > 0$.

1. Discrete symmetry:
   \[ J_c = -i\sigma_2 \otimes 1_{N_c} \in \text{SO}(2N_c) \]
   \[ J_i = -i\sigma_2 \otimes 1_{N_f/2} \in \text{SU}(2)_{\text{iso}} \]

2. Projection:
   \[ A^{\text{SO}}_{\mu} = J_c A^{\text{SO}}_{\mu} J_c^{-1}, \quad \psi^{\text{SO}} = J_c \psi^{\text{SO}} J_i^{-1} \]

3. Daughter theory: $U(N_c) \approx \text{SU}(N_c)$ gauge theory at $\mu_I > 0$.
   \[ A^{\text{proj}}_{\mu} = \begin{pmatrix} (A^U_{\mu})^C & 0 \\ 0 & A^U_{\mu} \end{pmatrix}, \quad \psi^{\text{proj}} = \begin{pmatrix} \psi^U_+ \\ \psi^- \end{pmatrix} \]

4. Orbifold equivalence: $\langle \bar{\psi}\psi \rangle^{\text{SO}} = \langle \bar{\psi}\psi \rangle^{\text{SU}}$ etc.

[Caution]
- $Z_4 \in \text{SU}(2)_{\text{iso}}$ unbroken from $\text{SO}(2N_c)$ to QCD at $\mu_I > 0$ everywhere.
Perturbative proof

Bershadsky-Johansen ('98)

- Insert $P(A^\text{SO}_\mu) = \frac{1}{2} (A^\text{SO}_\mu + J_c A^\text{SO}_\mu J_c^{-1})$ for each propagator.
- Take the same 't Hooft coupling.
- Difference comes from color factors.
- Condition: $\text{tr}(J^n_c) = 0$, when $J^n_c \neq \pm 1_{2N_c}$

$$\sum_{n_i=0,1} \left( \frac{1}{2} \right)^{N_P} \cdot \text{tr}(J^{-n_1} J^{n_4} J^{n_5}) \cdot \text{tr}(J^{-n_2} J^{-n_4} J^{n_6}) \cdot \text{tr}(J^{-n_3} J^{-n_5} J^{-n_6}) \cdot \text{tr}(J^{n_1} J^{n_2} J^{n_3}) = 2^{-6} \cdot 2^{6-3} \cdot 2^4 = 2$$

Generally, $2^{-N_P} \cdot 2^{N_P-(N_L-1)} \cdot 2^{N_L} = 2$ for any planar diagrams.
What are (aren’t) equivalent?

Not all the quantities are equivalent in the orbifold equivalence.

- Projection symmetry must be unbroken.
- Observables must keep the projection symmetry (neutral).
- Symmetry breaking patterns, quantum numbers of the condensates can be different, but their magnitudes are the same.
- Example: BCS gap (inside the BEC-BCS crossover)

\[
\begin{align*}
\Delta_{\mu_B}^{SU} &\sim \mu \exp \left( -\frac{\pi^2}{g} \sqrt{\frac{6N_c}{N_c + 1}} \right) \quad \to \quad 0 \\
\Delta_{\mu_I}^{SU} &\sim \mu \exp \left( -\frac{\pi^2}{g} \sqrt{\frac{6N_c}{N_c^2 - 1}} \right) \\
\Delta_{\mu_B}^{SO} &\sim \mu \exp \left( -\frac{\pi^2}{g} \sqrt{\frac{12}{2N_c - 1}} \right) \\
\Delta_{\mu_B}^{Sp} &\sim \mu \exp \left( -\frac{\pi^2}{g} \sqrt{\frac{12}{2N_c + 1}} \right)
\end{align*}
\]

‘t Hooft limit (large $N_c$, $g^2N_c$ fixed)
QCD inequality

- Correlation function:

\[ C_{\Gamma}(x, y) \equiv \langle M_{\Gamma}(x) M_{\Gamma}^\dagger(y) \rangle_{\psi, A} \]

\[ = -\langle \text{tr}[S_A(x, y) \Gamma S_A(y, x) \bar{\Gamma}] \rangle_A \]

\[ = \langle \text{tr}[S_A(x, y) \Gamma \gamma_5 S_A^\dagger(x, y) \gamma_5 \Gamma] \rangle_A \]

\[ \leq \sqrt{\langle \text{tr}[S_A S_A^\dagger] \rangle_A} \sqrt{\langle \text{tr}[\Gamma \gamma_5 S_A^\dagger \gamma_5 \Gamma (\Gamma \gamma_5 S_A^\dagger \gamma_5 \Gamma)^\dagger] \rangle_A} \]

\[ = \langle \text{tr}[S_A(x, y) S_A^\dagger(x, y)] \rangle_A, \]

Inequality saturated when \( \Gamma = i\gamma_5 \).

- Cauchy-Schwarz inequality:

\[ \text{tr}(AB^\dagger) \leq \sqrt{\text{tr} AA^\dagger} \sqrt{\text{tr} BB^\dagger} \]