- I would like to thank Colleagues in the World for Encouraging and Warm Voices, and many countries for the help.
- In West Japan, Osaka, Hiroshima, Fukuoka etc., as far as I know all Universities have no damage.
- In Tokyo, people are fine, but shortage of Electricity and Water
- At KEK, the network was down(now restarted), and workshops this week are cancelled, but people are fine.
- At RIKEN, all are fine.
- At J-PARC, there are damages due to the earthquake, but the Tunami was blocked, and all people are fine.
- At Tohoku, people of Nuclear Physics group, and ELPH lab are OK.
- Tokyo Univ. Computer center decided to stop their large cluster system because of the electric cut.
- We are worrying about the Nuclear reactors in Fukushima.

Finite Density QCD Simulations with Wilson Fermions

Keitaro NAGATA (Univ. Tokyo)
Atsushi NAKAMURA (Hiroshima Univ.)

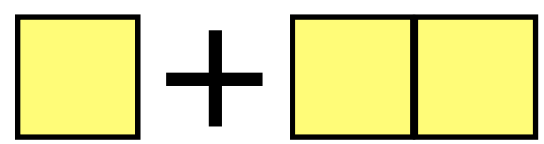
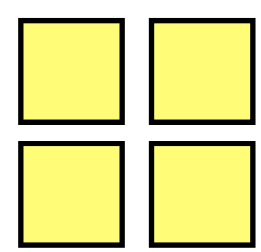
Quarks, Gluons, and Hadronic Matter under Extreme Conditions
15. Mar. 2011
St. Goar, Germany

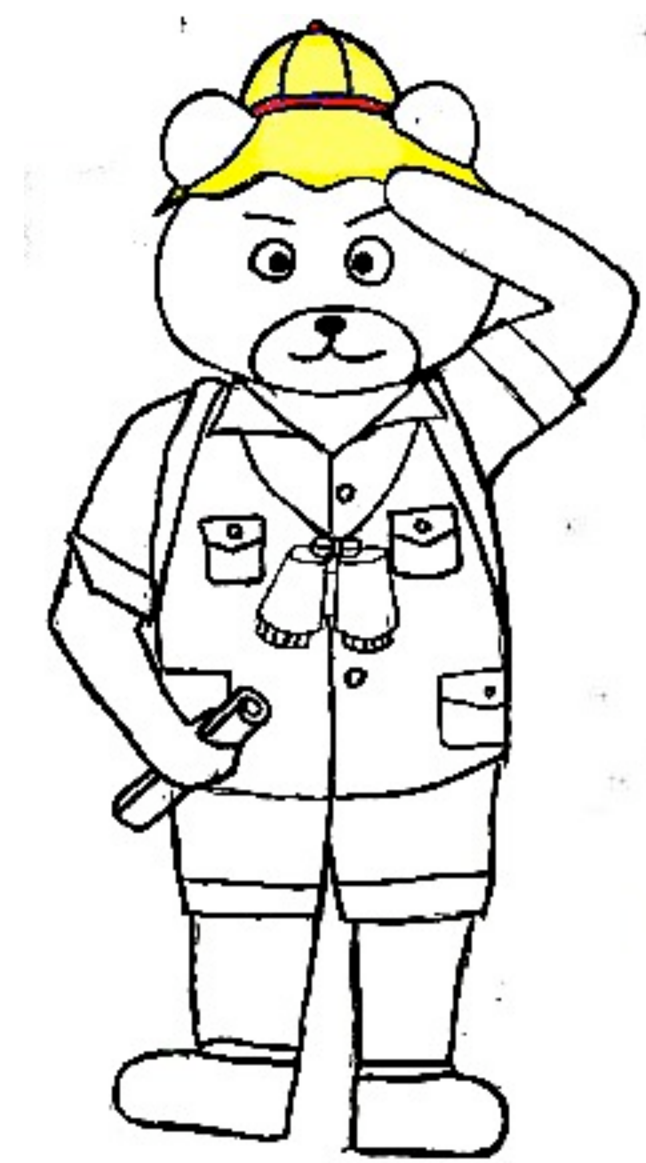
XQCD-J

1. Introduction
2. Reduction Formula
3. Imainary Chemical Potential

Study QCD at finite density !

Status Report

- Lattice QCD with Wilson Fermions
  - Improved actions
    - Gluonic part: 
    - Quark part: 
- Real and Imaginary Chemical Potentials



QCD at Finite T and ρ

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} \qquad \beta \equiv 1/kT$$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi$$
$$-\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

$$Z = \int \prod d\bar{\psi(x)}d\psi(x)dA_{\mu}(x)$$
$$\times e^{-\int_0^{\beta}(\mathcal{L}_{\mathrm{Gluon}}+\bar{\psi}\Delta\psi)}$$

$$Z = \int \prod dA_\mu(x) \det \Delta \, e^{-S_{\mathrm{Gluon}}}$$

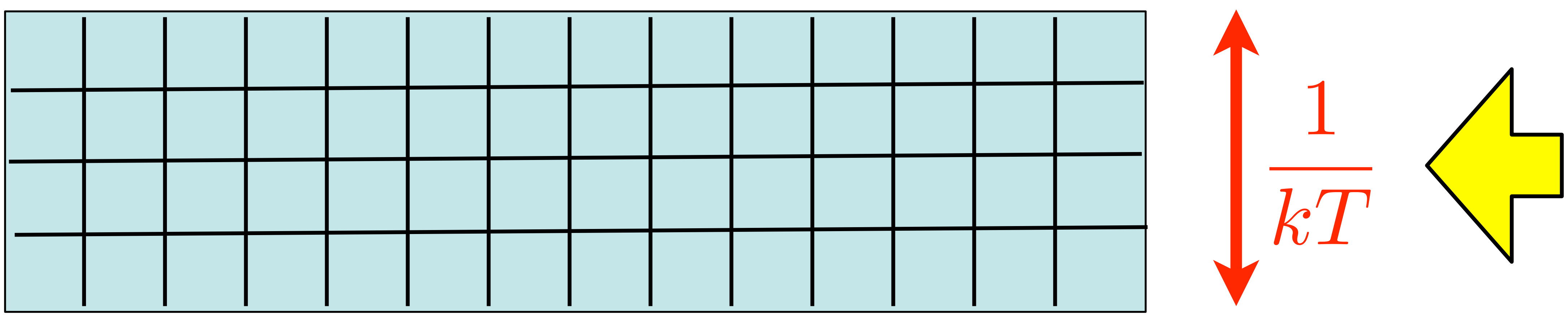
Quark Matrix $\Delta = i\gamma^\mu D_\mu - m - \mu\gamma_0$

Space-Time ➡ Lattice

$$Z = \int \prod dA_\mu(x) \det \Delta \, e^{-S_{\mathrm{Gluon}}}$$

Quark Matrix $\Delta = i\gamma^\mu D_\mu - m - \mu\gamma_0$

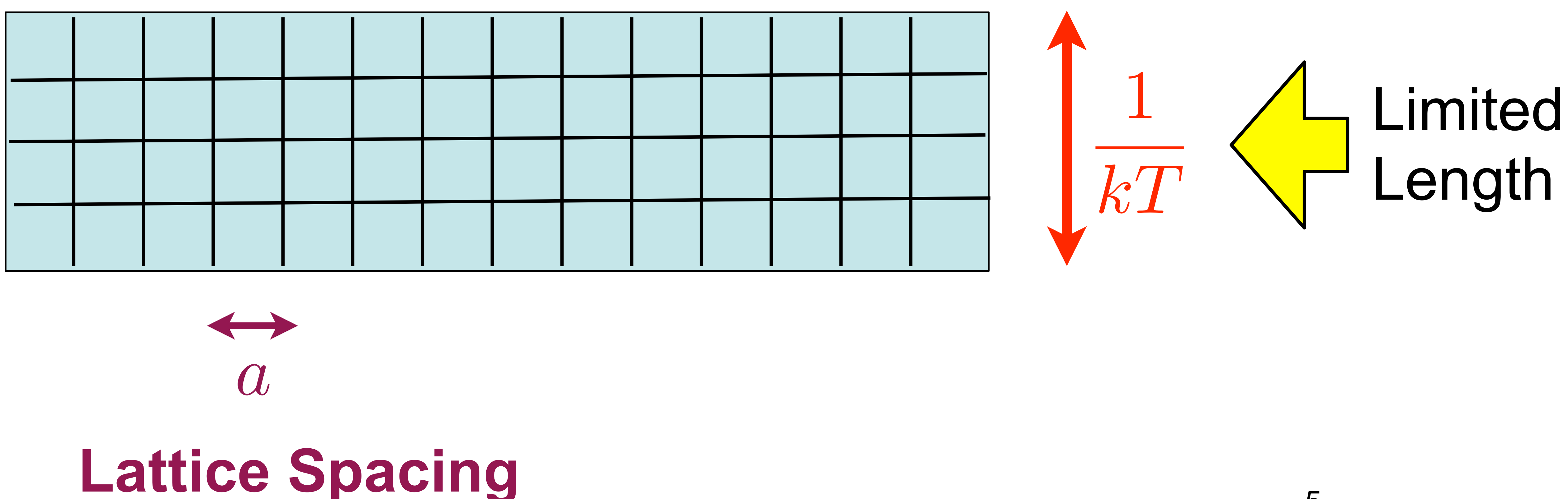
Space-Time ⇒ Lattice



$$Z = \int \prod dA_\mu(x)\,\det\Delta\,e^{-S_{\mathrm{Gluon}}}$$

Quark Matrix $\Delta = i\gamma^\mu D_\mu - m - \mu\gamma_0$

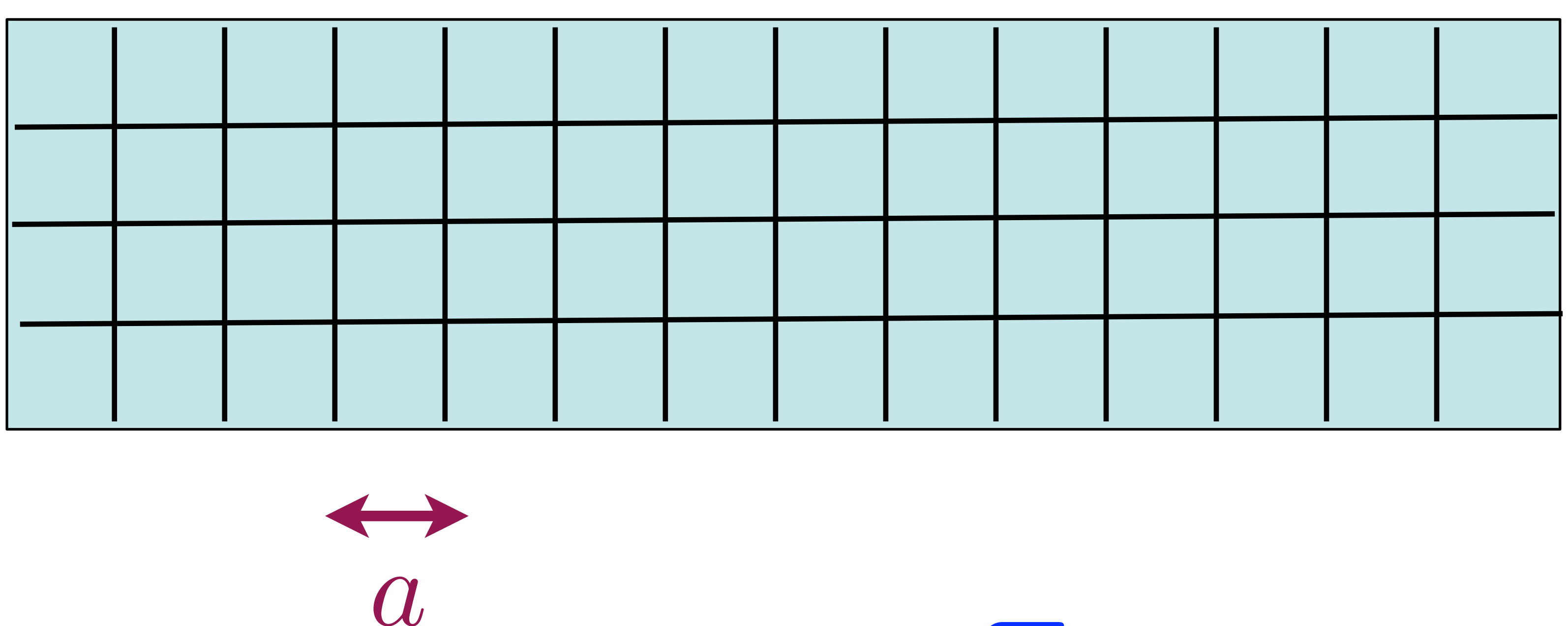
Space-Time ➡ Lattice



$$Z = \int \prod dA_\mu(x) \det \Delta \, e^{-S_{\mathrm{Gluon}}}$$

Quark Matrix $\Delta = i\gamma^\mu D_\mu - m - \mu\gamma_0$

Space-Time → Lattice



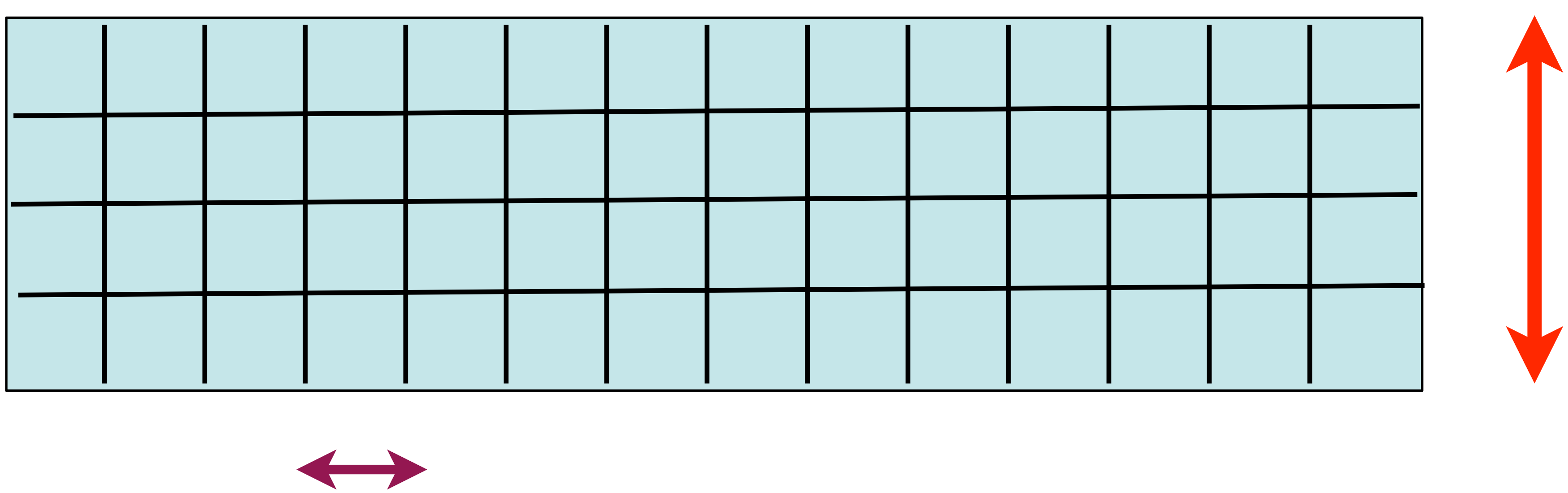
$\frac{1}{kT}$ ← Limited Length

Lattice Spacing $\frac{\pi}{a}$

$$Z = \int \prod dA_\mu(x) \det \Delta \, e^{-S_{\mathrm{Gluon}}}$$

Quark Matrix $\Delta = i\gamma^\mu D_\mu - m - \mu\gamma_0$

Space-Time → Lattice



$\frac{1}{kT}$ ← Limited Length

a

Lattice Spacing

$\frac{\pi}{a}$ Momentum Cut-Off of this field theory

Today's Talk

Still far from
the Real World.

- Small Lattice Size

  $4^4 \qquad 8^3 \times 4$

- Heavy quarks Mass

  $$\frac{m_\pi}{m_\rho} \sim 0.8$$

Sign Problem
in Finite Density QCD

Monte Carlo
Impossibe ?!

Finite Density QCD

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\beta S_G - \bar{\psi}\Delta\psi}$$

$$= \int \mathcal{D}U \prod_f \det \Delta(m_f)\, e^{-\beta S_G}$$

$$\Delta(\mu) = D_\nu \gamma_\nu + m + \mu\gamma_0$$

$$\Delta(\mu)^\dagger = -D_\nu\gamma_\nu + m + \mu^*\gamma_0 = \gamma_5 \Delta(-\mu^*)\gamma_5$$

$$(\det \Delta(\mu))^* = \det \Delta(\mu)^\dagger = \det \Delta(-\mu^*)$$

$(\det\Delta(\mu))^* = \det\Delta(\mu)^\dagger = \det\Delta(-\mu^*)$

For $\mu = 0$

$$(\det\Delta(0))^* = \det\Delta(0)$$

$\det\Delta$ ➡ *Real*

For $\mu \neq 0$ (in general)

$\det\Delta$ ➡ *Complex*

$$Z = \int \mathcal{D}U \prod_f \det\Delta(m_f, \mu_f) e^{-\beta S_G}$$

⬆

Complex ➡ Sign Problem

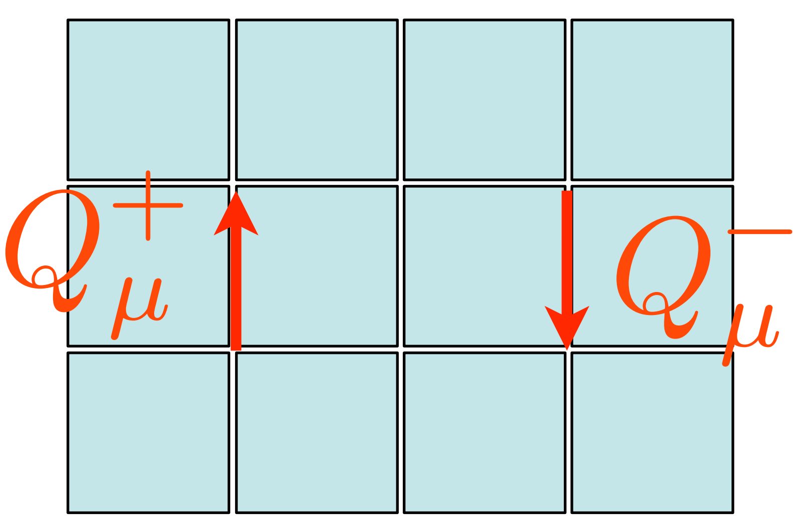
Physical Origin of Sign Problem

Wilson Fermions $\Delta = I - \kappa Q$

KS(Staggered) Fermions $\Delta = m - Q'$

$$= m(I - \frac{1}{m}Q)$$

$$Q = \sum_{i=1}^{3} (Q_i^+ + Q_i^-) + (e^{+\mu} Q_4^+ + e^{-\mu} Q_4^-)$$



$$Q_\mu^+ = **U_\mu(x)\delta_{x',x+\hat{\mu}}$$

$$Q_\mu^- = **U_\mu^\dagger(x')\delta_{x',x-\hat{\mu}}$$

$$\det \Delta = e^{\mathrm{Tr}\log \Delta} = e^{\mathrm{Tr}\log(I - \kappa Q)}$$

$$= e^{-\sum_n \frac{1}{n}\kappa^n \mathrm{Tr} Q^n}$$

Only closed loops survive.

Lowest

μ **-dependent terms**

$$\kappa^{N_t} e^{\mu N_t} \mathrm{Tr}(Q^+ \cdots Q^+)$$

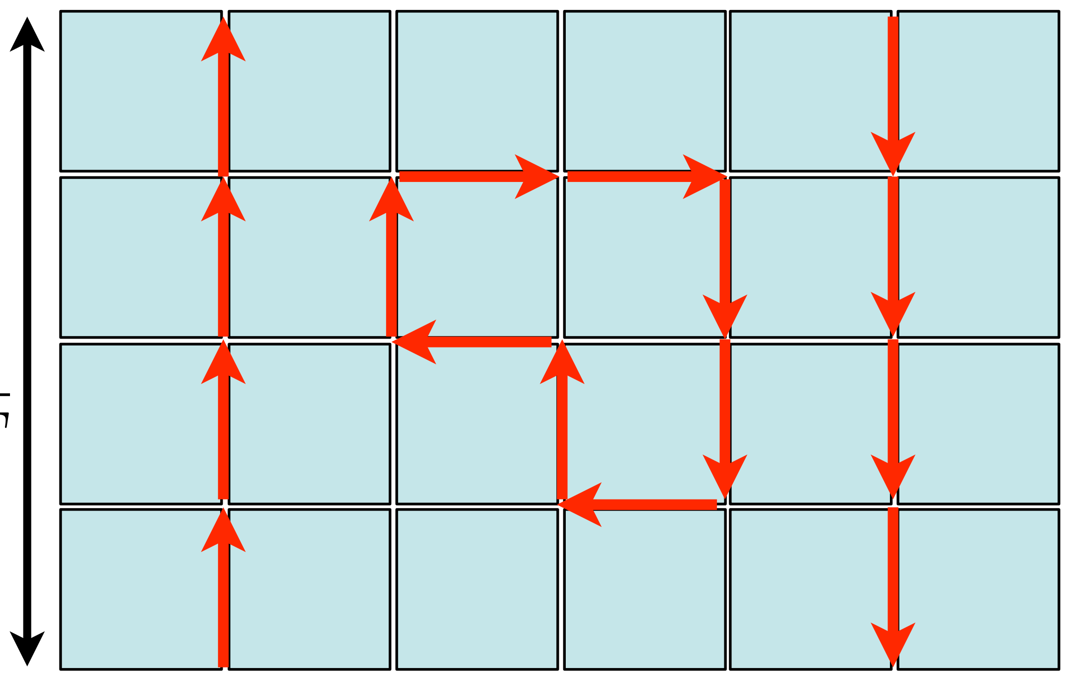
$$= ** \kappa^{N_t} e^{\mu/T} \mathrm{Tr} L$$

$$\kappa^{N_t} e^{-\mu N_t} \mathrm{Tr}(Q^- \cdots Q^-)$$

$$= ** \kappa^{N_t} e^{-\mu/T} \mathrm{Tr} L^\dagger$$

$\mathrm{Tr}L$: Polyakov Loop

$N_t = \frac{1}{T}$



Add both terms

$$** \kappa^{N_t} \left(\cosh \frac{\mu}{T} \Re \mathrm{Tr} L + i \sinh \frac{\mu}{T} \Im \mathrm{Tr} L \right)$$

There are cases where No Sign Problem

Think and then You can easily find them.

Really ?

They are not our final Solution, but informative.

No-Sign-Problem Cases

1. Imaginary Chemical Potential

$$(\det \Delta(\mu))^* = \det \Delta(-\mu^*)$$

$$\mu = i\mu_I \quad \Rightarrow \quad (\det \Delta(\mu_I))^* = \det \Delta(\mu_I)$$

2. Color SU(2) $\quad U_\mu^* = \sigma_2 U_\mu \sigma_2$

$$\det \Delta(U, \gamma_\mu)^* = \det \Delta(U^*, \gamma_\mu^*) = \det \sigma_2 \Delta(U, \gamma_\mu^*) \sigma_2$$

$$= \det \Delta(U, \gamma_\mu)$$

3. Iso-Vector Type（finite iso-spin)

$$\mu_d = -\mu_u$$

$$\det \Delta(\mu_u) \det \Delta(\mu_d) = \det \Delta(\mu_u) \det \Delta(-\mu_u)$$

$$= \det \Delta(\mu_u) \det \Delta(\mu_u)^* = \left| \det \Delta(\mu_u) \right|^2$$

(Phase Quench)

1. Introduction
2. Reduction Formula
3. Imainary Chemical Potential

Studies of Finite Density QCD (SU(3)) with Wilson Fermions

- H.-S. Chen, X.-Q. Luo
  - Phys.Rev. D72 (2005) 0345041
  - hep-lat/0411023
- A.Li, X. Meng, A. Alexandru, K-F. Liu
  - PoS LAT2008:032 and 178 (arXiv:0810.2349, arXiv: 0811.2112)
- C. Gattringer and L. Liptak
  - arXiv:0906.1088
- J. Danzer, C. Gattringer, L. Liptak and M. Marinkovic
  - arXiv:0907.3084 and LAT2009: 185 (2009) (arXiv: 0910.3541)

In the finite density lattice QCD,

- we should often handle the fermion determinant, directly,
  - e.g.
    - Multi-parameter Re-weighting by Fodor-Katz

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \, O \, \det \Delta(\mu) \, e^{-\beta S_G}$$

$$= \frac{1}{Z} \int \mathcal{D}U \, O \, \underbrace{\det \Delta(0) \, e^{-\beta_0 S_G}}_{\text{Measure}} \, \underbrace{\frac{\det \Delta(\mu)}{\det \Delta(0)} \, e^{(\beta_0 - \beta) S_G}}_{\text{Reweighting Factor}}$$

Measure

Reweighting Factor

For KS Fermions, a Trick behind

- Gibbs Formula(*)
  - P.E.Gibbs, Phys.Lett. B172 (1986) 53-61

$$\det \Delta = z^{-N} \begin{vmatrix} -B(-V) - z & 1 \\ -V^2 & -z \end{vmatrix}$$

$$= \left| \begin{pmatrix} BV & 1 \\ -V^2 & 0 \end{pmatrix} - zI \right|$$

$$= \det (P - zI)$$

$$= \prod (\lambda_i - z)$$

P

$$z \equiv e^{-\mu}$$

- P is $(2 \times N_c \times N_x \times N_y \times N_z)^2$ (Matrix Reduction)
- Determinant for any value of μ

*) A similar formula was developped by Neuberger (1997) for a chiral fermion and applied by Kikukawa(1998).

A Reduction Formula for Wilson Fermions

- Keitaro Nagata and Atsushi Nakamura
  Wilson Fermion Determinant in Lattice QCD
  Phys. Rev. D82,094027 (arXiv:1009.2149)
- A. Alexandru and U. Wenger
  arXiv:1009.2197
- Budapest-Wuppertal group also obtained
  a similar result.

The same matrix transformation like KS case cannot be employed, due to the fact that

$r \pm \gamma_4$ have no inverse, if the Wilson term $r = 1$.

Gibbs started to multiply V to the fermion matrix Δ .

