Is there a critical end point in the QCD phase diagram?

Is it connected to a chiral phase transition?

Imaginary chemical potential: rich phase structure, benchmark for models!
The conference location...
The conference location...

...a critical point for Rhine navigation
The QCD phase diagram established by experiment:

Nuclear liquid gas transition with critical end point

15 MeV

~940 MeV

$\mu_B$
QCD phase diagram: theorist’s view

QGP and colour SC at asymptotic $T$ and densities by asymptotic freedom!

Until 2001: no finite density lattice calculations, sign problem!

Expectation based on models: NJL, NJL+Polyakov loop, linear sigma models, random matrix models, ...
The Monte Carlo method, zero density

QCD partition fcn: \[ Z = \int DU \prod_f \det M(\mu_f, m_f; U) e^{-S_{gauge}(\beta; U)} \]

links=gauge fields

lattice spacing \( a \ll \) hadron \( \ll L \) !

thermodynamic behaviour, large \( V \) !

Monte Carlo by importance sampling

Continuum limit: \( N_t \to \infty, a \to 0 \)

Here: \( N_t = 4 - 10 \) \( \quad a \sim 0.1 - 0.3 \) fm

staggered fermions
Theory: how to calculate p.t., critical temperature

definition/chiral phase transition $\rightarrow$ quark gluon plasma

“order parameter”:
chiral condensate $\langle \bar{\psi} \psi \rangle$

generalized susceptibilities:
$\chi = V(\langle O^2 \rangle - \langle O \rangle^2)$

$\Rightarrow \chi_{max} = \chi(\beta_c) \Rightarrow T_c$

only pseudo-critical on finite $V$!
Order of transition:
finite volume scaling
$\chi_{max} \sim V^\sigma$

$\sigma = 1$ 1st order
$\sigma = \text{crit. exponent}$ 2nd order
crossover

$\sigma = 0$ crossover
The order of the p.t., arbitrary quark masses \( \mu = 0 \)

**deconfinement p.t.:**
breaking of global \( Z(3) \) symmetry

**chiral p.t.:**
restoration of global symmetry in flavour space

\[ SU(2)_L \times SU(2)_R \times U(1)_A \]

**anomalous**

**crossover**

**1st**

**2nd order \( O(4) \) ?**

**2nd order \( Z(2) \)**

**N\(_{\text{f}}\) = 2**

**Pure Gauge**

**deconfinement critical line**

**chiral critical line**
How to identify the order of the phase transition

$$B_4(\bar{\psi}\psi) \equiv \frac{\langle(\delta \bar{\psi}\psi)^4\rangle}{\langle(\delta \bar{\psi}\psi)^2\rangle^2} \quad \mu \rightarrow \infty \quad \begin{cases} 1.604 & 3d \text{ Ising} \\ 1 & \text{first-order} \\ 3 & \text{crossover} \end{cases}$$

$$\mu = 0: \quad B_4(m, L) = 1.604 + bL^{1/\nu}(m - m_0^c), \quad \nu = 0.63$$

Parameter along phase boundary, $T = T_c(x)$
Hard part: order of p.t., arbitrary quark masses $\mu = 0$

- Physical point: crossover in the continuum
  - $N_f = 2$ chiral $O(4)$ vs. 1st still open
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- Chiral critical line on $N_t = 4, a \sim 0.3$ fm

- Consistent with tri-critical point at $m_{u,d} = 0, m_{s}^{\text{tric}} \sim 2.8T$

- But: $N_f = 2$ chiral $O(4)$ vs. 1st still open
  - $U_A(1)$ anomaly!

- $Aoki$ et al. 06
- de Forcrand, O.P. 07
- $N_f = 2+1$
- Physical point
- $m_s^{\text{tric}} - C m_{u,d}^{2/5}$
- $N_t = 4, a \sim 0.3 \text{ fm}$
- Consistent with tri-critical point at $m_{u,d} = 0, m_{s}^{\text{tric}} \sim 2.8T$
- But: $N_f = 2$ chiral $O(4)$ vs. 1st still open
  - $U_A(1)$ anomaly!
- $Di Giacomo$ et al. 05, Kogut, Sinclair 07
- Chandrasekharan, Mehta 07
The nature of the transition for phys. masses

...in the staggered approximation...in the continuum...is a crossover!
The ‘sign problem’ is a phase problem

\[ Z = \int DU \left[ \det M(\mu) \right]^f e^{-S_g[U]} \]

importance sampling requires positive weights

Dirac operator: \[ \bar{D}(\mu) = \gamma_5 D(-\mu^*) \gamma_5 \]

\[ \Rightarrow \det(M) \text{ complex for SU(3), } \mu \neq 0 \]
\[ \Rightarrow \text{real positive for SU(2), } \mu = i \mu_i \]
\[ \Rightarrow \text{real positive for } \mu_u = -\mu_d \]

N.B.: all expectation values real, imaginary parts cancel, but importance sampling config. by config. impossible!

Same problem in many condensed matter systems!
Finite density: methods to evade the sign problem

- **Reweighting:**
  \[ Z = \int DU \det M(0) \frac{\det M(\mu)}{\det M(0)} e^{-S_g} \]
  \(~\exp(V)\) statistics needed, overlap problem

  Optimal: use \(|\det|\) in measure, reweight in phase

- **Taylor expansion:**
  \[ \langle O \rangle(\mu) = \langle O \rangle(0) + \sum_{k=1} \frac{\mu^k}{\pi^k T^k} \]

- **Imaginary** \(\mu = i\mu_i\): no sign problem, fit by polynomial, then analytically continue
  \[ \langle O \rangle(\mu_i) = \sum_{k=0}^N c_k \left( \frac{\mu_i}{\pi T} \right)^{2k}, \quad \mu_i \to -i\mu \]

All require \(\mu/T < 1\)!
Comparing approaches: the critical line

$N_t = 4, N_f = 4$; same actions (unimproved staggered), same mass

$\beta, a\mu, \mu/T, T/T_c$ diagrams showing

- Imaginary $\mu$
- 2 param. imag. $\mu$
- Double reweighting, LY zeros
- Same, susceptibilities canonical

Agreement for $\mu/T \lesssim 1$

Test of methods: comparing $T_c(\mu)$

de Forcrand, Kratochvila 05
The calculable region of the phase diagram

- need $\mu/T \lesssim 1$ ($\mu = \mu_B/3$)
- Upper region: equation of state, screening masses, quark number susceptibilities etc. under control
The (pseudo-) critical temperature

\[ \frac{T_c(\mu)}{T_c(0)} = 1 - \kappa(N_f, m_q) \left( \frac{\mu}{T} \right)^2 + \ldots \]

Curvature rather small

\[ \kappa \propto \frac{N_f}{N_c} \]  Toublan 05

de Forcrand, O.P. 03  
D’Elia, Lombardo 03
Phase boundary in the chiral limit

\[ \frac{\chi_{m,q}}{T} = \frac{\partial^2 \langle \bar{\psi} \psi \rangle T^3}{\partial (\mu_q/T)^2} = \frac{2\kappa T}{t_0 m_s} h^{-(1-\beta)/\beta \delta} \frac{df_G(z)}{dz} \]

scaling form of magnetic EoS

\[ t \equiv \frac{1}{t_0} \left( \frac{T - T_c}{T_c} + \kappa_q \left( \frac{\mu_q}{T} \right)^2 \right) \]

\[ h \equiv \frac{1}{h_0} \frac{m_l}{m_s} \]

\[ z = \frac{h^{1/\beta \delta}}{t} \]

- Curvature of crit. line from Taylor expansion 2+1 flavours, Nt=4, 8 improved staggered
- Extrapolation to chiral limit assuming O(4),O(2) scaling of magn. EoS
- Consistent with determinations at finite mass with imag. chem. pot.

\[ \frac{T_c(\mu)}{T_c(0)} = 1 - 0.059(2)(4) \left( \frac{\mu}{T} \right)^2 + O \left( \left( \frac{\mu}{T} \right)^4 \right) \]
Phase boundary for physical masses

Curvature of crit. line from Taylor expansion
2+1 flavours, Nt=6,8,10 improved staggered

Observables \( \bar{\psi} \psi_r, \chi_s \)