Instead, we multiply $P = (c_a r_- + c_b r_+ V z^{-1})$

Here,

$$V = \left(\begin{array}{c|c|cccc|c} 0 & U_4(t=1) & 0 & \cdots & & & 0 \\ \hline 0 & 0 & U_4(t=2) & \cdots & & & 0 \\ \hline 0 & 0 & 0 & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ & & & \cdots & U_4(t=N_t-2) & & 0 \\ \hline 0 & 0 & & \cdots & 0 & & U_4(t=N_t-1) \\ \hline -U_4(t=N_t) & 0 & & \cdots & 0 & & 0 \end{array} \right)$$

c_a and c_b are arbitary non-zero numbers.

$$\det P = (c_a c_b z^{-1})^{N/2}$$

if we take the following trick, Borici (2004)

$$r_+ r_- = \frac{r^2 - 1}{4} = \epsilon \to 0$$

where $r_\pm \equiv \dfrac{r \pm \gamma_4}{2}$

After very long calculation (See Nagata-Nakamura arXiv:1009.2149), we get

$$\det \Delta(\mu) = (c_a c_b)^{-N/2} z^{-N/2}$$

$$\times \left(\prod_{i=1}^{N_t} \det(\alpha_i) \right) \det \left(z^{N_t} + Q \right)$$

$$\frac{\det \Delta(\mu)}{\det \Delta(0)} = \frac{\det(\xi + Q)}{\det(1+Q)}$$

$$\xi \equiv e^{-\mu/T}$$ (fugacity)

Q is $(4N_cN_xN_yN_z) \times (4N_cN_xN_yN_z)$ matrix.

No N_t !

($(2N_cN_xN_yN_z) \times (2N_cN_xN_yN_z)$ ⇐ KS case)

Diagonalize Q,

$$Q \rightarrow \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_{N_{red}} \end{pmatrix}$$

$$\det(\xi + Q) = \prod(\xi + \lambda_n)$$

λ_n does not depend on μ.

⇒ Once we calculate λ_n, we can evaluate $\det\Delta(\mu)$ for any μ.

Symmetry in Eigen Values

$$(\det \Delta(\mu))^* = \det \Delta(-\mu^*)$$

$$\det \Delta(\mu) \propto \prod (\xi + \lambda_n)$$

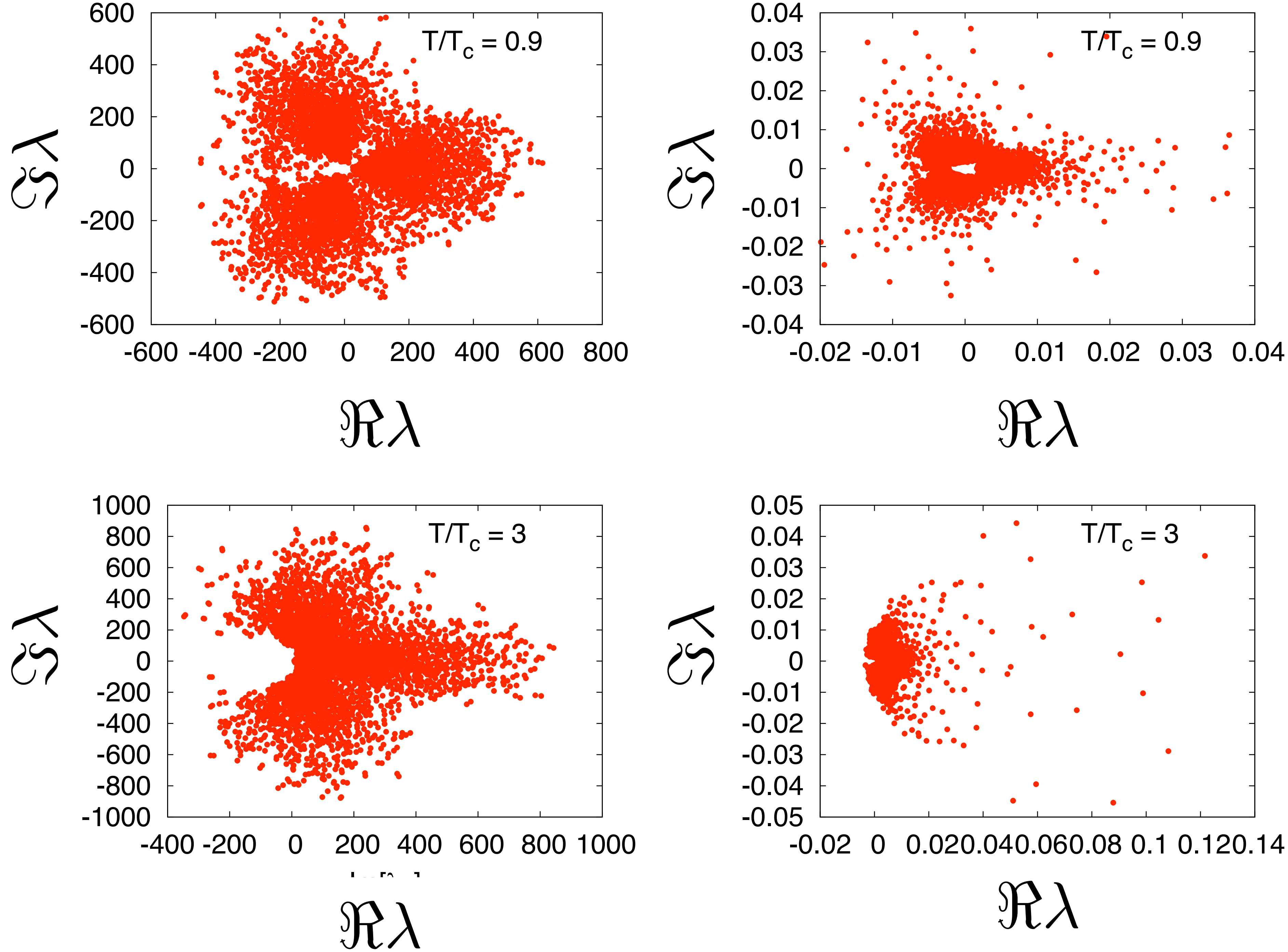
$$\xi \equiv e^{-\mu/T}$$

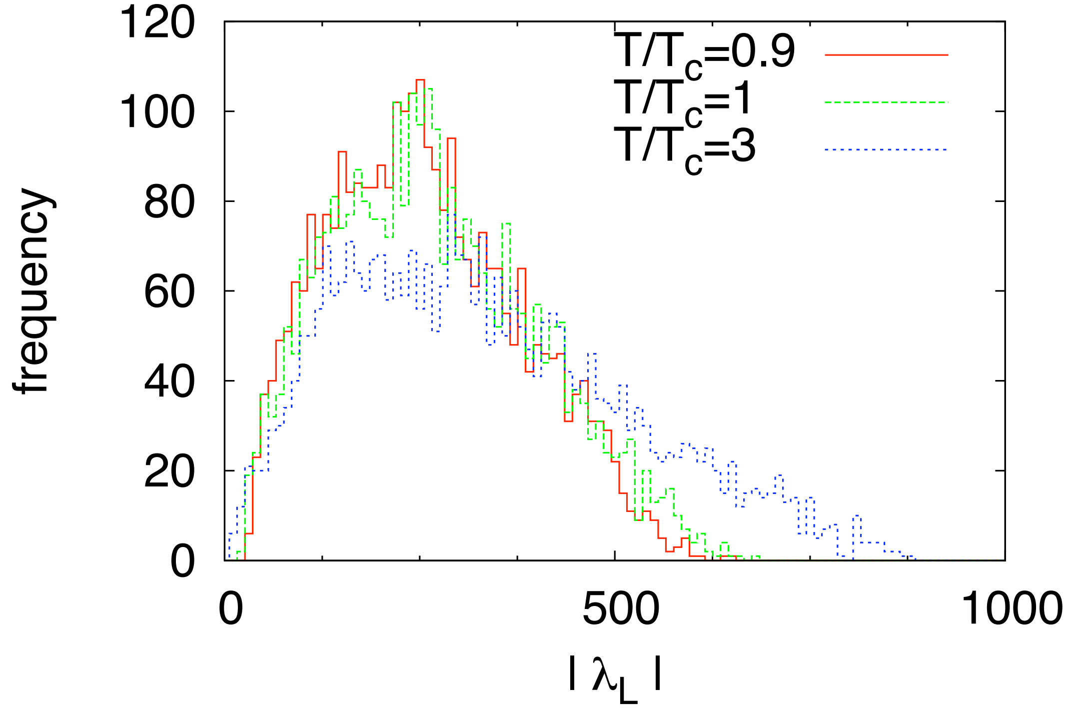
$$\lambda_{n'} = \frac{1}{\lambda_n^*}$$

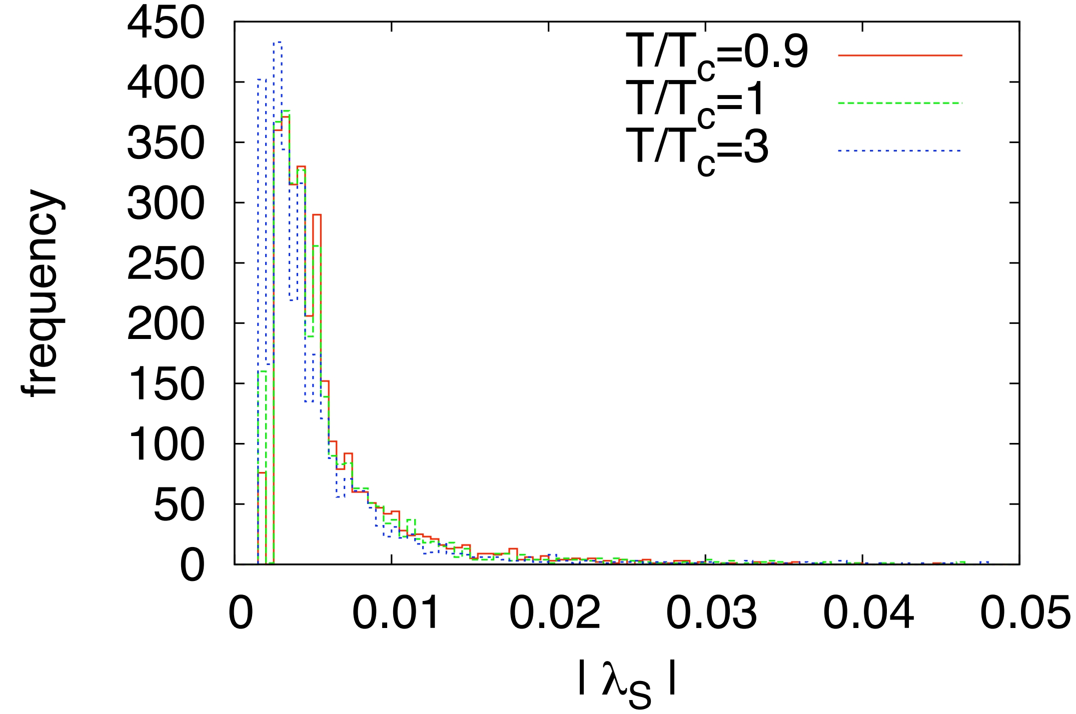
Eigen Value Distributions

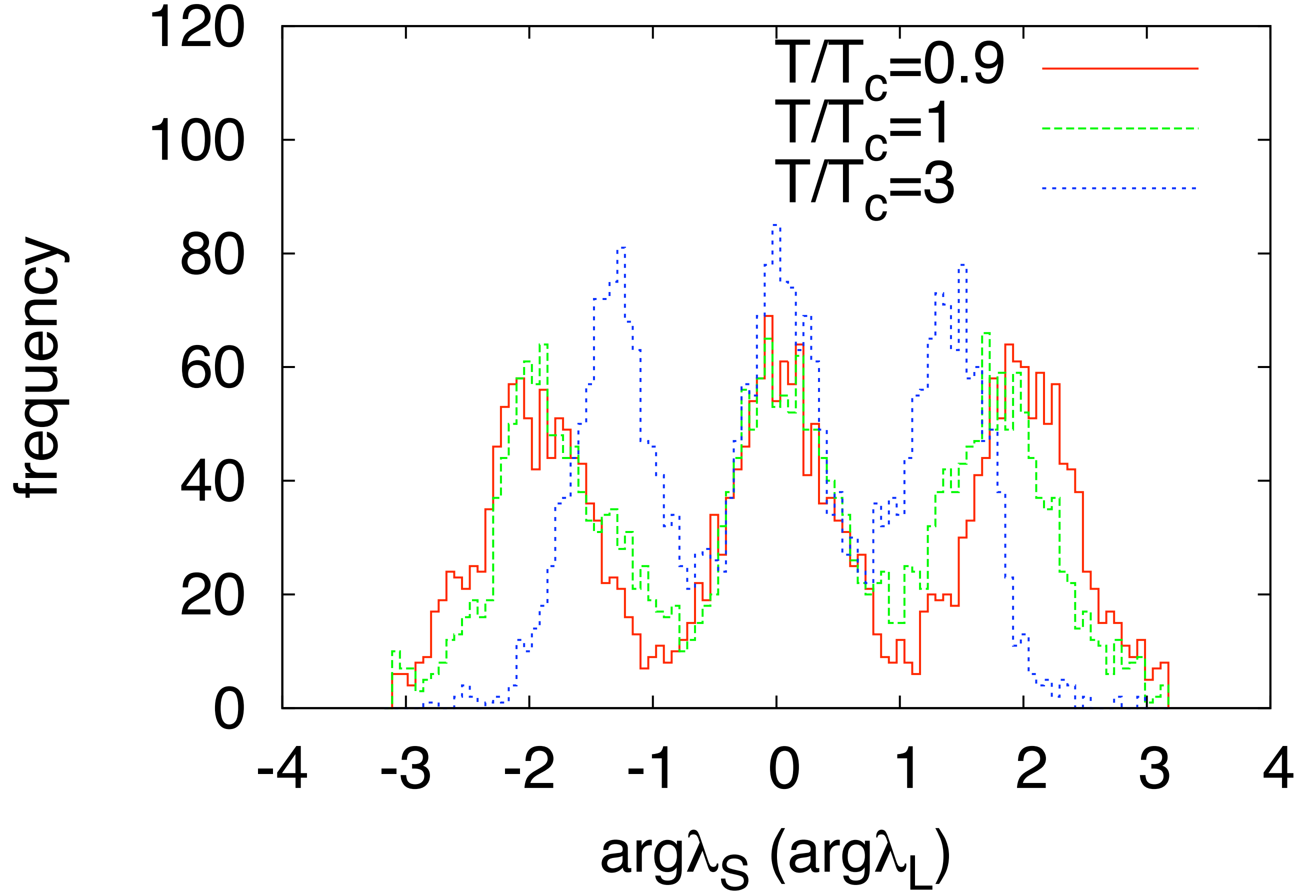
Large Eigenvalues

Small Eigenvalues




T/T_c=0.9
T/T_c=1
T/T_c=3
frequency
| λ_L |
0
20
40
60
80
100
120
0
500
1000


T/T_c=0.9
T/T_c=1
T/T_c=3
frequency
| λ_S |
0
50
100
150
200
250
300
350
400
450
0
0.01
0.02
0.03
0.04
0.05

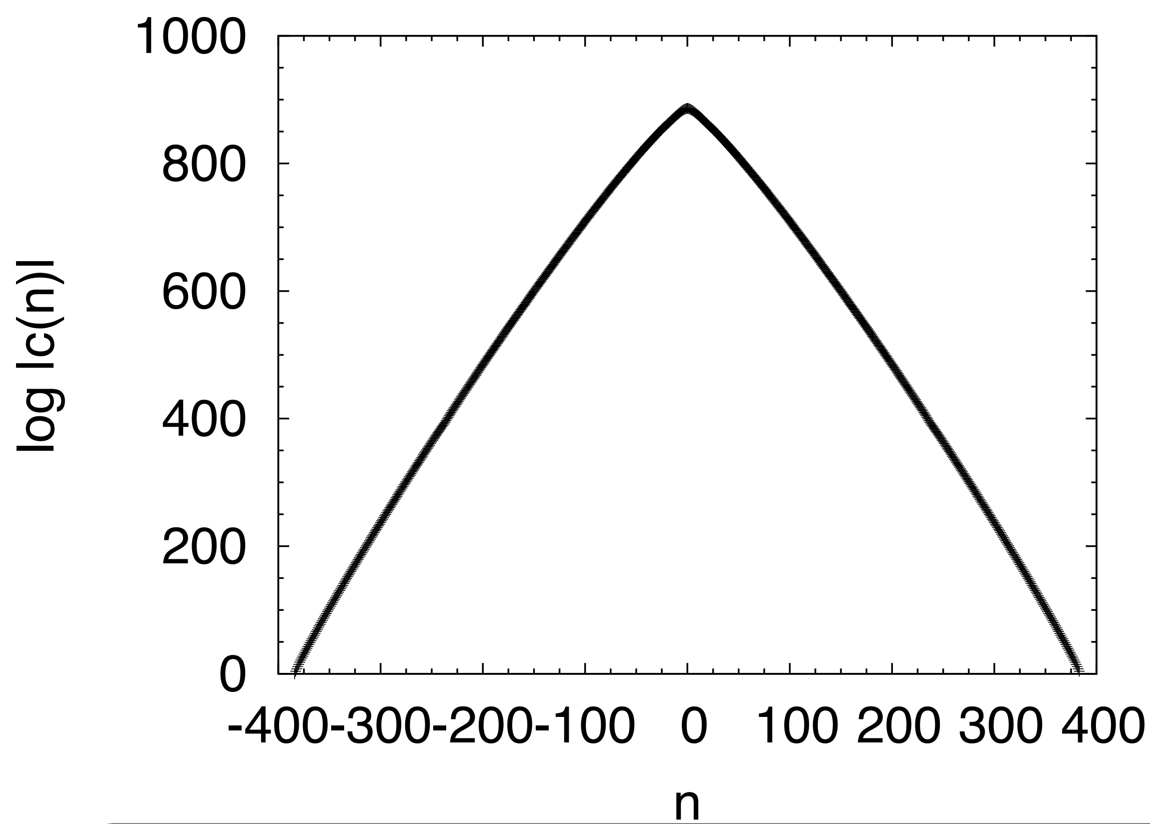

T/T_c=0.9
T/T_c=1
T/T_c=3
frequency
argλ_S (argλ_L)
0
20
40
60
80
100
120
-4
-3
-2
-1
0
1
2
3
4

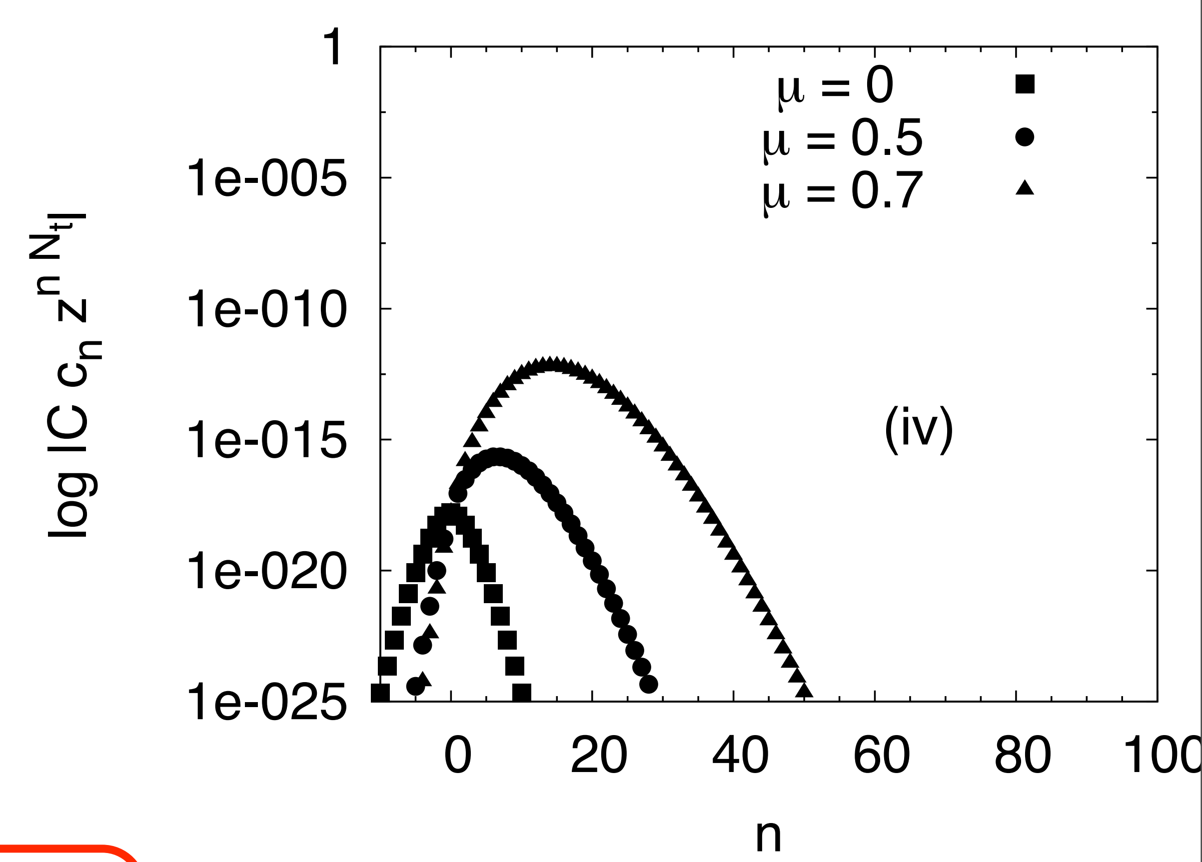
$$\det(\xi + Q) = \prod(\xi + \lambda_k) = \sum C_n \xi^n$$

$$\beta = 1.85 \quad 4^4$$

$\log|C_n|$

$\log|C_n|\left(e^{\mu/T}\right)^n$

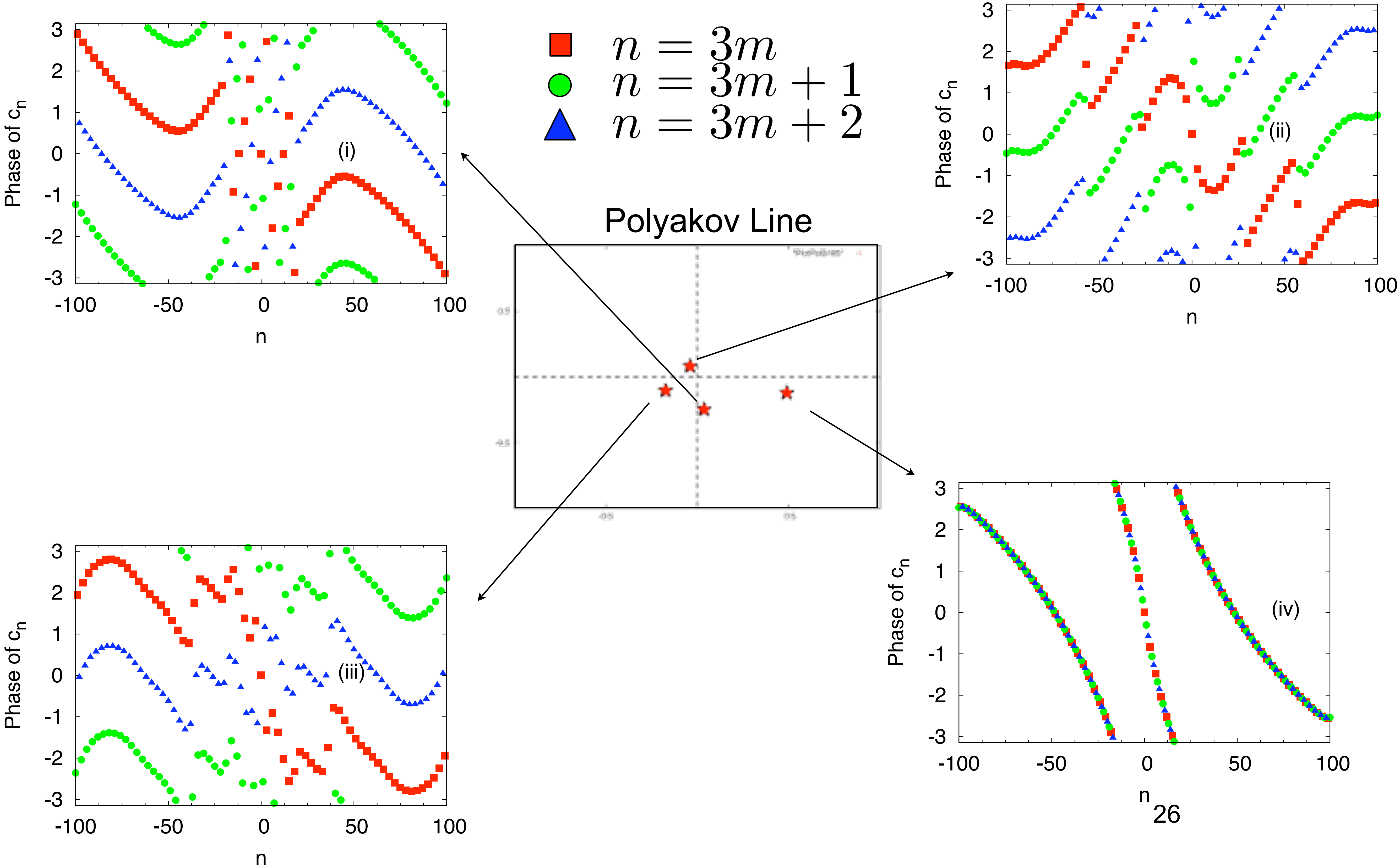




To handle these very wide range numbers, we need special care on the accuracy.