Continuum extrapolation:

\[
\kappa(\bar{\psi} \psi_r) = 0.0066(20) \quad \kappa(\chi_s/T^2) = 0.0089(14)
\]

\[
\kappa(T; N_t) = \kappa(T_c; \text{cont}) + c_0 \cdot t + c_1 \cdot t^2 + c_2/N_t^2 + c_3 \cdot t/N_t^2
\]

\[
t = \frac{T - T_c}{T_c}
\]
Comparison with freeze-out curve

$T_c(\mu)$ considerably flatter than freeze-out curve (factor $\sim 3$ in $\left. \frac{d^2 T_c}{d\mu^2} \right|_{\mu=0}$)
Much harder: is there a QCD critical point?
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Two strategies:
1. follow vertical line: $m = m_{\text{phys}}$, turn on $\mu$
Much harder: is there a QCD critical point?

Two strategies:
1 follow vertical line: $m = m_{\text{phys}}$, turn on $\mu$
2 follow critical surface: $m = m_{\text{crit}}(\mu)$
$N_t = 4, N_f = 2 + 1$ physical quark masses, unimproved staggered fermions

Lee-Yang zero:

$\left( \mu_E^q, T_E \right) = (120(13), 162(2))$ MeV

abrupt change: physics or problem of the method?

Splittorf 05, Stephanov 08
Approach 1b: CEP from Taylor expansion

\[ \frac{p}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left( \frac{\mu}{T} \right)^{2n} \]

Nearest singularity = radius of convergence

\[ \frac{\mu_E}{T_E} = \lim_{n \to \infty} \sqrt{\left| \frac{c_{2n}}{c_{2n+2}} \right|}, \quad \lim_{n \to \infty} \left| \frac{c_0}{c_{2n}} \right|^{\frac{1}{2n}} \]

Gavai, Gupta

\( N_f = 2 \)

Bielefeld-Swansea-RBC improved staggered

\( N_t = 4 \)
Predictivity?

Different definitions agree only for \( n \to \infty \) not \( n=1,2,3,... \)
CEP may not be nearest singularity, control of systematics?

\[
\rho_n[\chi_B] = \sqrt{\frac{n(n-1)}{(n+1)(n+2)}} \rho_n[p/T^4]
\]

Hadron resonance gas

C. Schmidt, hotQCD 09

Radius of convergence necessary condition, but can it proof the existence of a CEP?
Approach 2: follow chiral critical line

\[ m_c(\mu) = m_c(0) \left( 1 + \sum_{k=1}^{\infty} c_k \left( \frac{\mu}{\pi T} \right)^{2k} \right) \]

1. Tune quark mass(es) to \( m_c(0) \): 2nd order transition at \( \mu = 0, T = T_c \) known universality class: 3d Ising

2. Measure derivatives \( \frac{d^k m_c}{d\mu^{2k}} \bigg|_{\mu=0} \):

   Turn on imaginary \( \mu \) and measure \( \frac{m_c(\mu)}{m_c(0)} \)

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dee Forcrand, O.P. 08,09
Finite density: chiral critical line \(\rightarrow\) critical surface

Finite density:

- Chiral critical line
- Critical surface

Phys. point

- \(N_f = 2\)
- \(N_f = 3\)
- \(N_f = 1\)

\[ m_{c}(\mu) = m_{c}(0) + \sum_{k=1}^{c_1 > 0} c_k \left( \frac{\mu}{\pi T} \right)^{2k} \]

- Standard scenario: transition strengthens
- Exotic scenario: transition weakens

Real world

- Heavy quarks

\[ \mu \]

Heavy quarks

- \(m > m_{\gamma}(0)\)

QGP

- Confined
- Color superconductor

Pure Gauge

\(2nd\) order

- O(4)?
- Z(2)

1st order

\(QCD\) critical point

\(crossover\)

\(1st\) order

\(Z(2)\)

\(2nd\) order

\[ T_c \]

\[ m > m_{\gamma}(0) \]
Curvature of the chiral critical surface

\[ N_f = 3 \]

\[ N_f = 2 + 1, \ m_s = m_s^{\text{phys}} \]

consistent $8^3 \times 4$ and $12^3 \times 4$, $\sim 5 \times 10^6$ traj.

\[
\frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3) \left( \frac{\mu}{\pi T} \right)^2 - 47(20) \left( \frac{\mu}{\pi T} \right)^4 - \ldots
\]

\[ 8\text{th derivative of } P \]

16$^3 \times 4$, Grid computing, $\sim 10^6$ traj.

\[
\frac{m_c^{ud}(\mu)}{m_c^{ud}(0)} = 1 - 39(8) \left( \frac{\mu}{\pi T} \right)^2 - \ldots
\]

de Forcrand, O.P. 08,09
On coarse lattice exotic scenario: no chiral critical point at small density

Weakening of p.t. with chemical potential also for:

- Heavy quarks  
  de Forcrand, Kim, Takaishi 05

- Light quarks with finite isospin density  
  Kogut, Sinclair 07

- Electroweak phase transition with finite lepton density  
  Gynther 03
Towards the continuum: \( N_t = 6, a \sim 0.2 \text{ fm} \)

\[
\frac{m_c^c(N_t = 4)}{m_c^c(N_t = 6)} \approx 1.77 \quad N_f = 3
\]

Physical point deeper in crossover region as \( a \to 0 \)

Cut-off effects stronger than finite density effects!

Preliminary: curvature of chiral crit. surface remains negative de Forcrand, O.P. 10

No chiral critical point at small density, other crit. points possible
Same statement with different methods

Study suitably defined width of crossover region

\[
\frac{1}{W} \frac{\partial W}{\partial (\mu^2)} = - \frac{1}{T_c} \frac{\partial \kappa}{\partial T} \Bigg|_{T=T_c}
\]

Endrödi et al., 11 find weakening of crossover

strengthening of transition
Chiral/deconf. and $\mathbb{Z}(3)$ transitions at imaginary $\mu$

Nf=4: D’Elia, Di Renzo, Lombardo 07  
Nf=2: D’Elia, Sanfilippo 09  
Nf=3: de Forcrand, O.P. 10

Strategy: fix $\frac{\mu_i}{T} = \frac{\pi}{3}, \pi$, measure $\text{Im}(L)$, order parameter at $\frac{\mu_i}{T} = \pi$

determine order of $\mathbb{Z}(3)$ branch/end point as function of $m$
Results:

\[ B_4(\beta, L) = B_4(\beta_c, \infty) + C_1(\beta - \beta_c) L^{1/\nu} + C_2(\beta - \beta_c)^2 L^{2/\nu} \ldots \]

B4 at intersection has large finite size corrections (well known), \( \nu \) more stable
\( \nu = 0.33, 0.5, 0.63 \)

for 1st order, tri-critical, 3d Ising scaling

On infinite volume, this becomes a step function, smoothness due to finite \( L \)

Phase diagram at fixed \( \frac{\mu_i}{T} = \frac{\pi}{3}, \pi \)
Critical lines at imaginary $\mu$

$\mu = 0$

$\mu = i \frac{\pi T}{3}$

- Connection computable with standard Monte Carlo!
- Here: heavy quarks in eff. theory
Critical surfaces

shape, sign of curvature determined by tric. scaling!

Similar chiral crit. surface: tric. line renders curvature negative!
$m \to \infty$: QCD $\to$ theory of Polyakov lines $\to$ universality class of 3d 3-state Potts model (3d Ising, $Z(2)$)

small $\mu/T$: sign problem mild, doable for real $\mu$!

de Forcrand, Kim, Kratochvila, Takaishi

Potts:

QCD, $N_t=1$, strong coupling series: Langelage, O.P. 09

\[
m_c / T \left( \frac{\mu}{T} \right)^2 = m_{tric} / T + K \left[ \left( \frac{\pi}{3} \right)^2 + \left( \frac{\mu}{T} \right)^2 \right]^{2/5}
\]

tri-critical scaling: exponent universal
Conclusions

- Reweighting, Taylor: indications for critical point, systematics?
- Chiral crit. surface, deconfinement crit. surface: Transitions *weaken with chemical potential*
- For lattices $a \approx 0.3, 0.2$ fm no chiral critical point for $\mu/T \lesssim 1$
- Still possible: chiral critical point at large chemical potential, non-chiral critical point(s)

\[ \text{cf. Ejiri 0812.1534} \rightarrow \left( \frac{\mu}{T} \right)_{\text{CEP}} \sim 2.4 \]