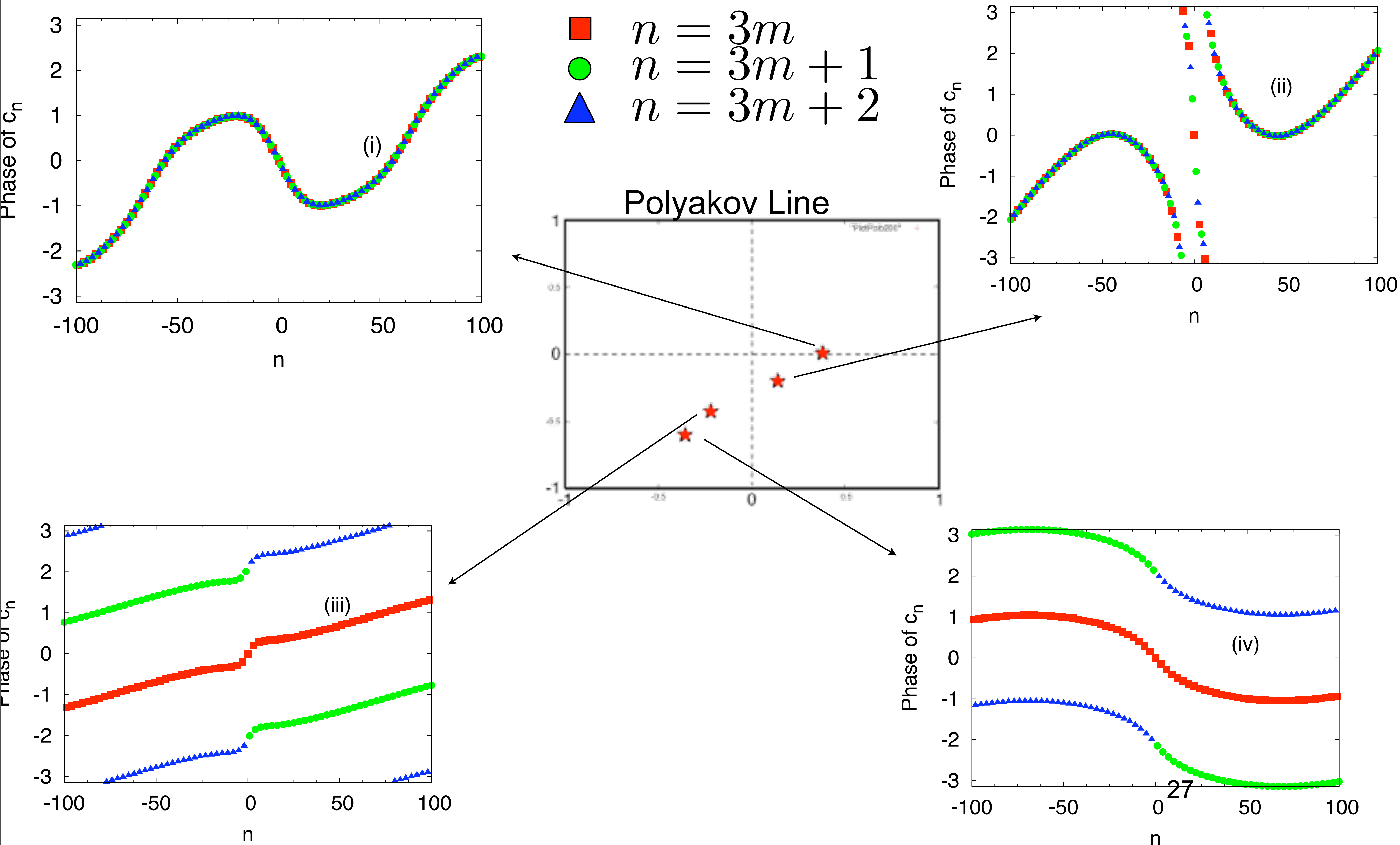
Phase of C_n

$$4^4 \quad \beta = 1.85 \quad (T \sim T_c)$$



Phase of

$$4^4 \quad \beta = 2.0 \qquad (T > T_c)$$



Reweighting Factor

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \, O \, \det \Delta(\mu) \, e^{-\beta S_G}$$

$$= \frac{1}{Z} \int DU O \det{}^2\Delta(0) e^{-\beta_0 S_G} \mathcal{R}.\mathcal{F}. = \frac{< O \times R.F. >_0}{< R.F. >_0}$$

Measure Reweighting Factor

Here

$$\mathcal{R}.\mathcal{F}. = \left(\frac{\det \Delta(\mu)}{\det \Delta(0)} \right)^2 \times e^{(\beta_0 - \beta) S_G} \equiv e^{2\theta} e^{F} e^{G}$$

Reweighting Factor should be "LARGE".

"LARGE" ? It is a function of *U*.

Large Contribution ?

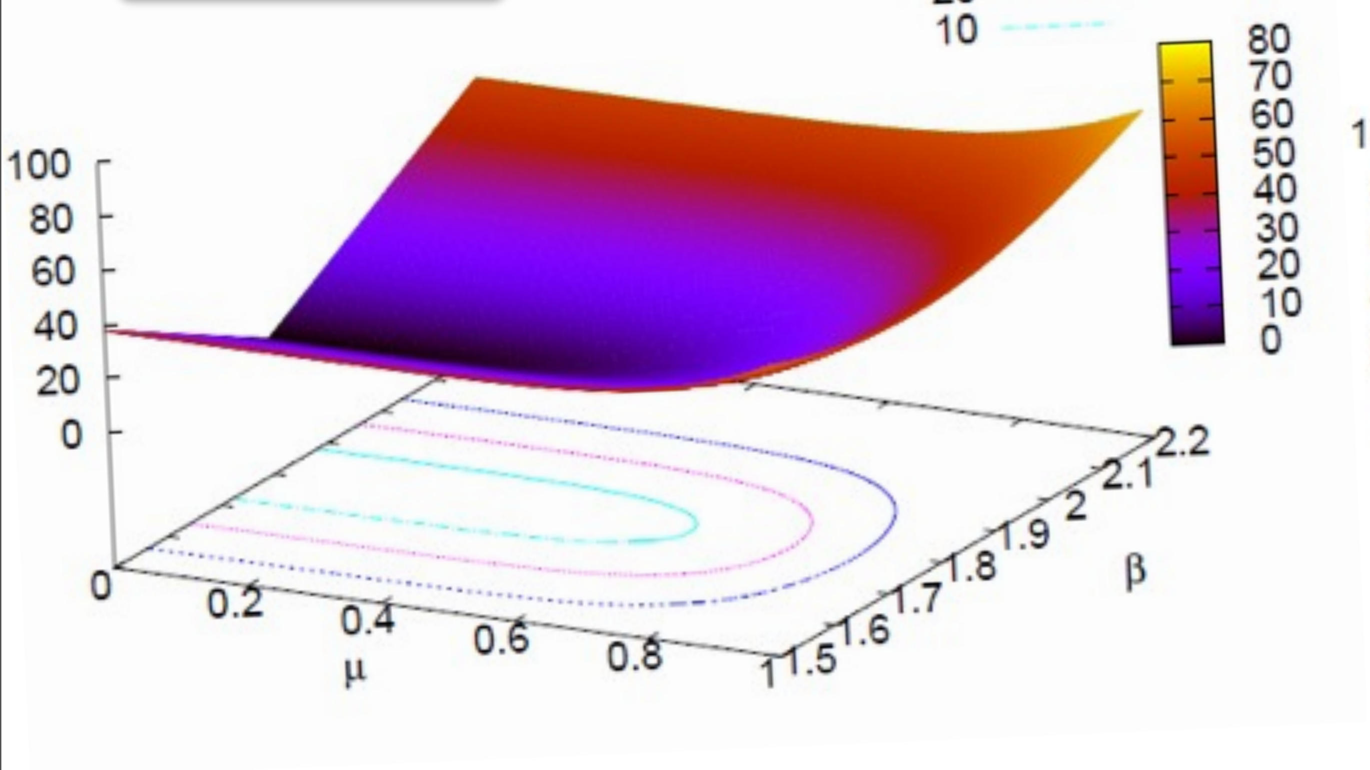
Small Fluctuation ?

S.Ejiri, Phys.Rev. D69 (2004) 094506
hep-lat/0401012

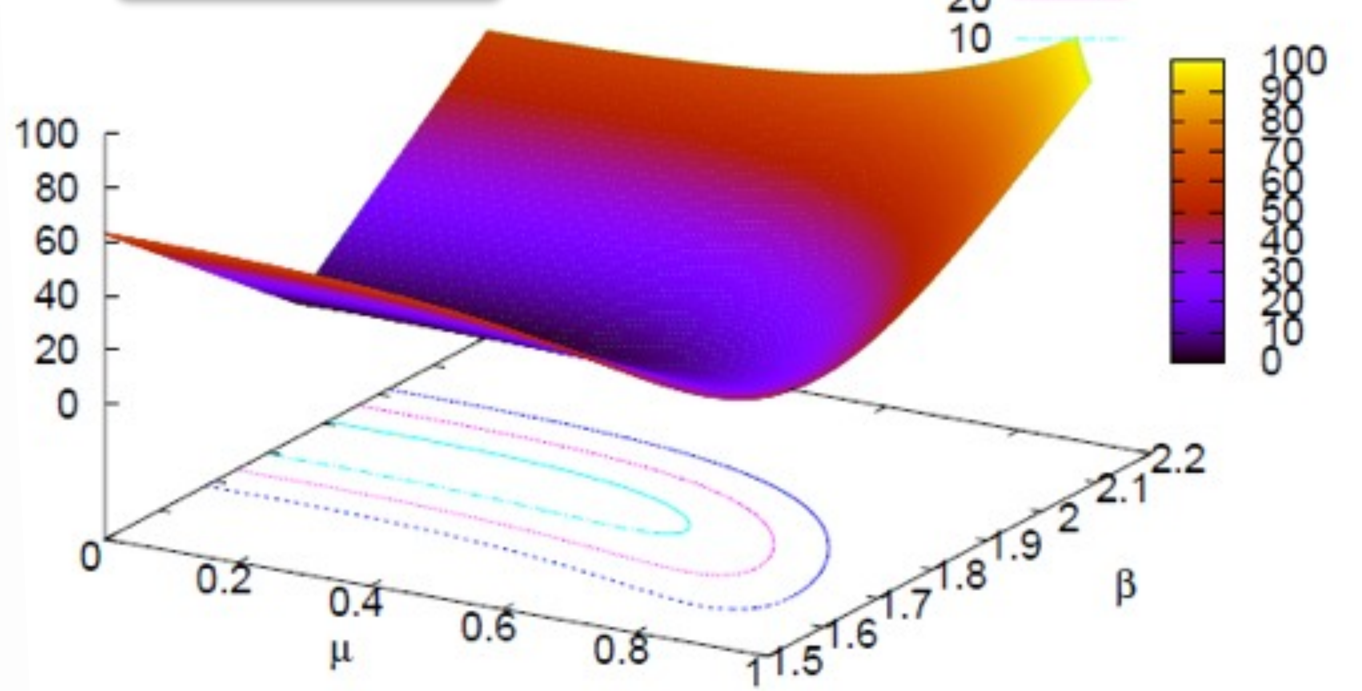
$$((F + G) - \langle F + G \rangle)^2 \qquad 8^3 \times 4$$

Preliminary

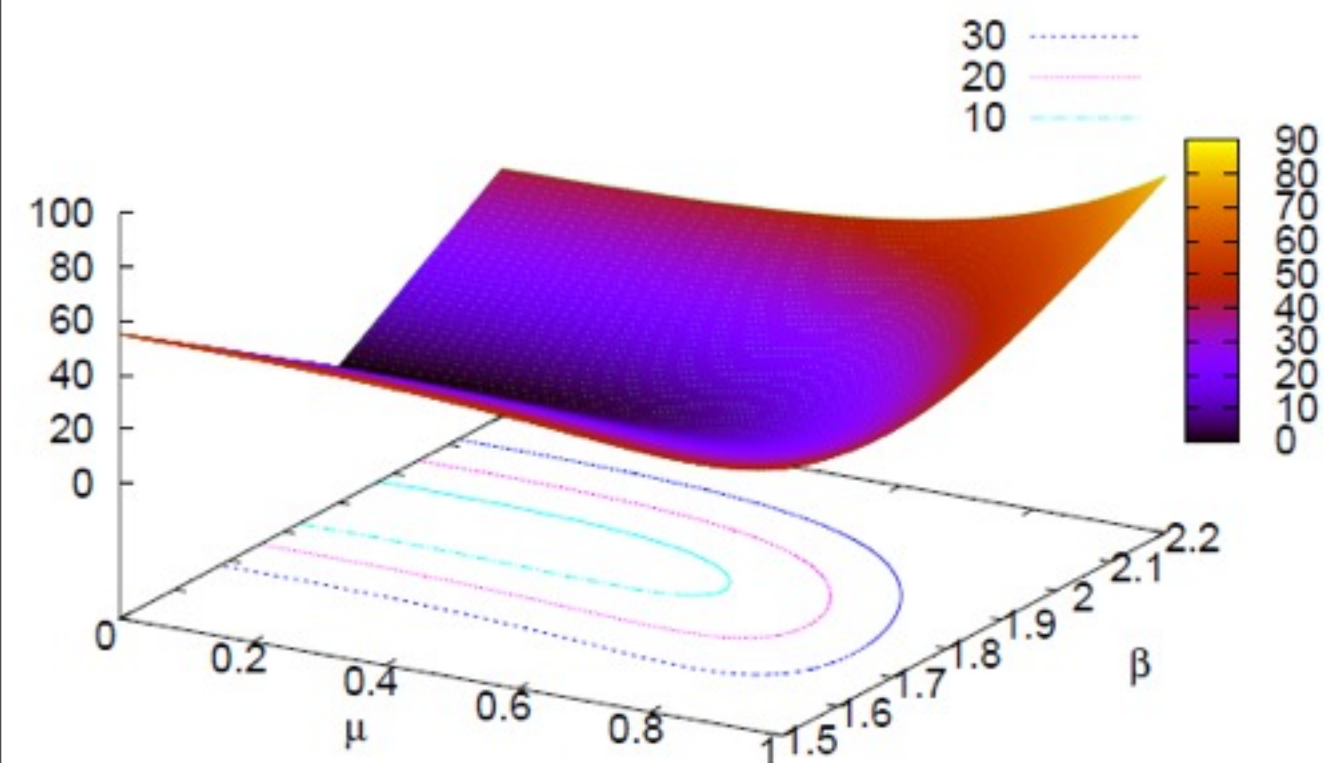
β0=1.80



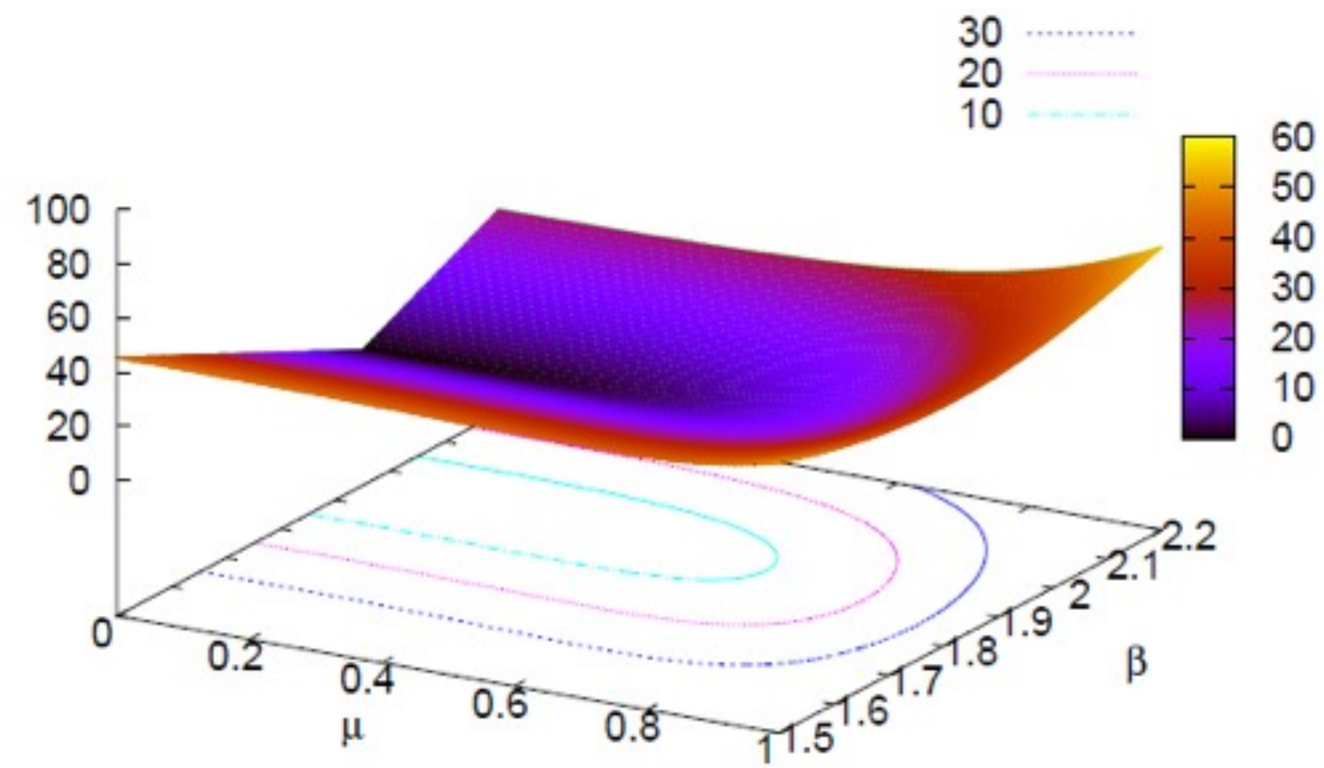
β0=1.85



β0=1.90



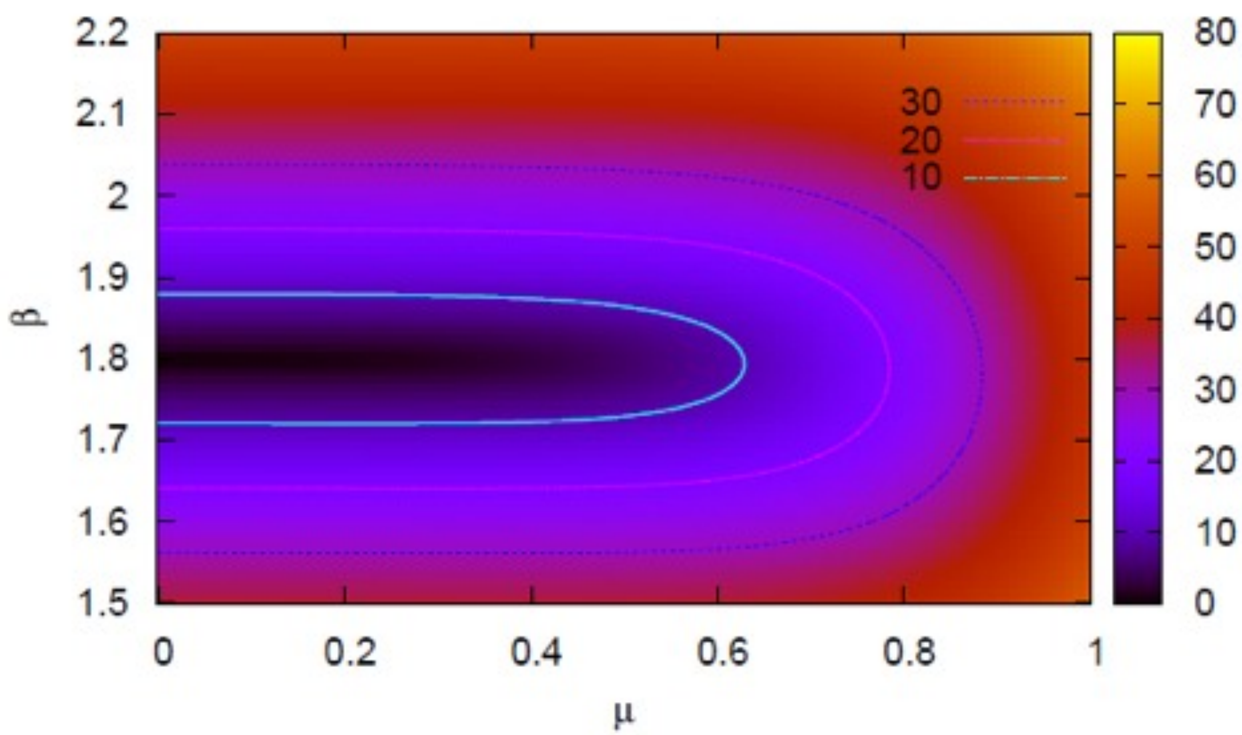
β0=1.95



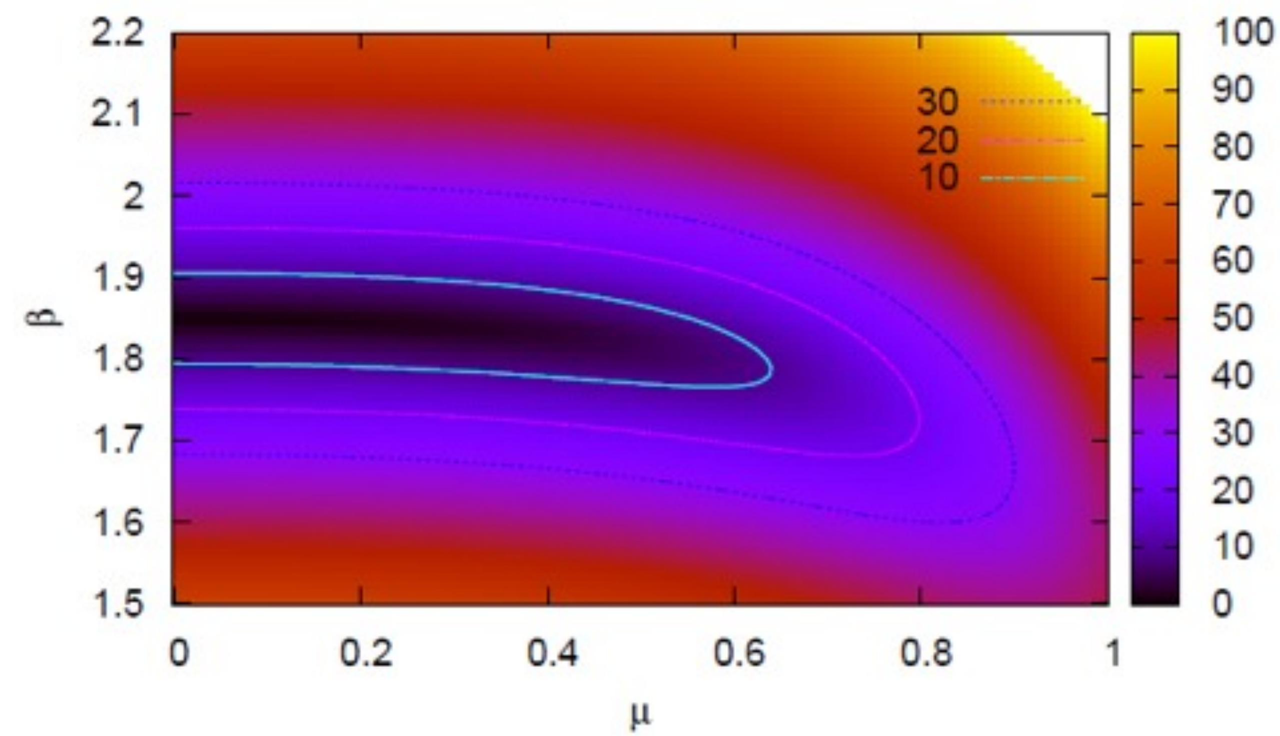
$$\langle\ (\ (F+G) - \langle F+G \rangle\ )^2 \rangle \quad 8^3 \times 4$$

Preliminary

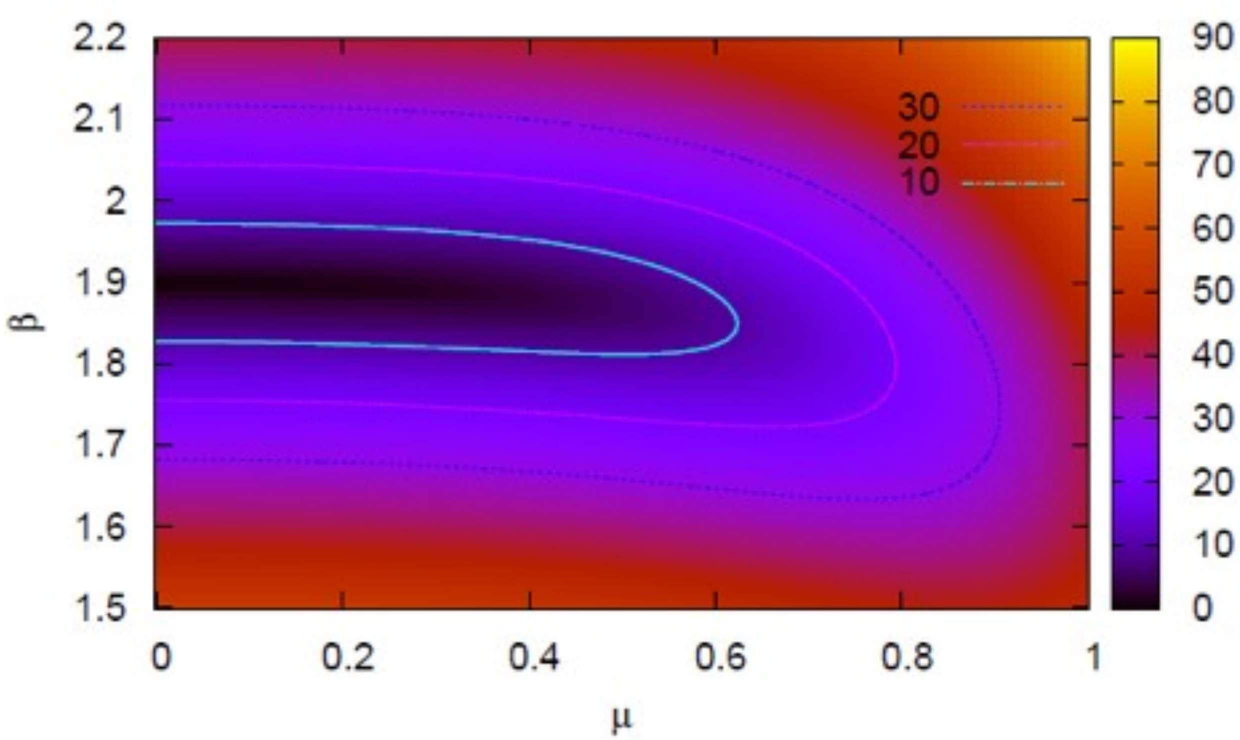
β0=1.80



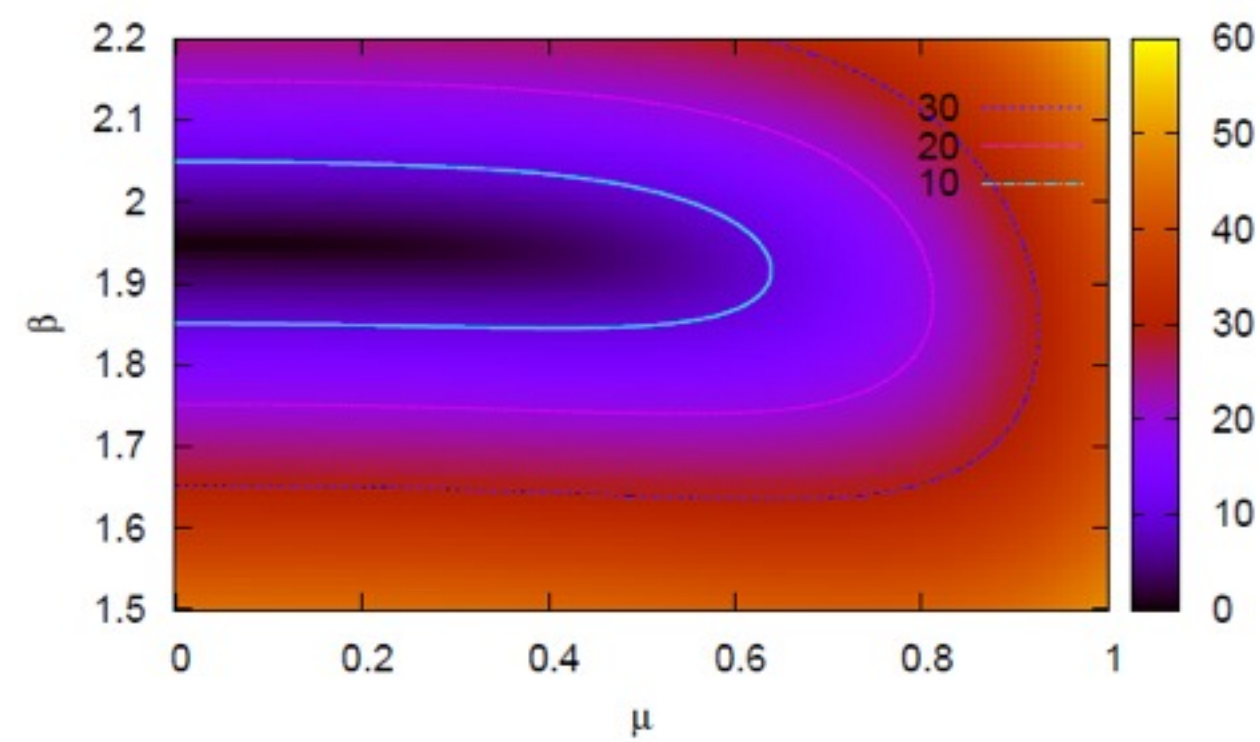
β0=1.85



β0=1.90

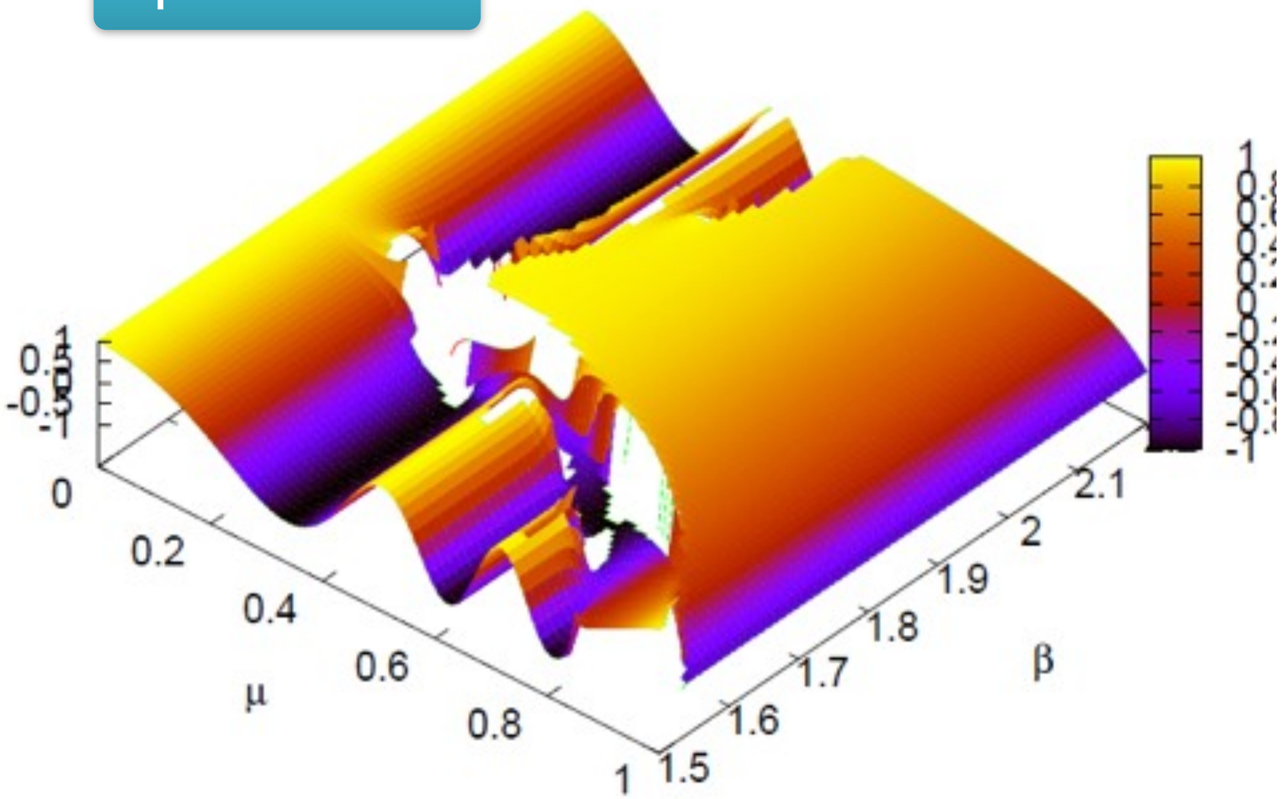


β0=1.95

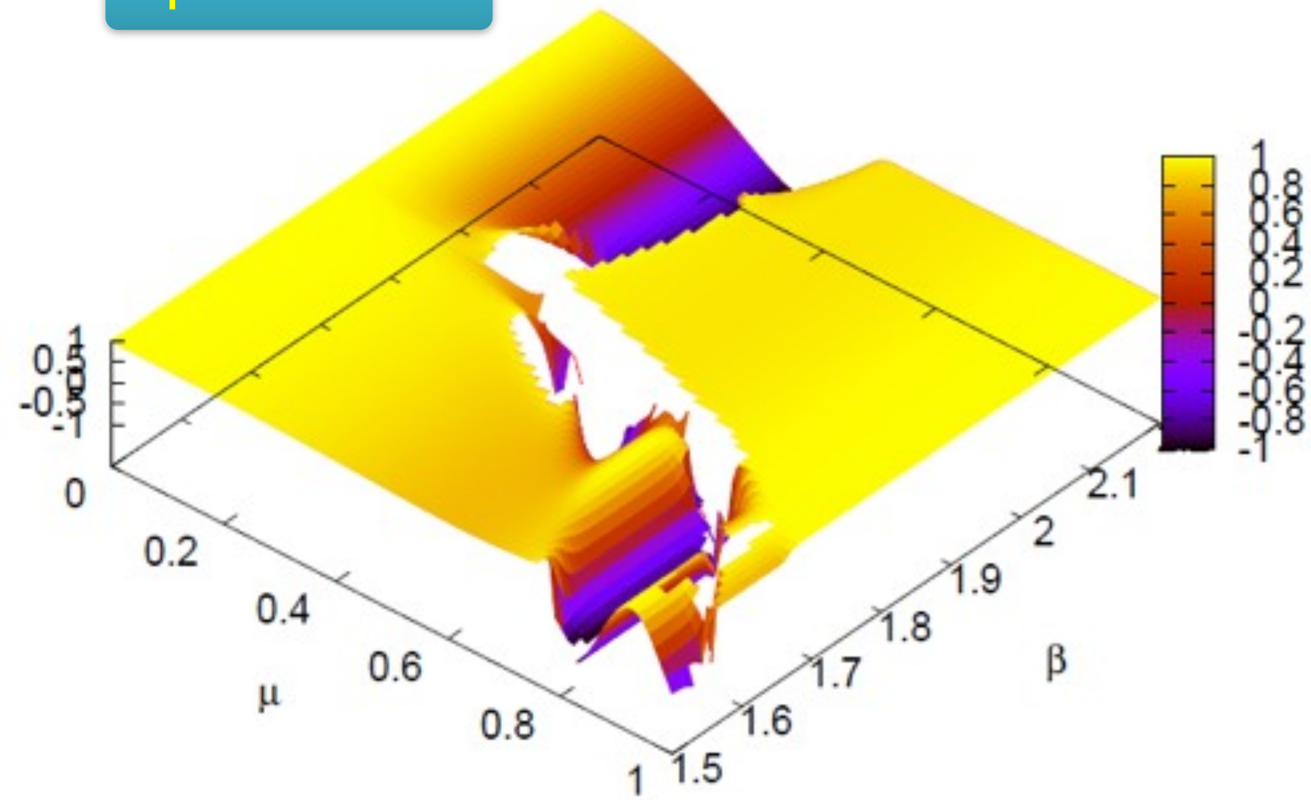


$$\langle \cos 2\theta \rangle = \langle \cos 2\theta \;\; R.F \rangle_0 / \langle R.F \rangle_0$$

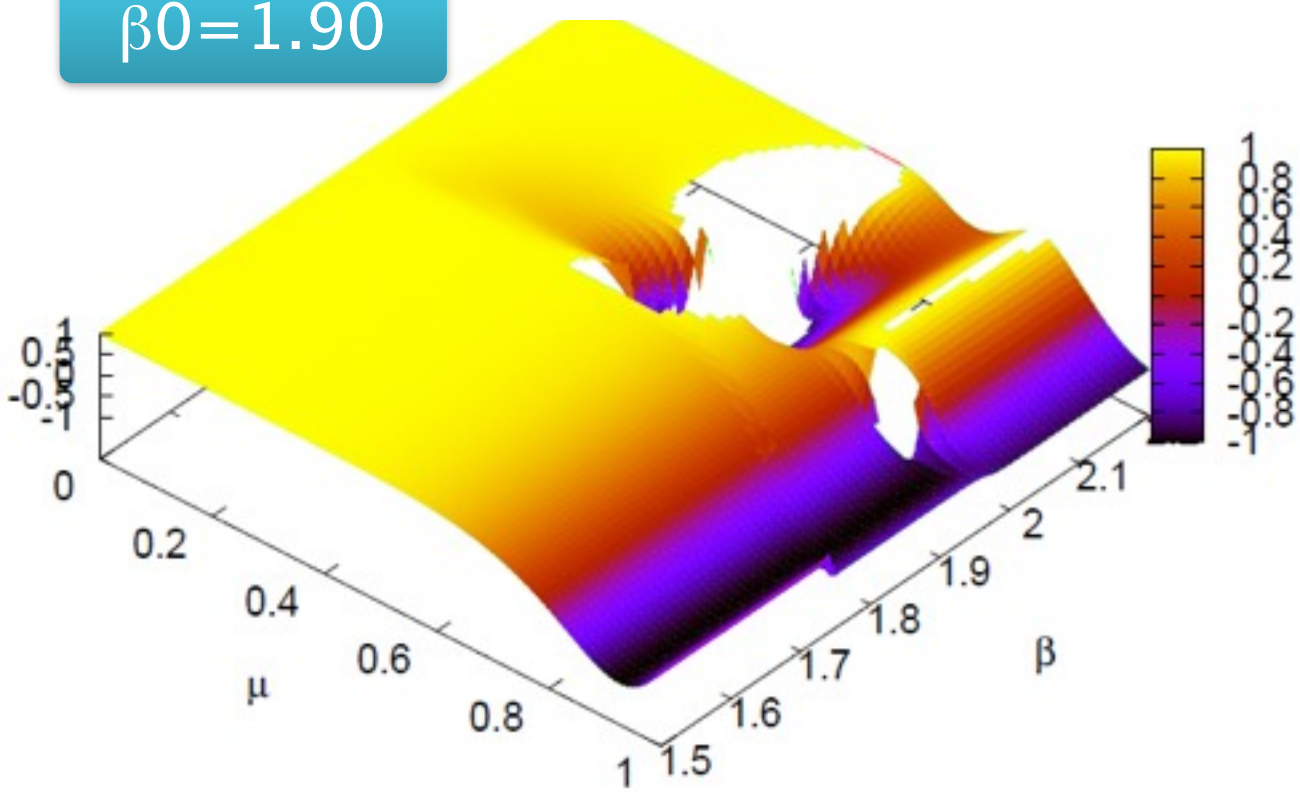
β0=1.80



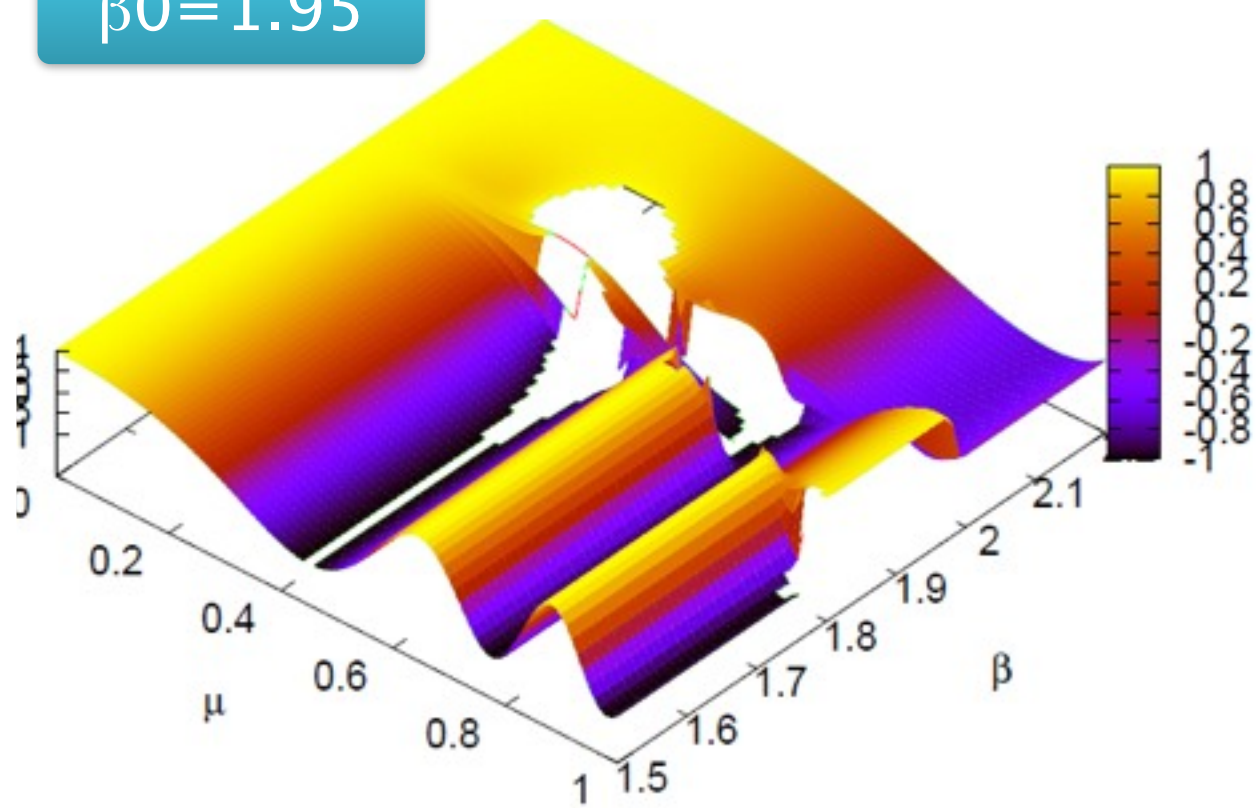
β0=1.85



β0=1.90

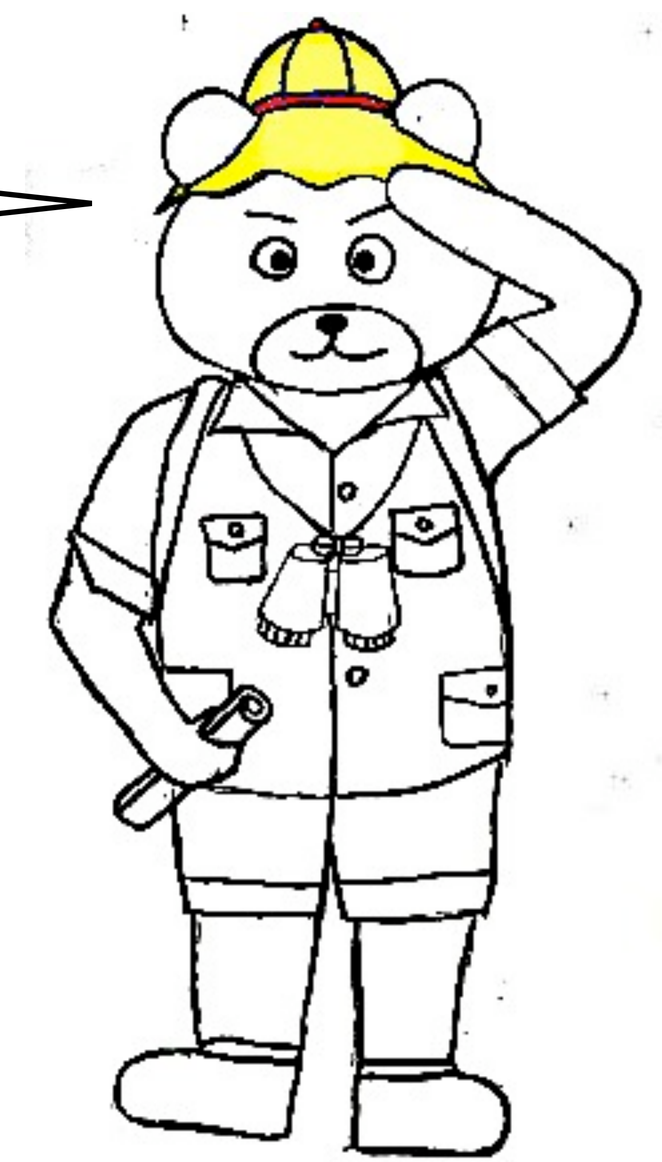


β0=1.95



Imaginary Chemical Potential

Let's Explore Imaginary World !

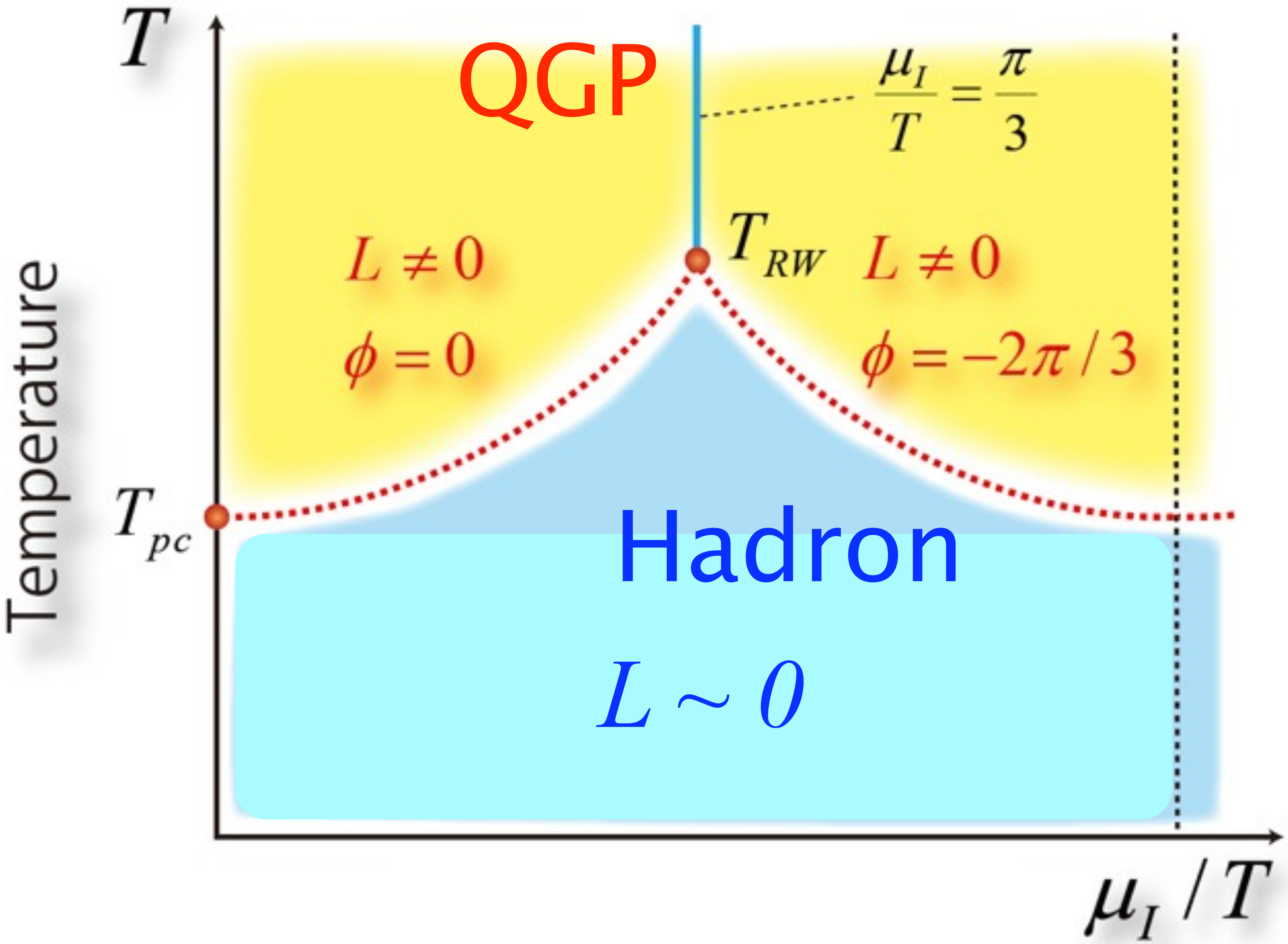


1. Introduction
2. Reduction Formula
3. Imainary Chemical Potential

Expected

Phase diagram in μI regions

a la Reberge-Weiss



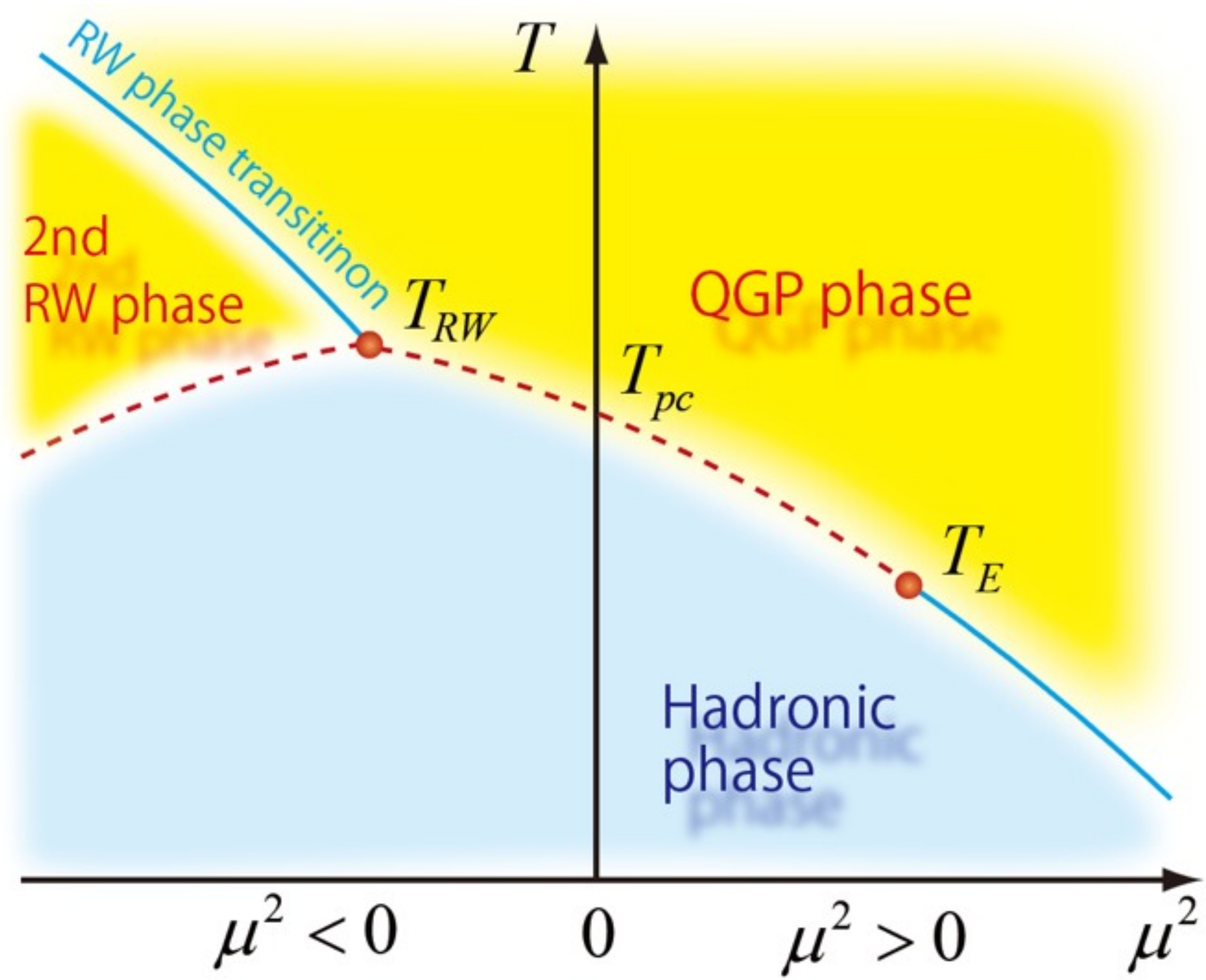
Polyakov loop

$$P = L_P \exp(i\phi_P)$$

If μ is pure imaginary

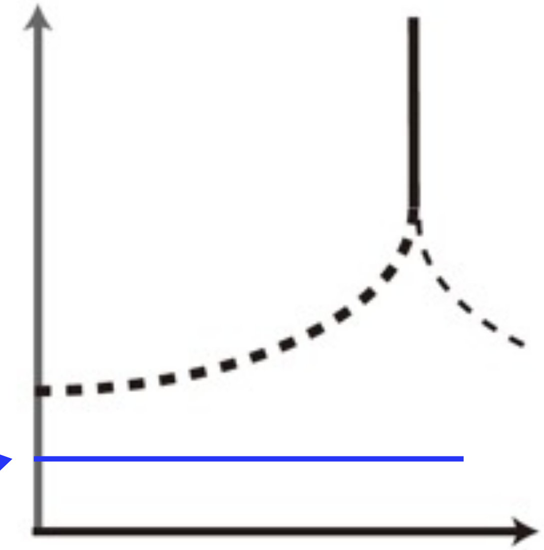
there is no sign problem.

Imaginary to real chemical potential

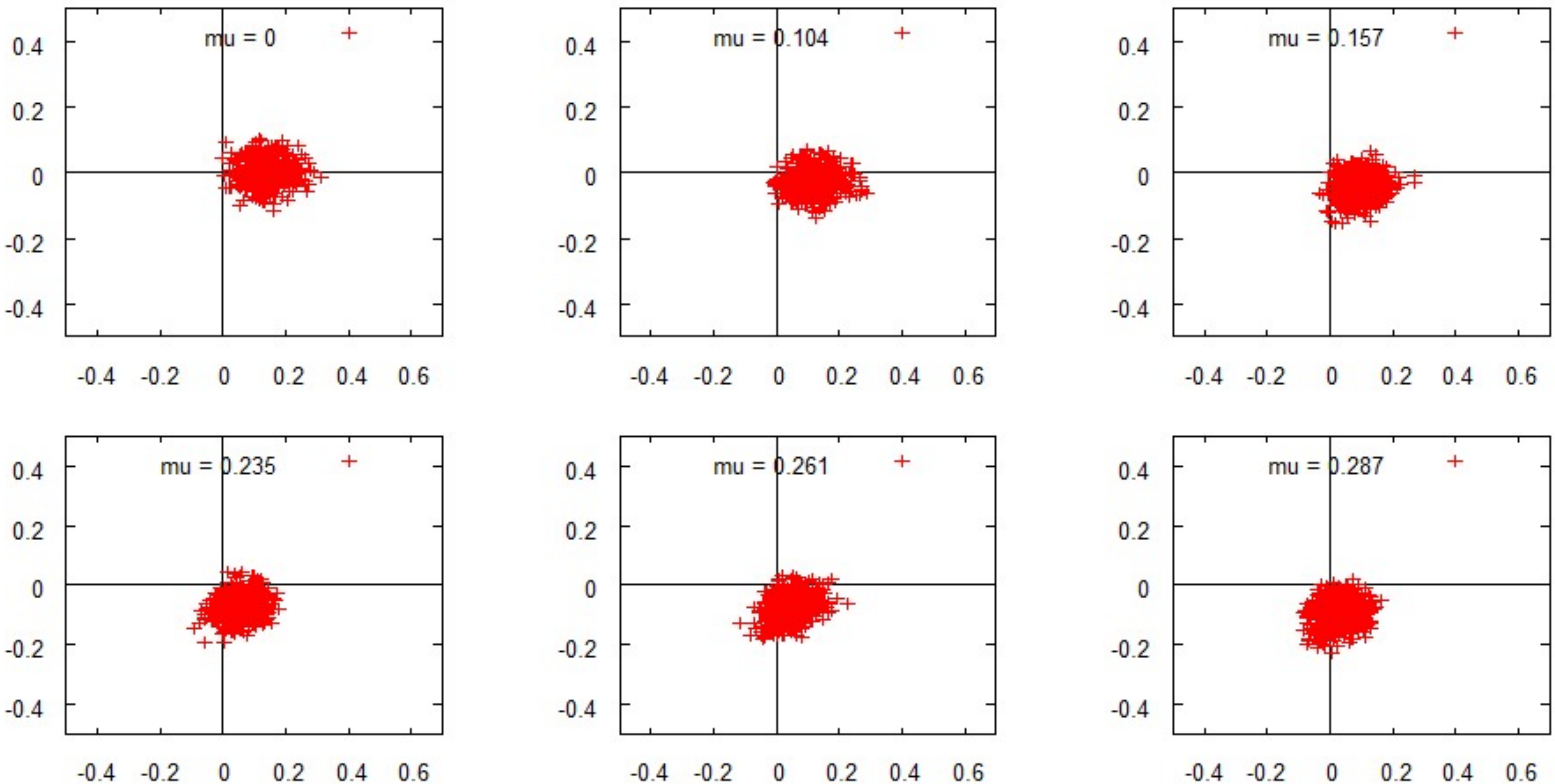


Polyakov loop scatter plot

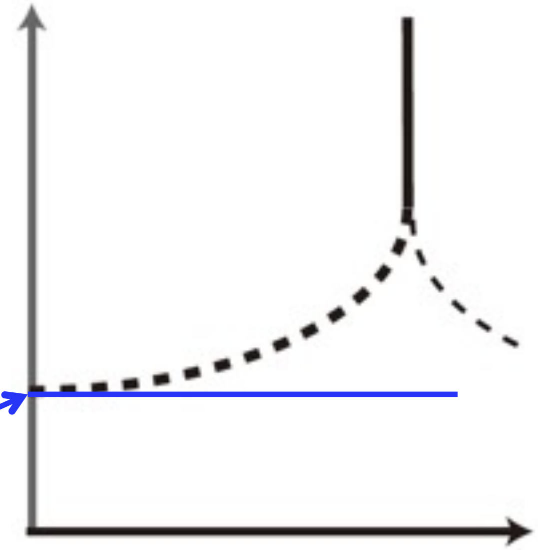
T=0.93Tc
beta = 1.80



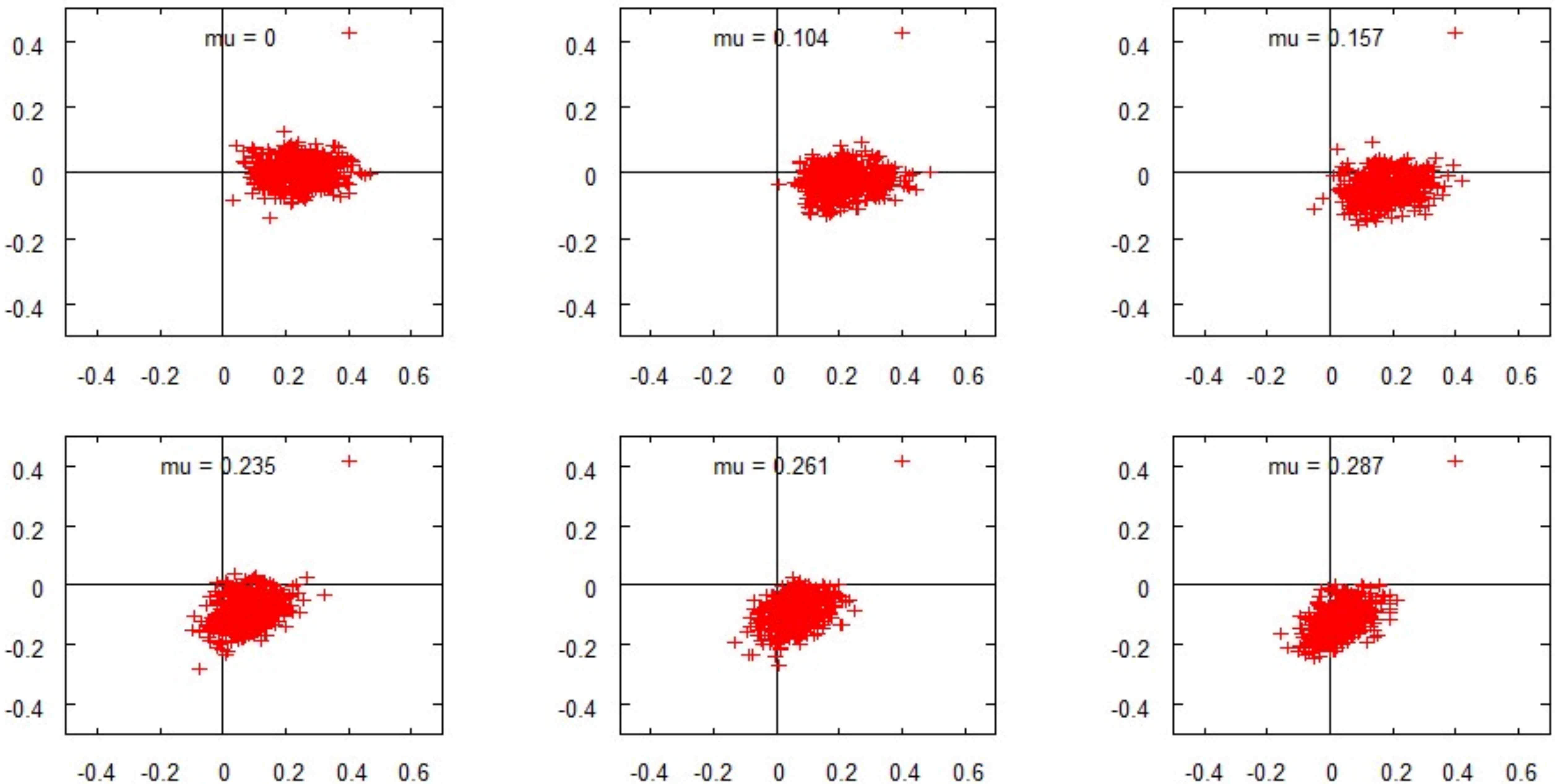
ϕ changes continously



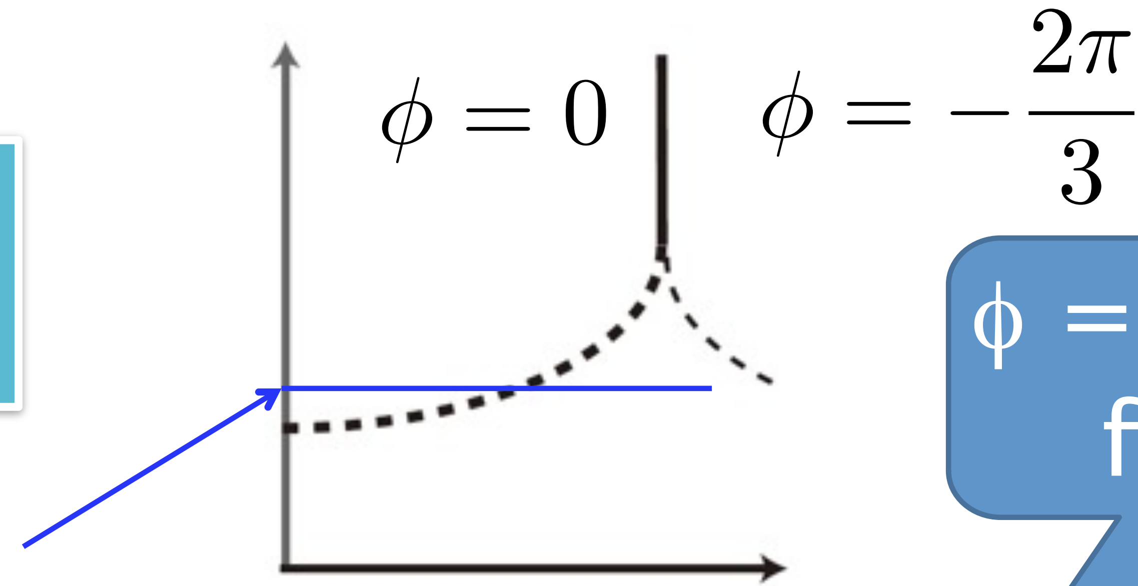
T=0.99Tc
beta = 1.85



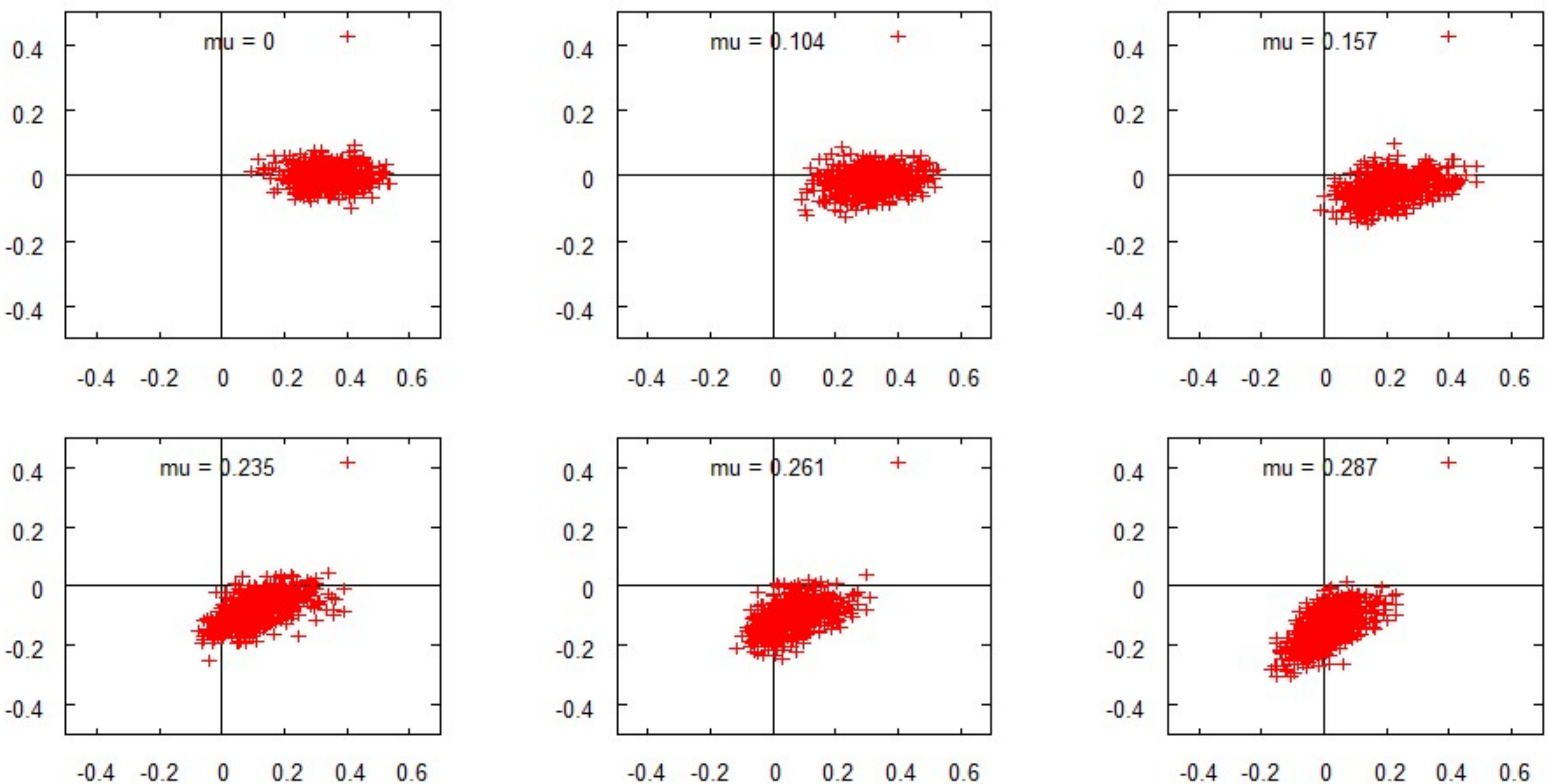
ϕ changes continously



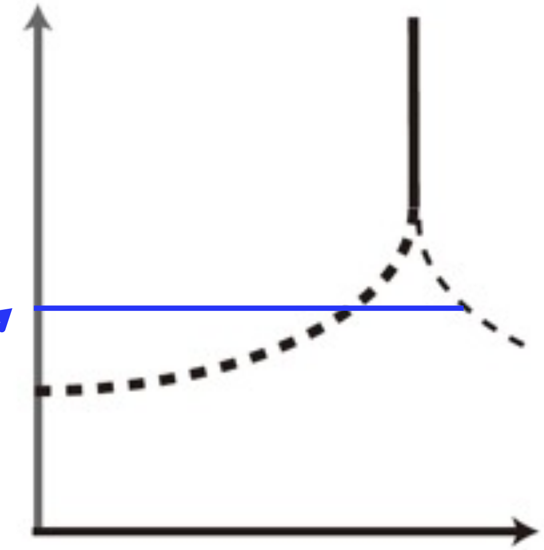
T=1.02Tc
beta = 1.87



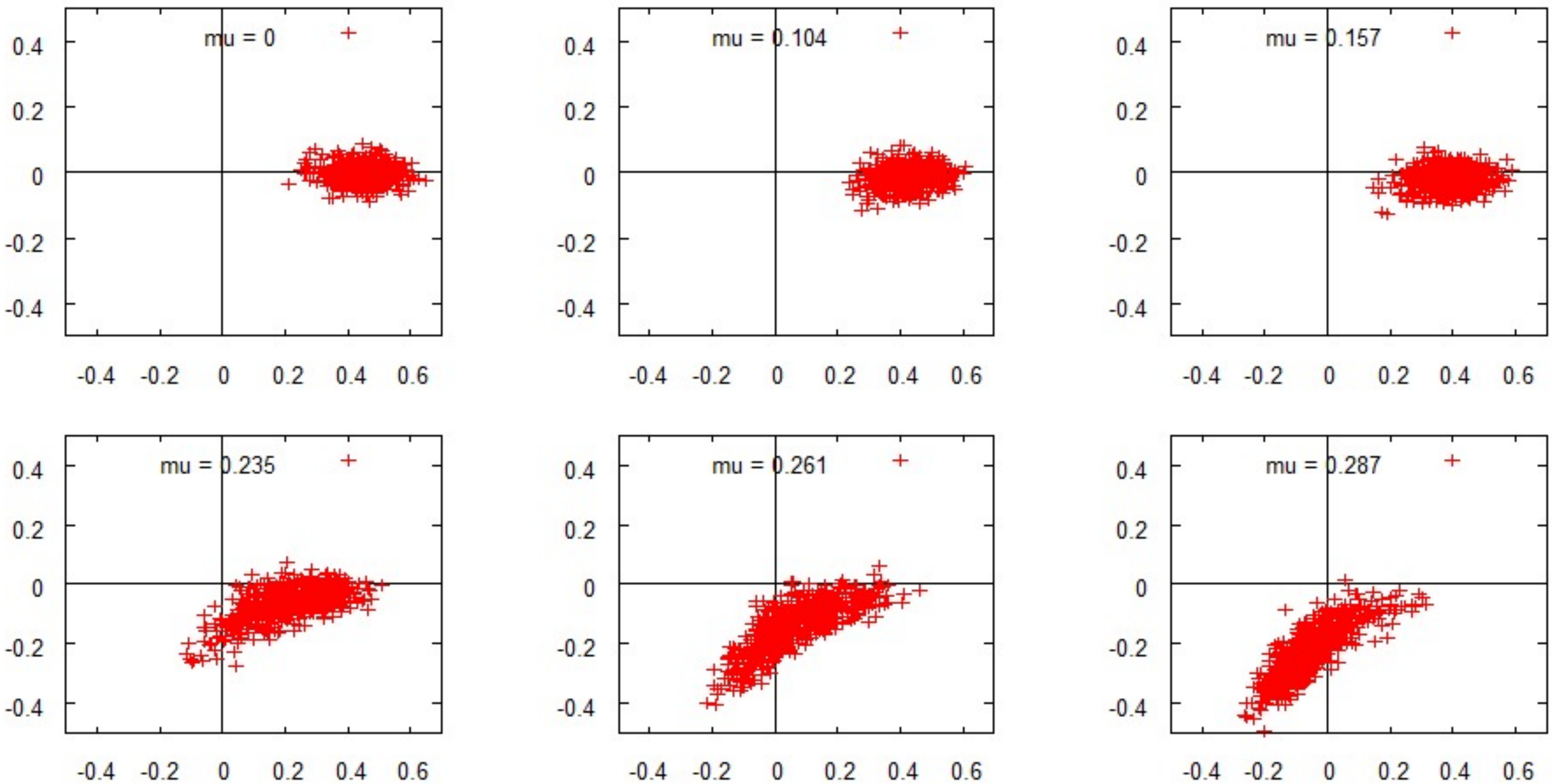
ϕ = 0 for μ<0.1
finite for μ>0.1



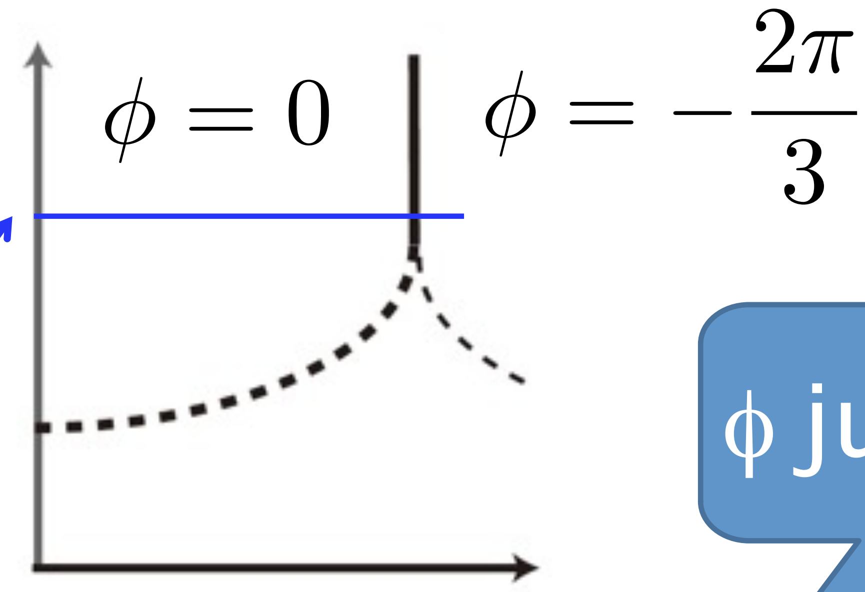
T=1.08Tc
beta = 1.90



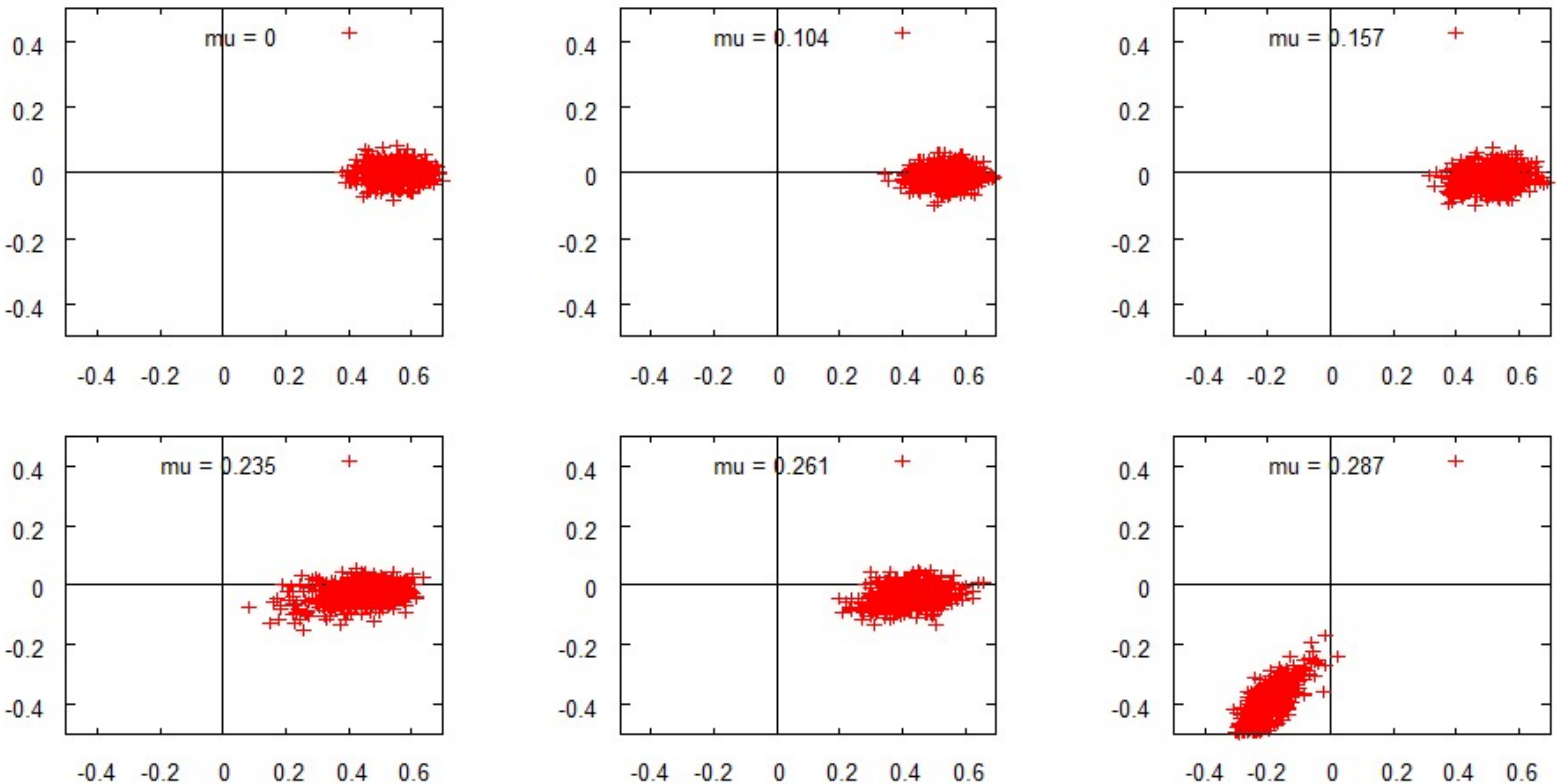
ϕ = 0 for μ<0.2
finite for μ>0.2



T=1.20Tc
beta = 1.95

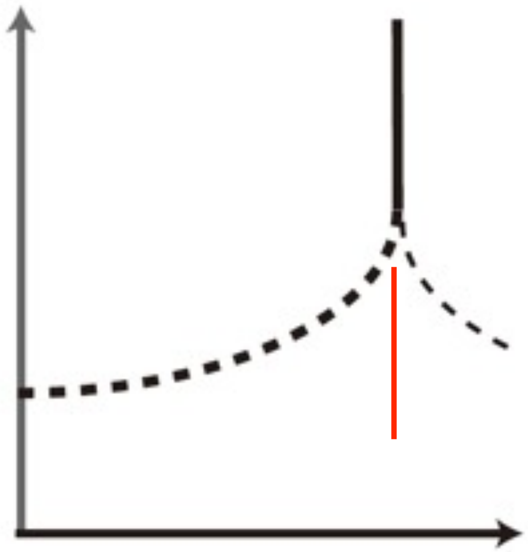


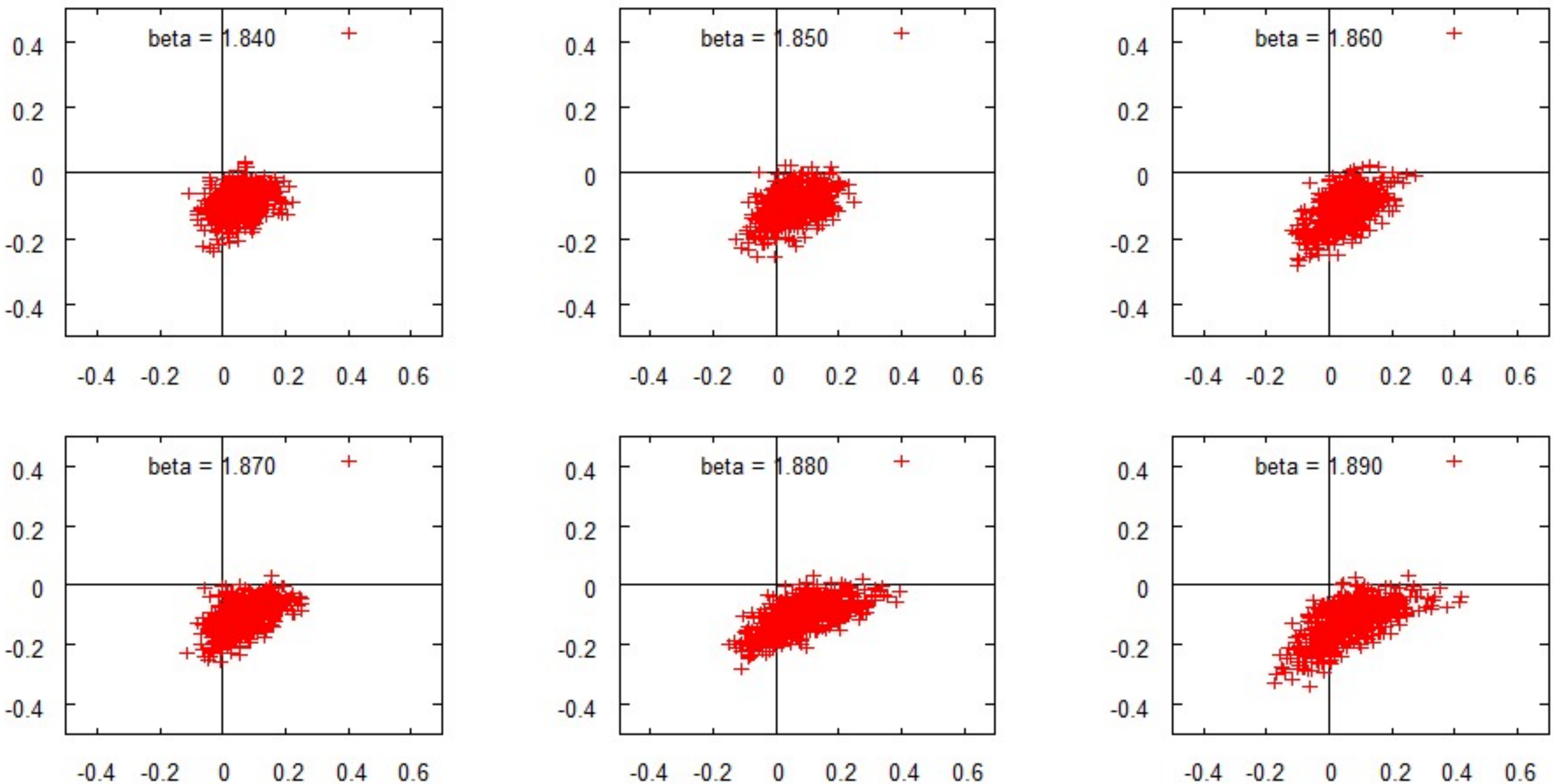
ϕ jumps at μ=0.26



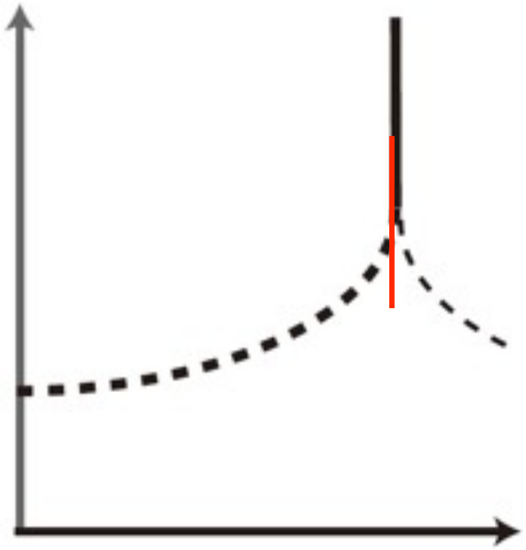
Polyakov loop scatter plot

β = 1.84-1.89

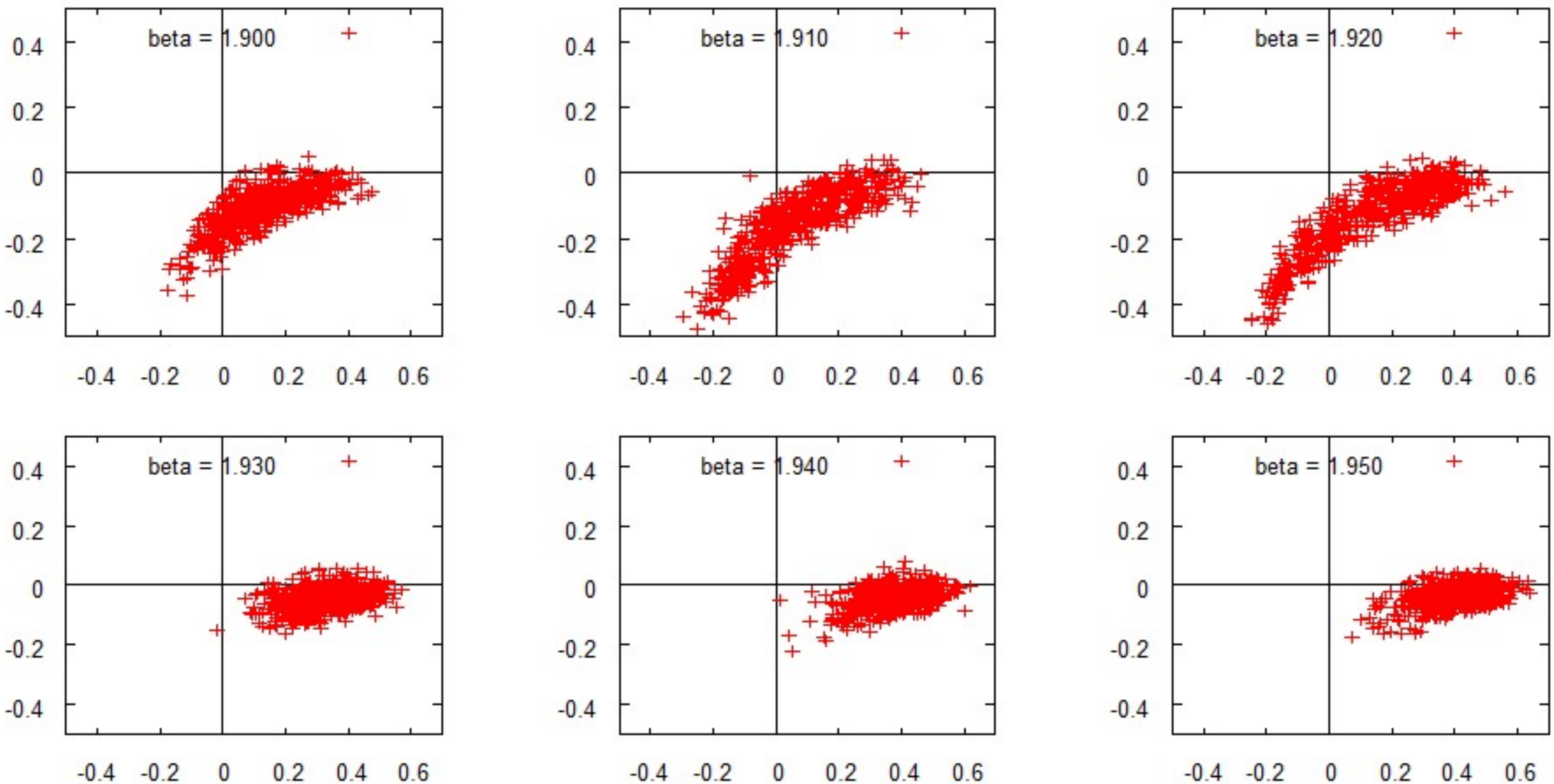




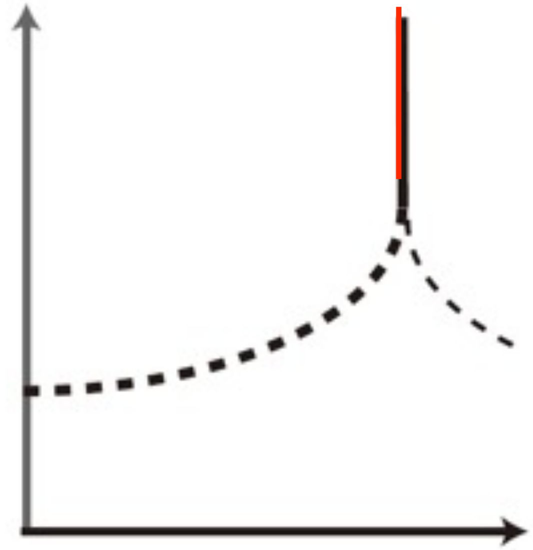
β = 1.90-1.95



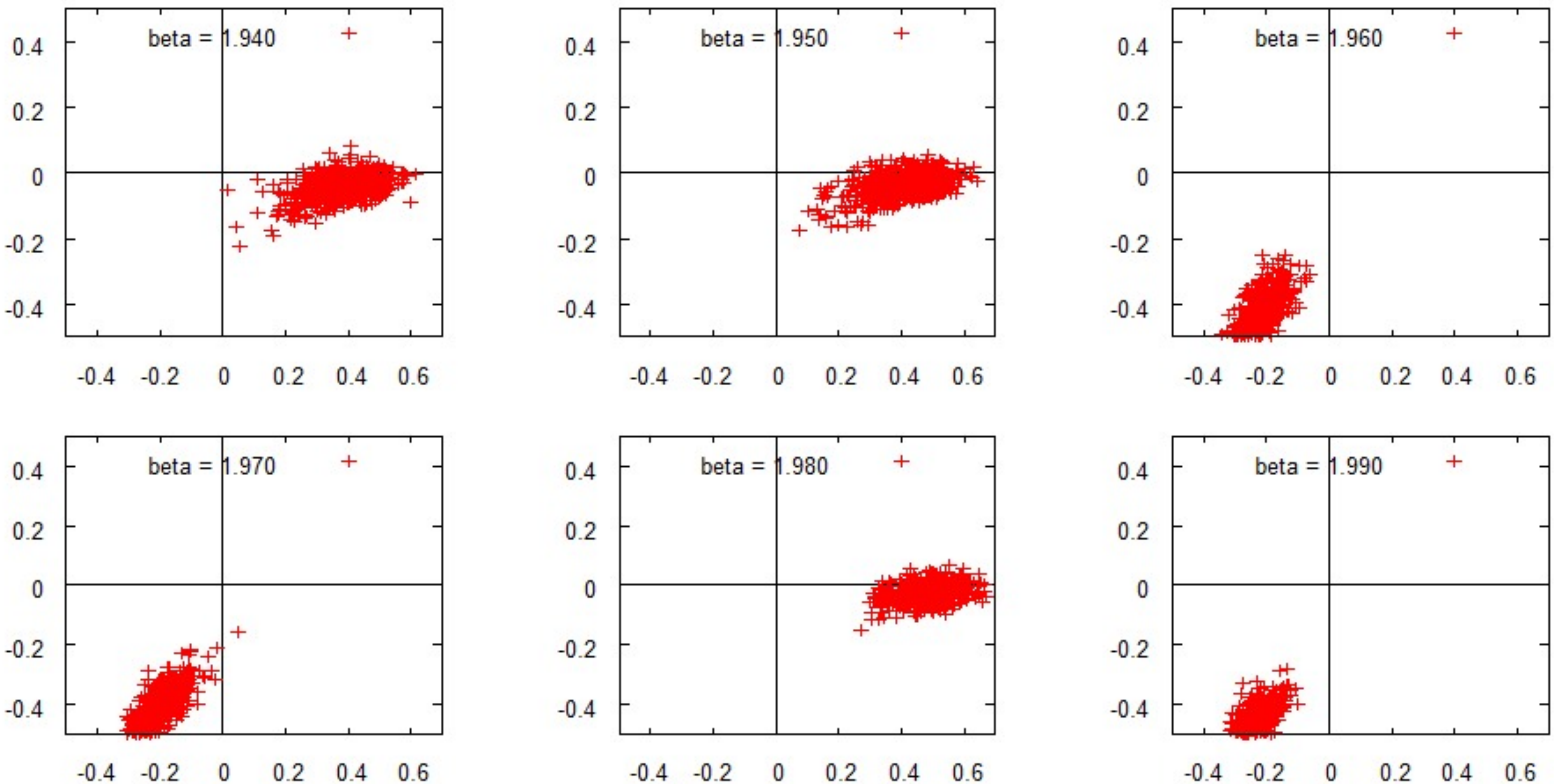
2-peak behaviour around β=1.92



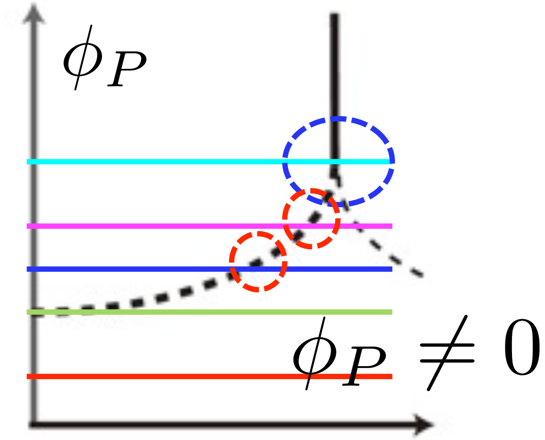
β = 1.94-1.99

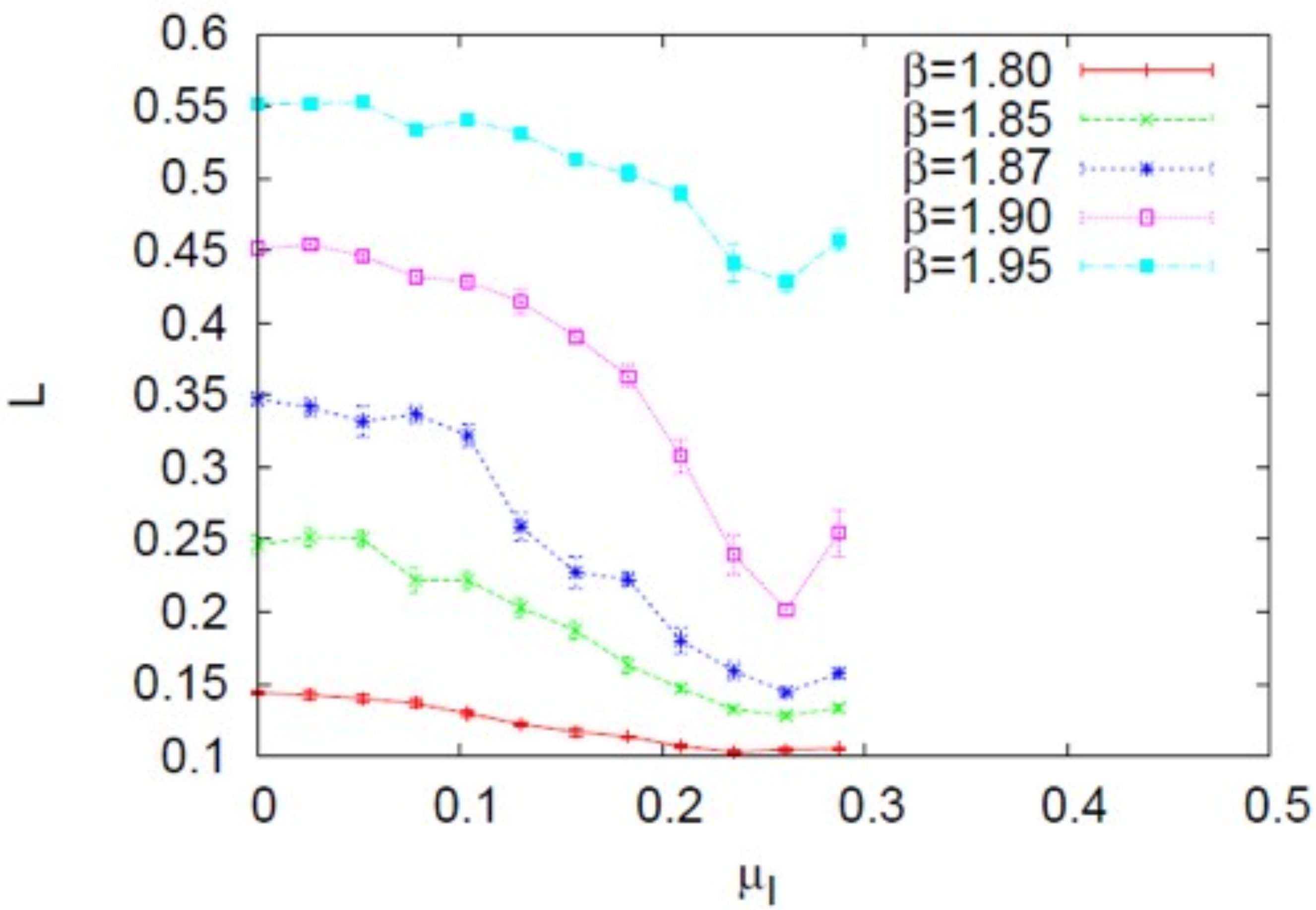


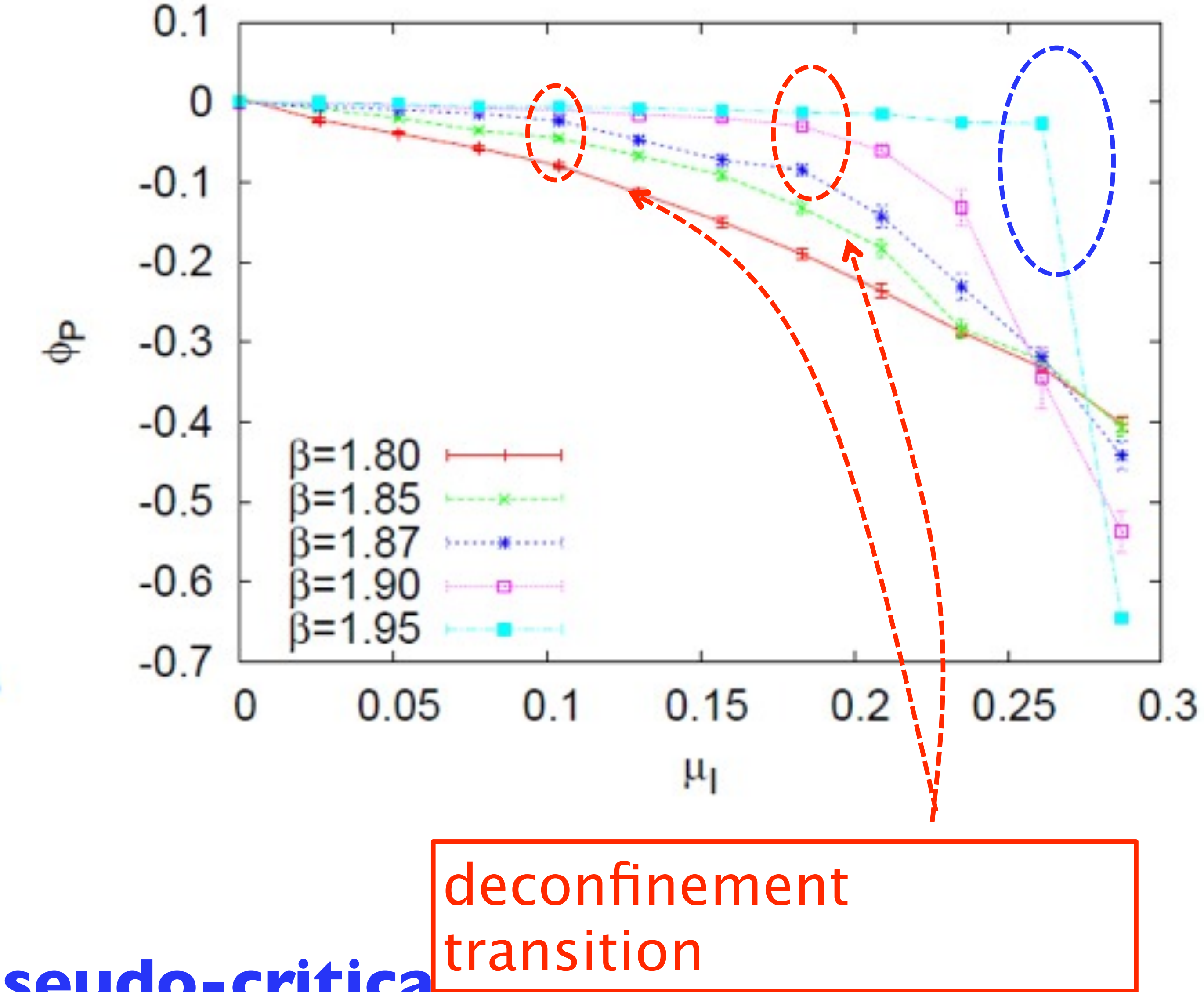
spontanous breaking in high T



Polyakov Loop (L: absolute, r: phase) μ-dependence



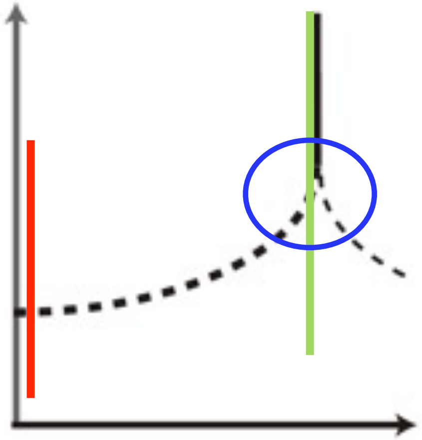


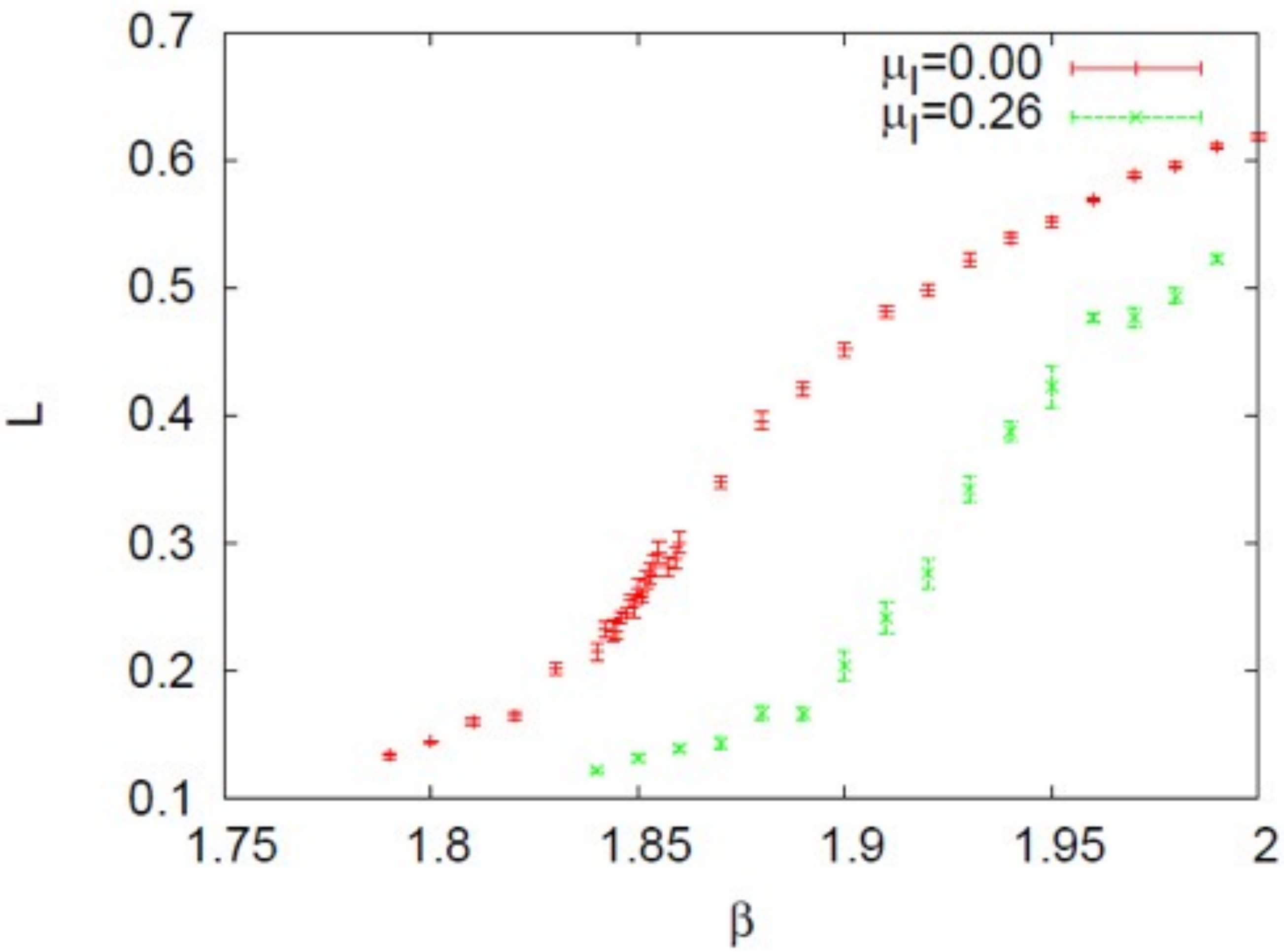


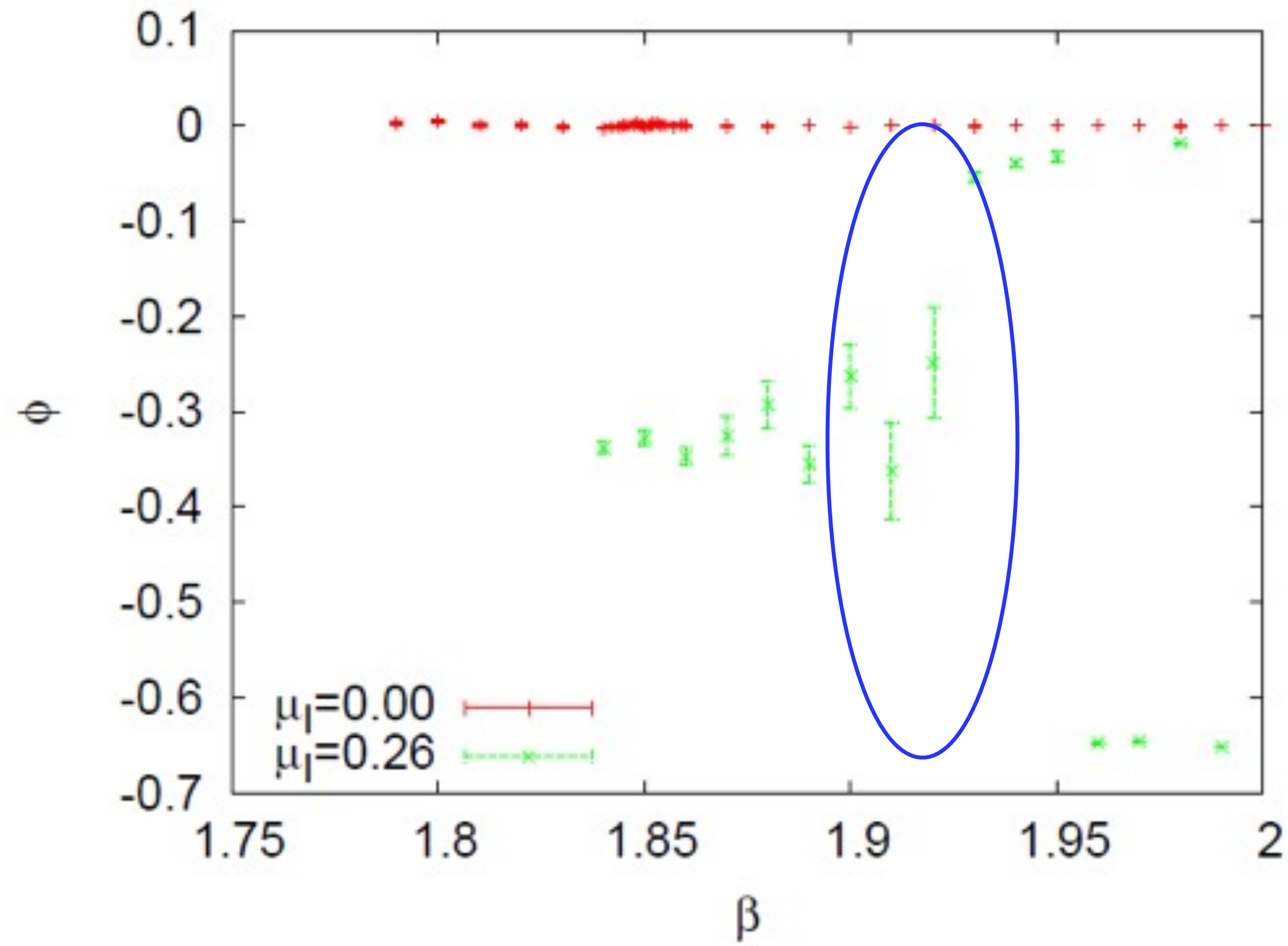
- **We determine the shape of the pseudo-critical line,**
- **assuming the phase of PL is also an order parameter of the deconfinement transition in μI region.**

Polyakov Loop T-dependence

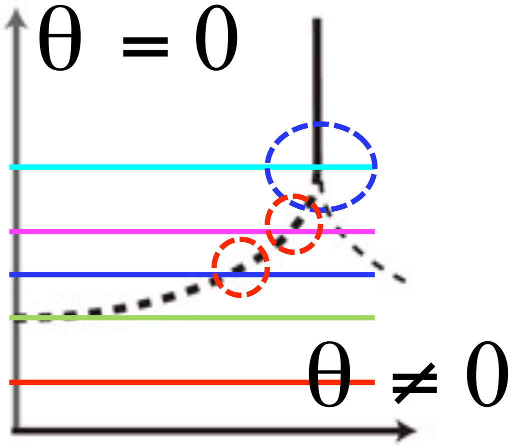
RW End point

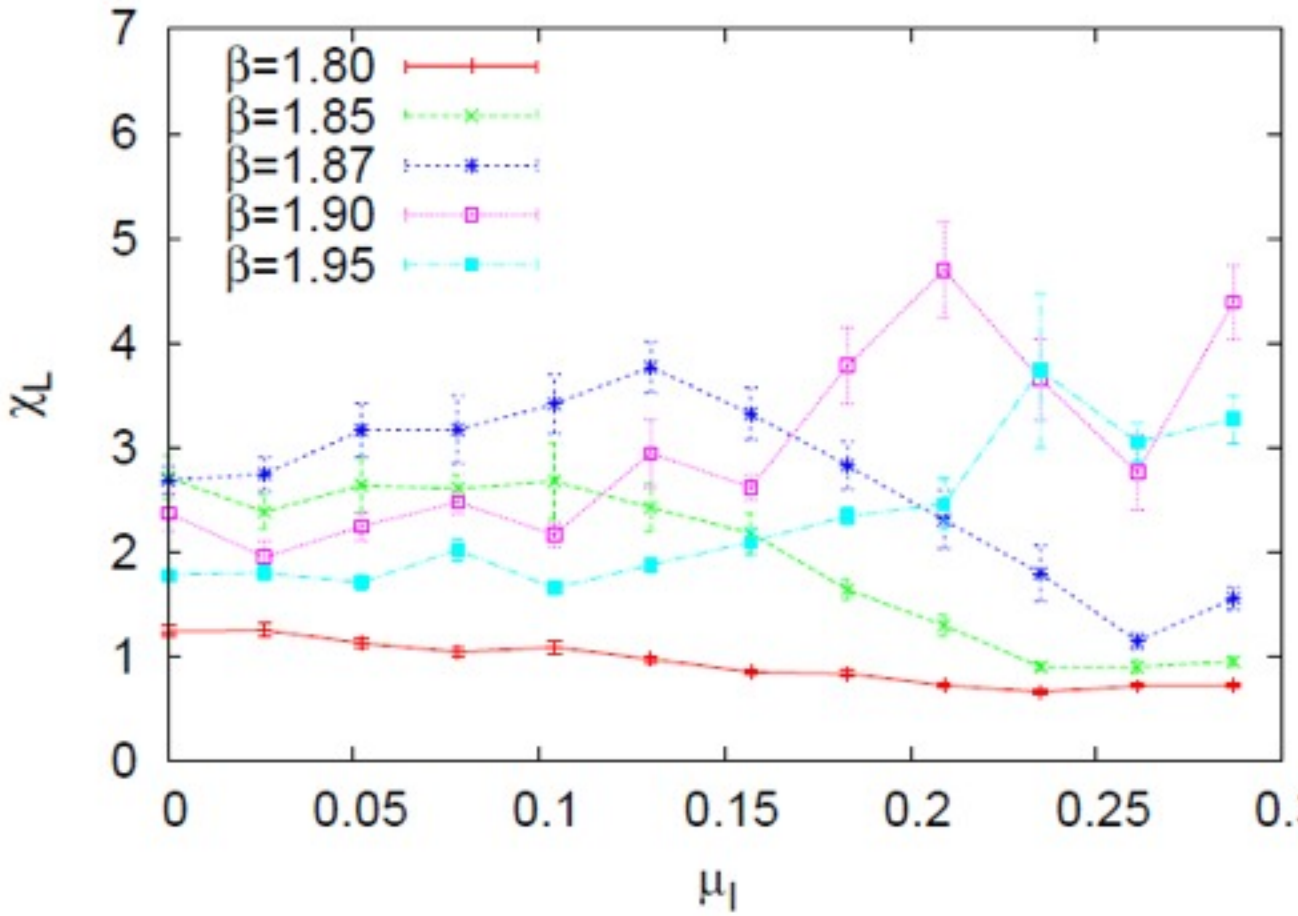


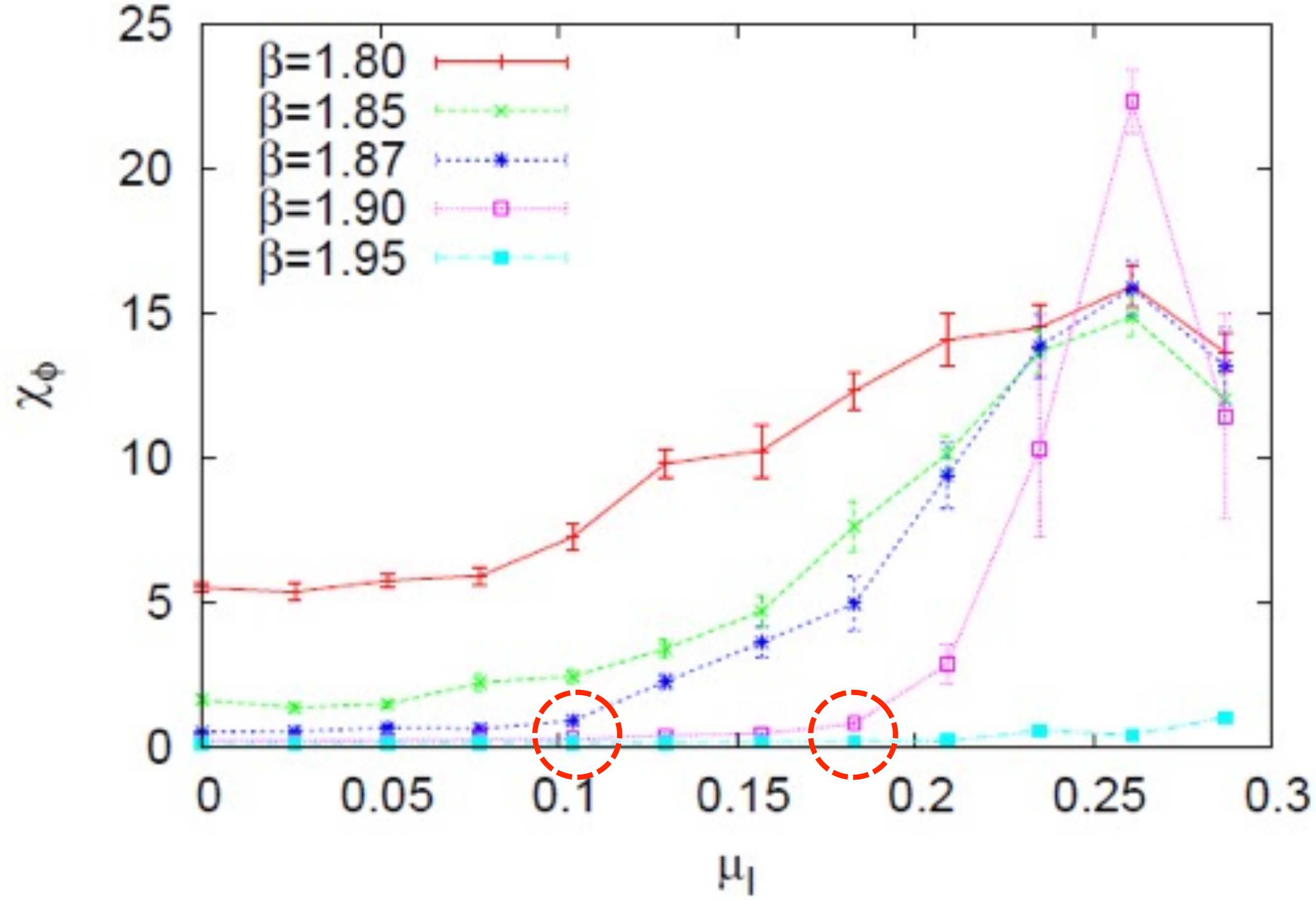




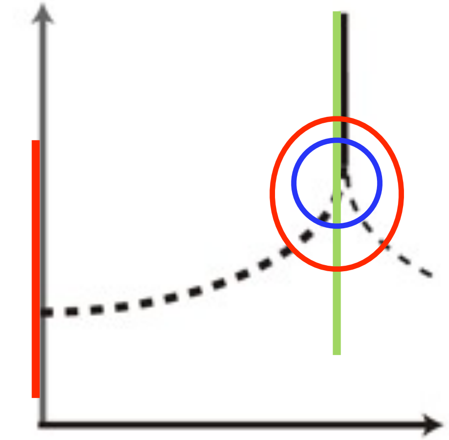
Susceptibility
μ-dependence

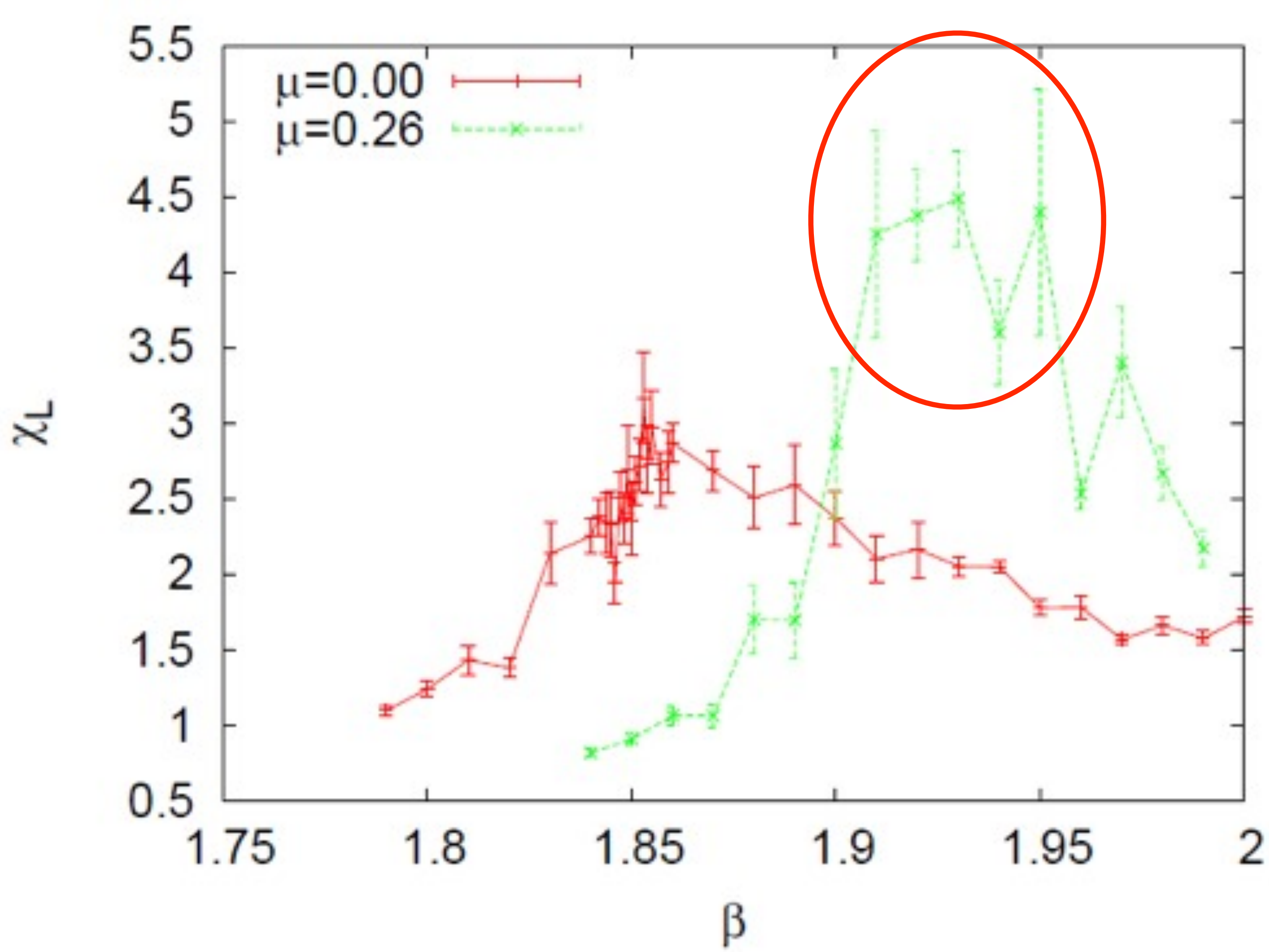


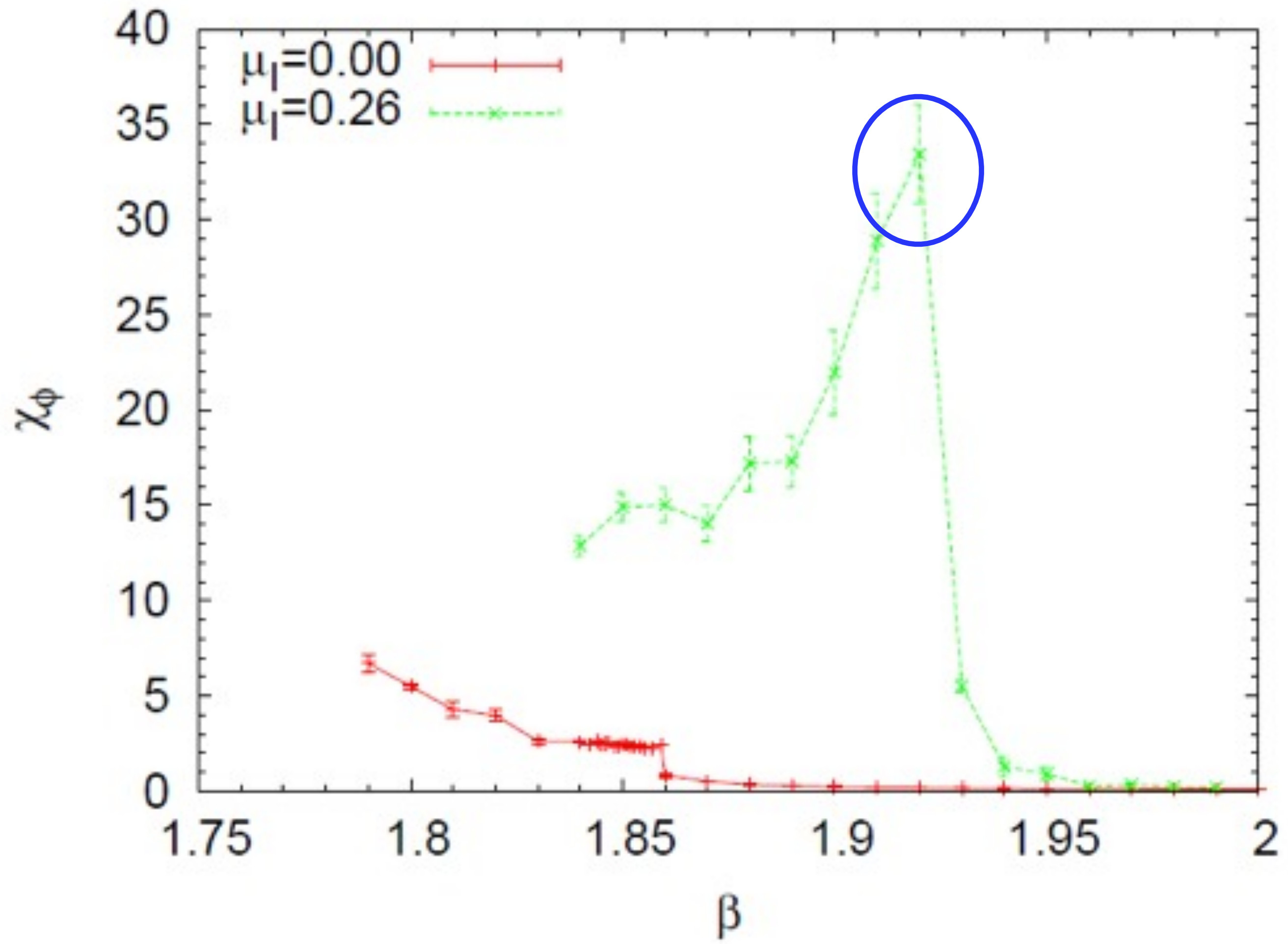




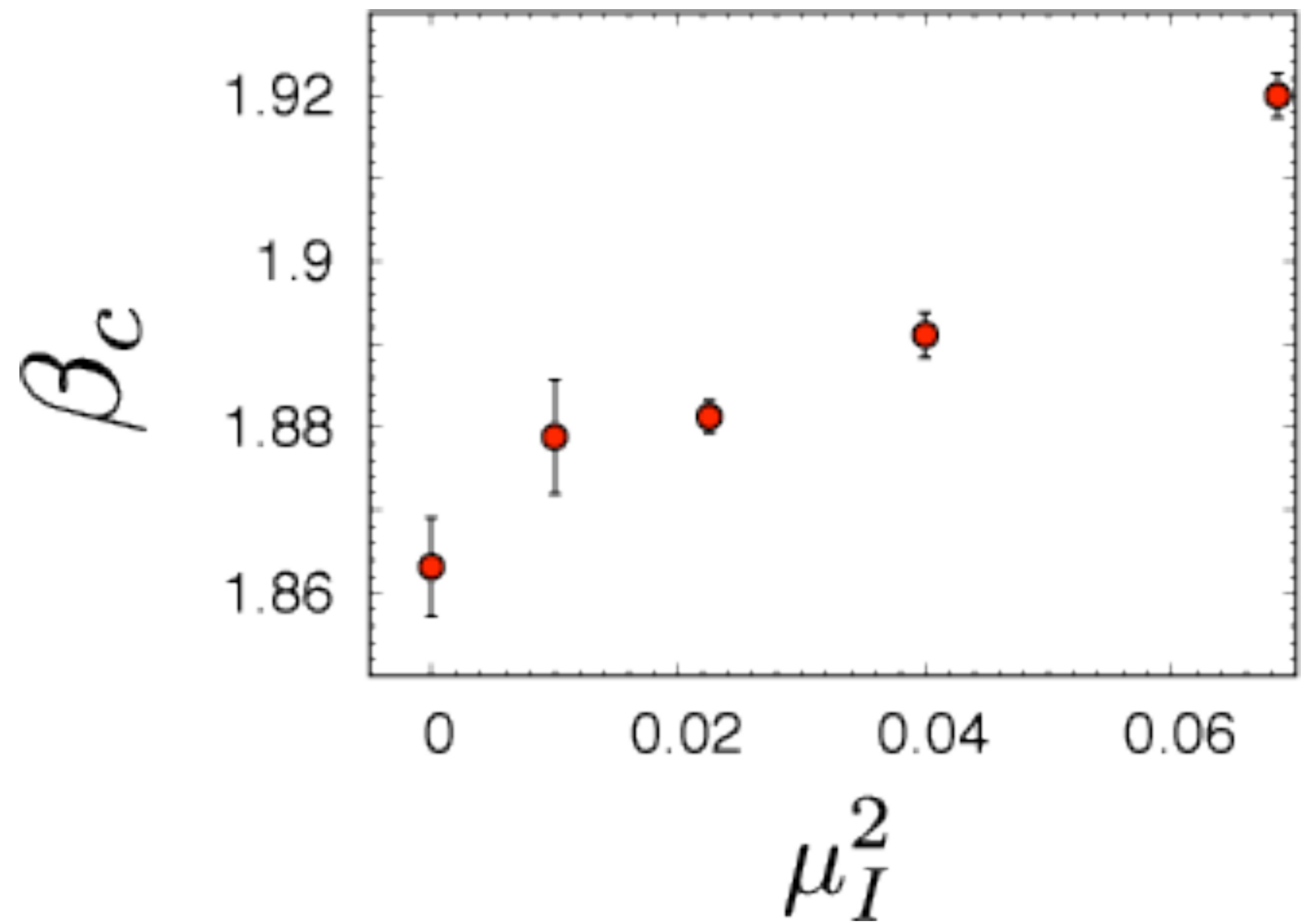
Susceptibility T-dependence



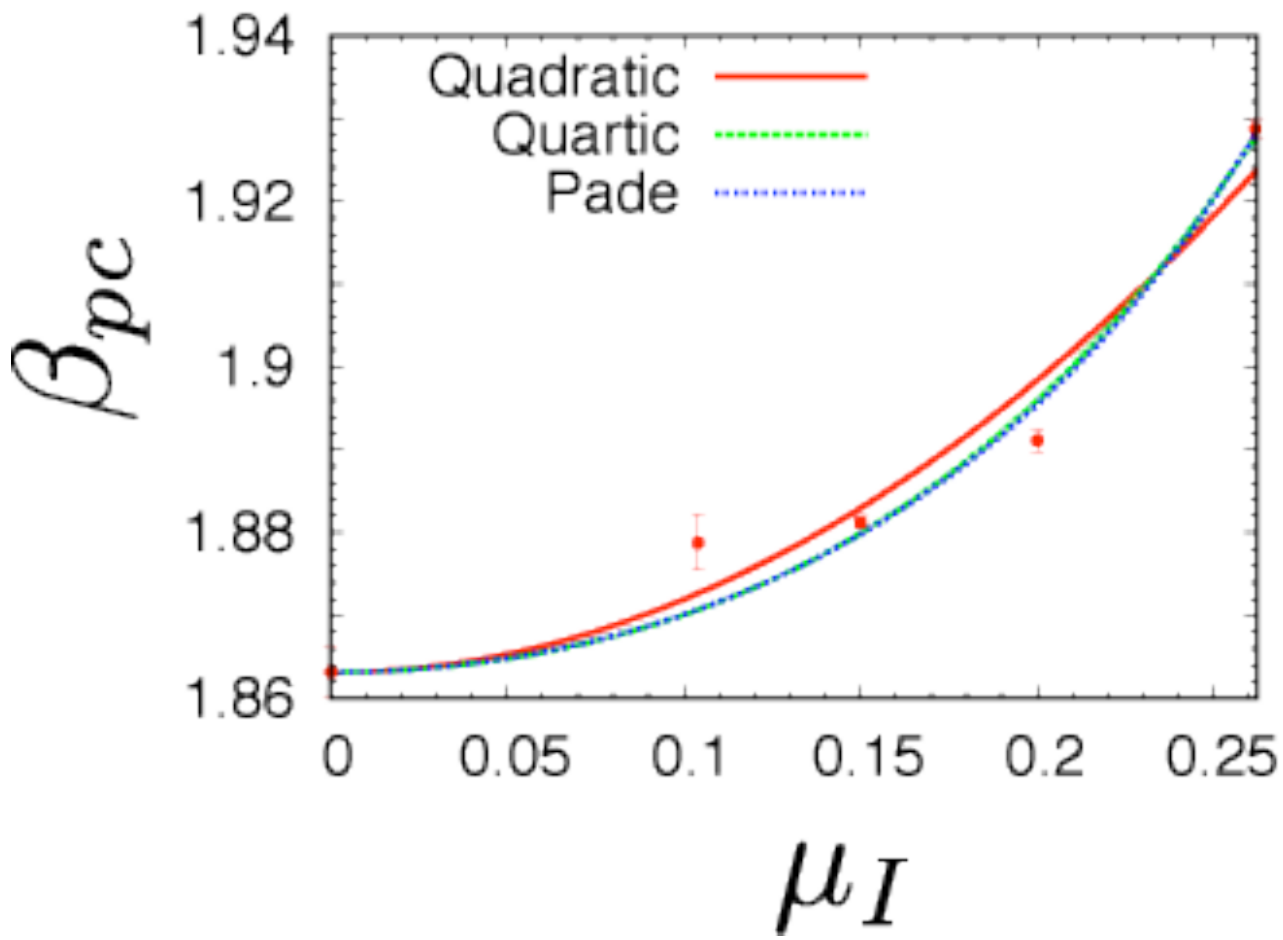


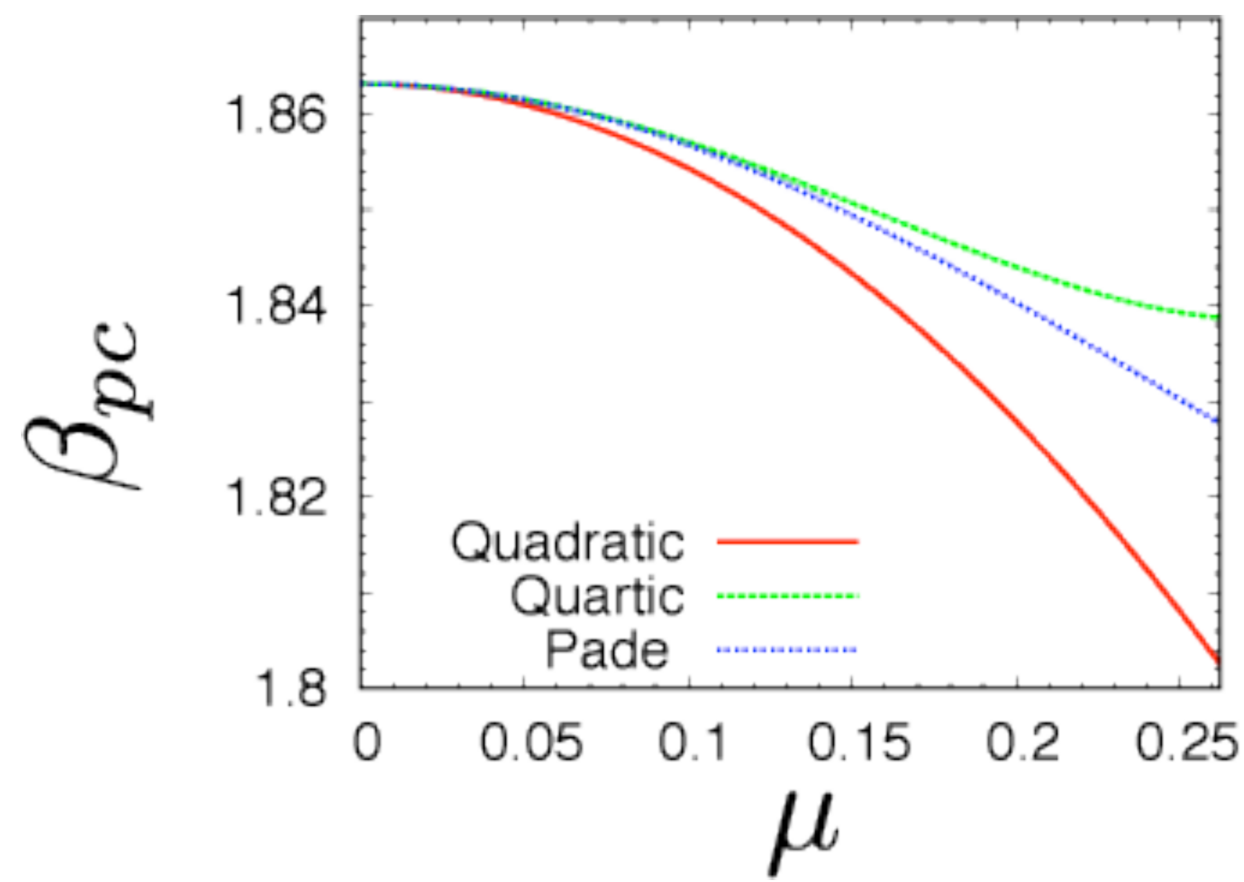


- **Deconfinement and RW transition lines end at the same point within errors.**
- **The RW end-point shows the 2nd order behavior.**

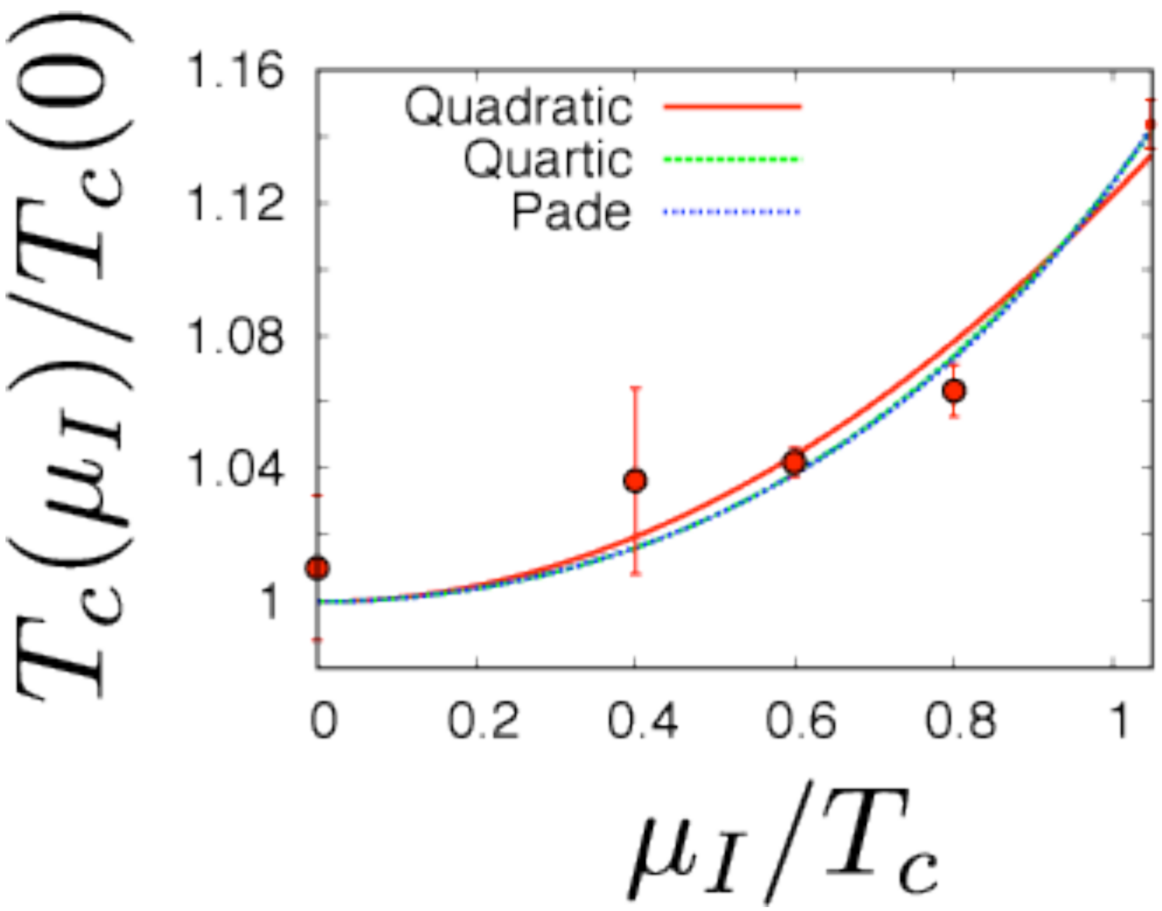


$$\frac{T_c(\mu_I)}{T_c(0)} = 1 + d_1 \left(\frac{\mu}{T_c}\right)^2 \qquad d_1 = 0.062 \pm 0.010$$

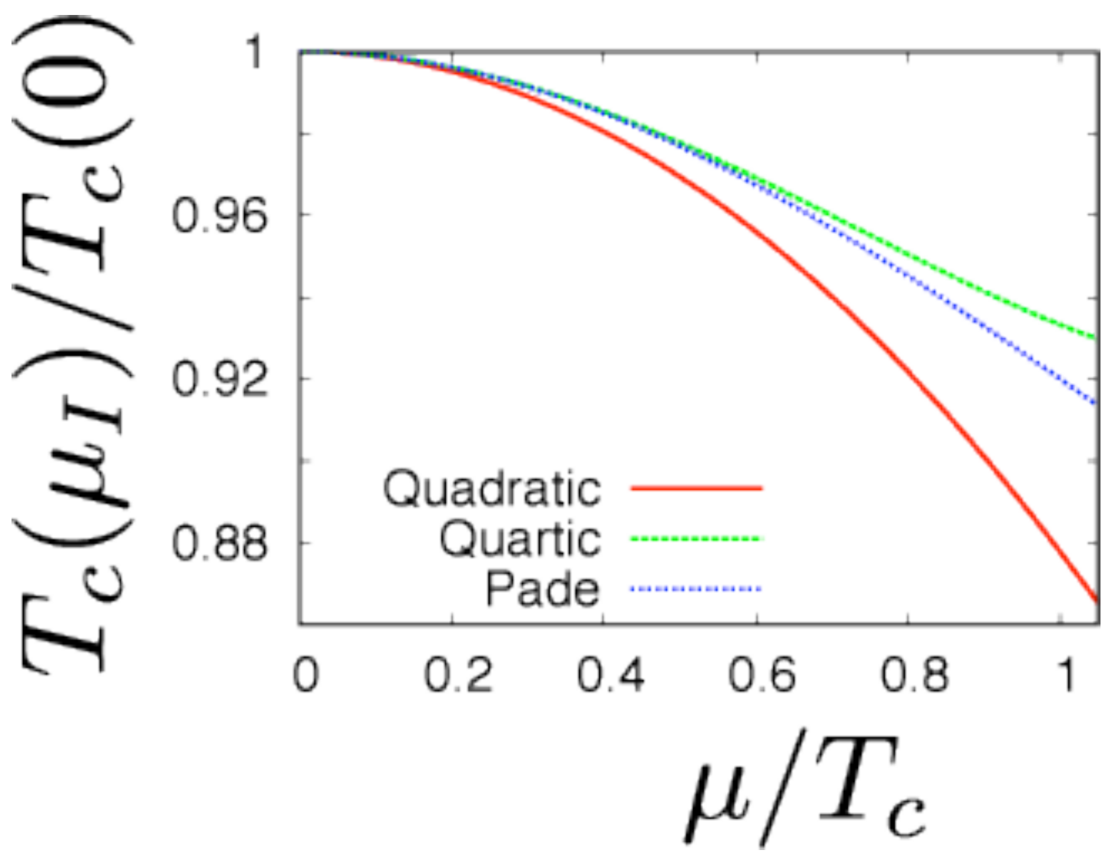




Pade $\beta_{pc} = c_0 \dfrac{1 + c_1 \mu_I^2}{1 + c_2 \mu_I^2}$

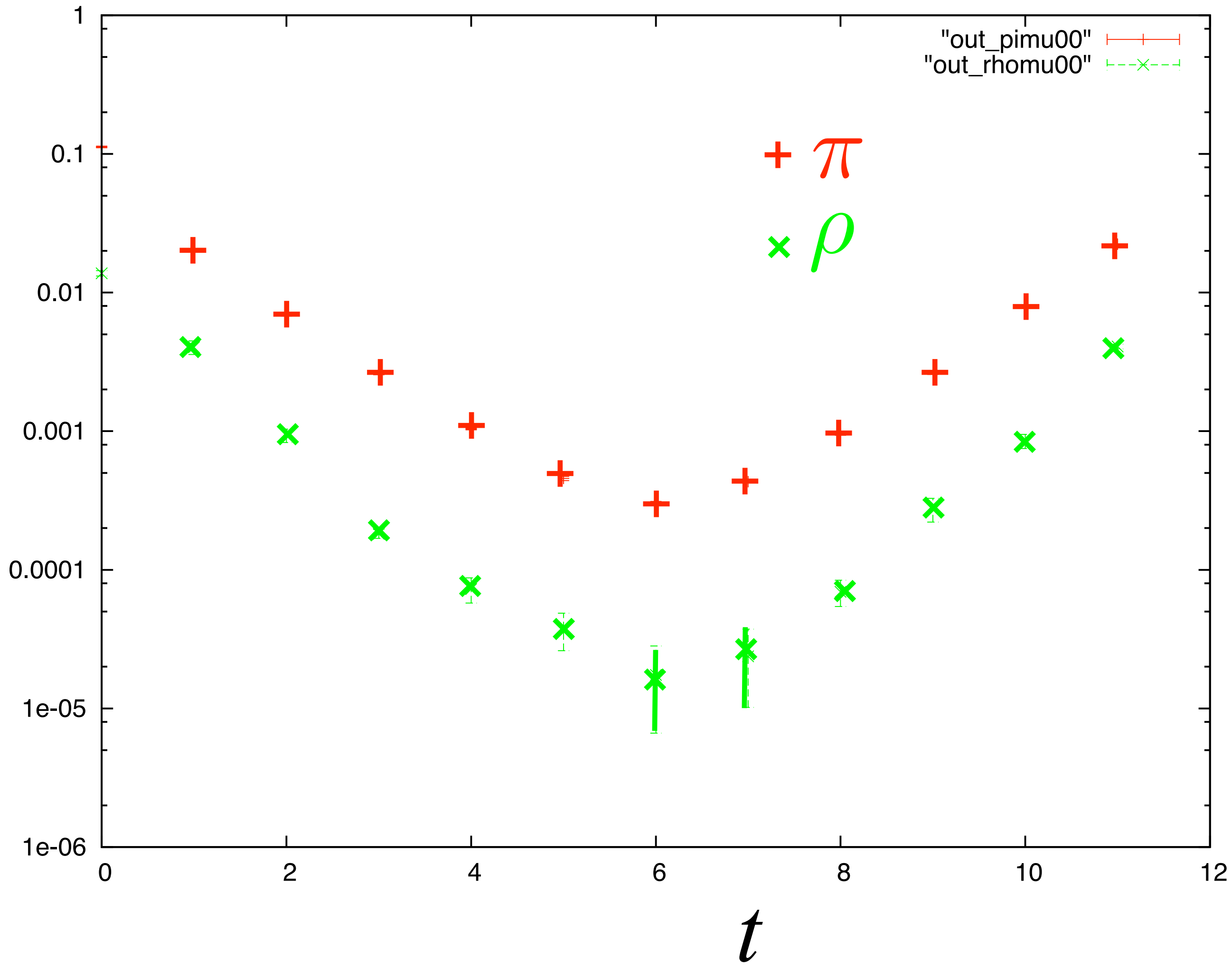


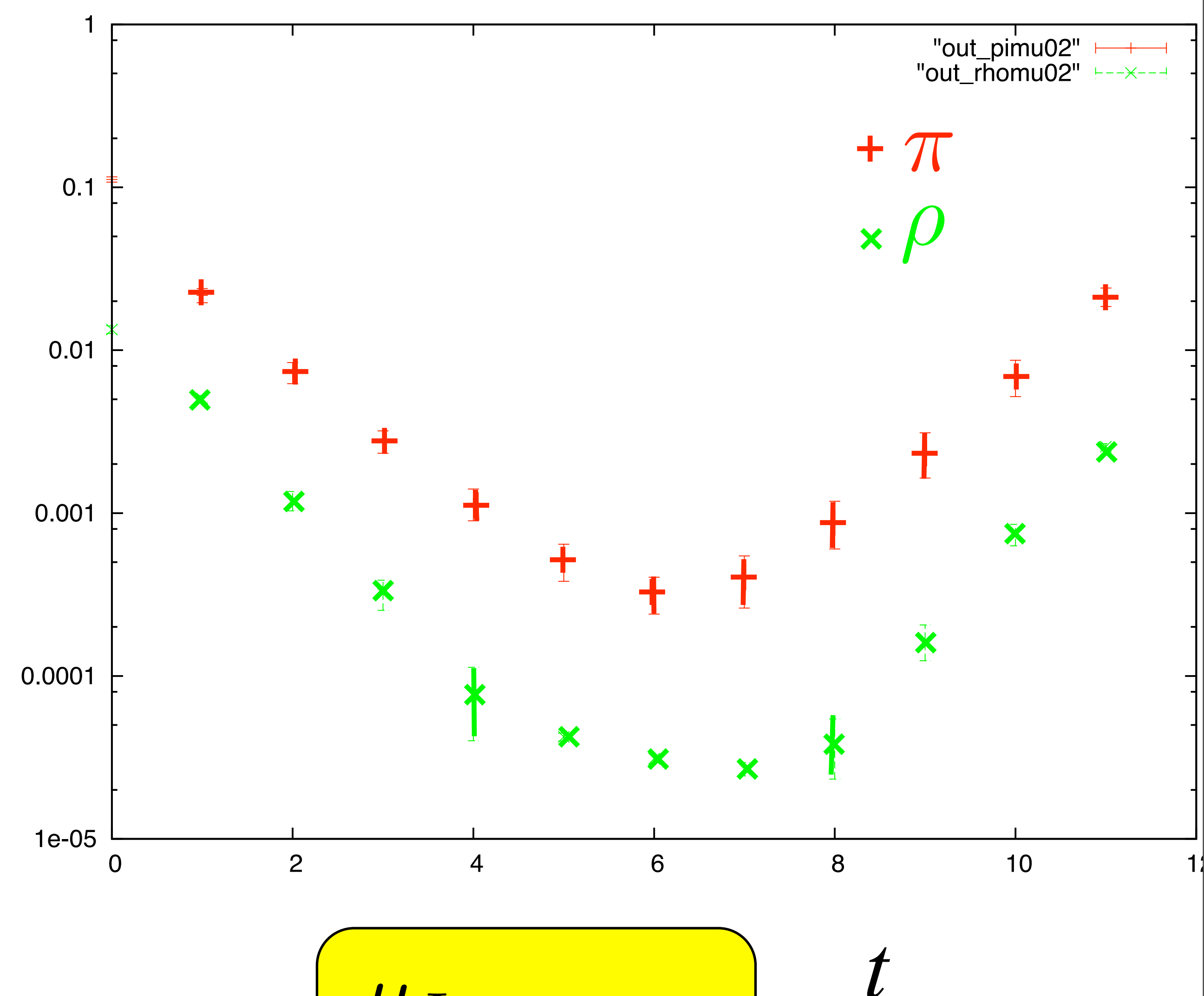
$T_c(\mu_I)/T_c(0)$
1.16
1.12
1.08
1.04
1
Quadratic
Quartic
Pade
0
0.2
0.4
0.6
0.8
1
μ_I/T_c



$T_c(\mu_I)/T_c(0)$
1
0.96
0.92
0.88
Quadratic
Quartic
Pade
0
0.2
0.4
0.6
0.8
1
μ/T_c

pi and rho propagators



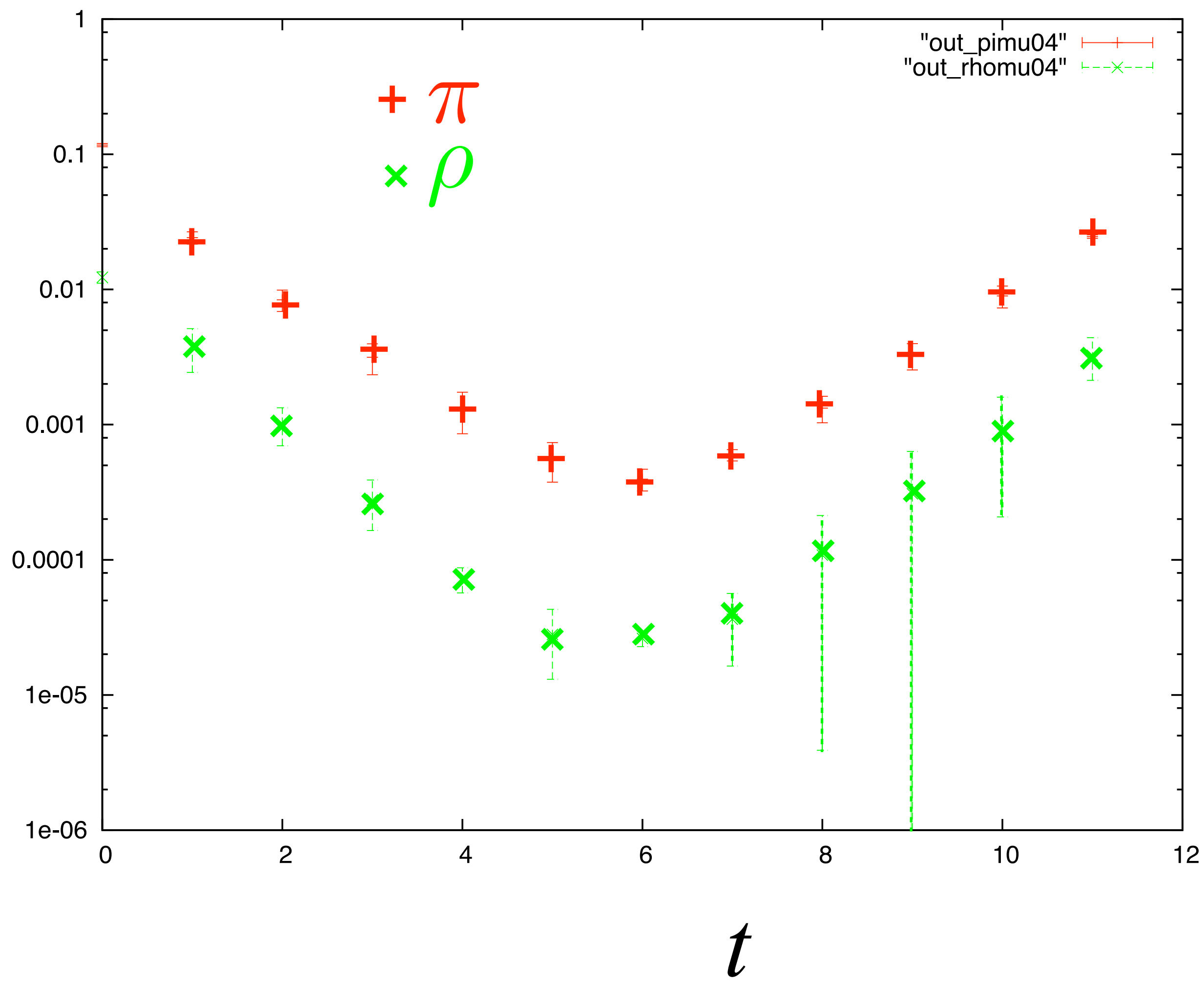


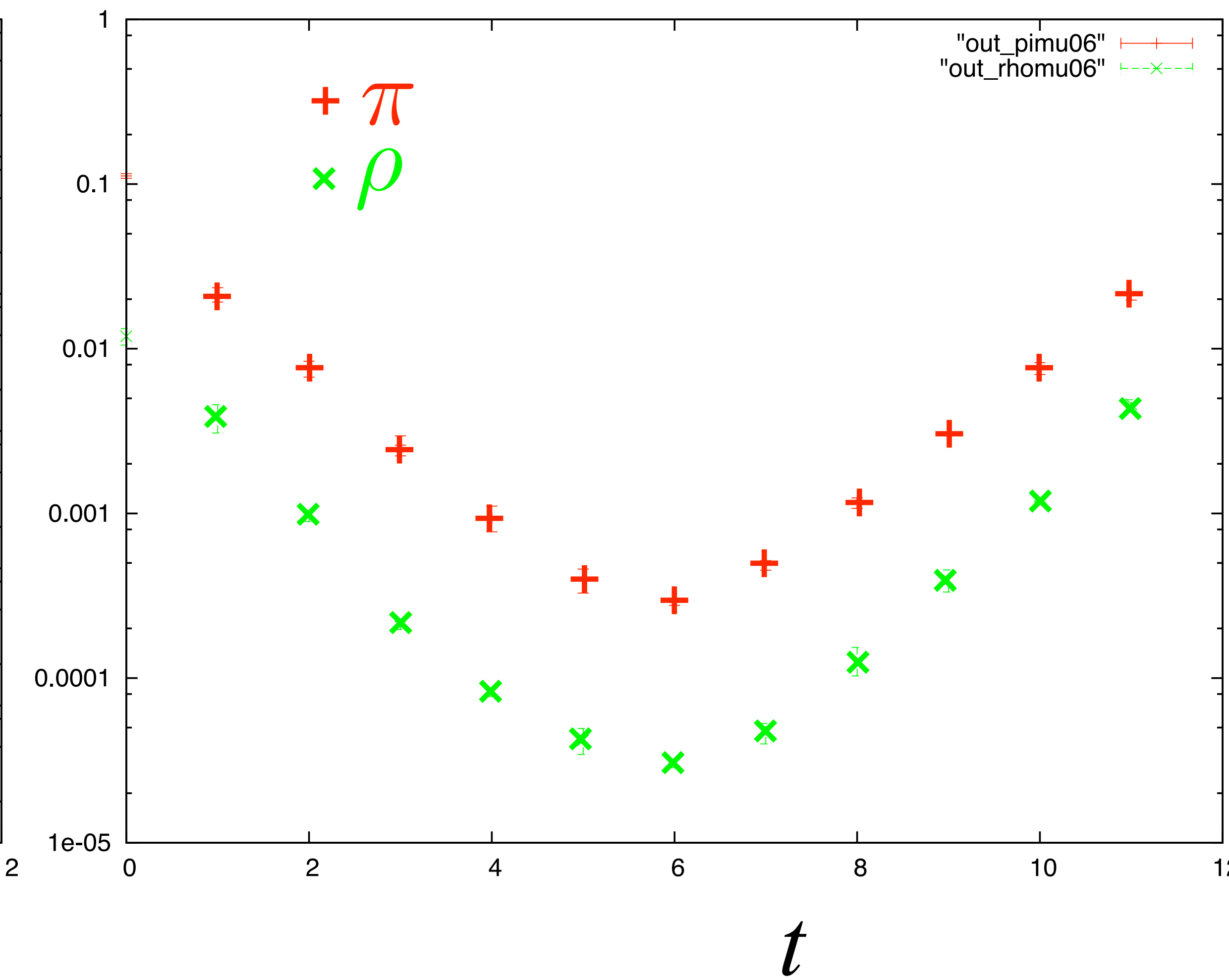
$$\frac{\mu_I}{T} = 0.0$$

$$\frac{\mu_I}{T} = 0.24$$

$$6 \times 6 \times 6 \times 12 \quad T \sim \frac{1}{3} T_c \qquad m_{PS}/m_V = 0.65 \sim 0.8$$

pi and rho propagators (2)





$$\frac{\mu_I}{T} = 0.48$$

$$\frac{\mu_I}{T} = 0.72$$

$$6 \times 6 \times 6 \times 12$$

To Do List (1)

We (Nagata and I) must learn how to find good Reweighting parameters (often only a few out of 100 configurations contribute), and how to be sure the overlap is large enough.

We must check the fluctuation of Canonical Ensamble (obtained by the Fugacity expansion) coefficients is under controle.

Trials to increase Signal/Noise ration,

☆ e.g., Fugacity expansion, then set $C_n = 0$ for n=3m+1, 3m+2.

To Do List (2) (after List (1))

- Larger Volume $12^3 \times 6$
- Smaller quark mass
- To Investigate QCD Phase using both real and imaginary chemical potential.
- Canonical Ensamble approach