Polyakov-Loop-Extended NJL Model
meeting
Schwinger-Dyson Equations

Thomas Hell
Kouji Kashiwa, Nino Bratovic, Wolfram Weise

Quarks, Gluons, and Hadronic Matter
under Extreme Conditions
Schlosshotel Rheinfels, St. Goar
March 15 – March 18, 2011
QCD Phase Diagram: Expectation

- Hadron gas ($\langle \bar{q}q \rangle \neq 0$), $\bar{q}q$ condensation
- Cross-over critical point
- Quark-gluon plasma ($\langle \Phi \rangle \sim 1$)
- Color superconductor ($\langle qq \rangle \neq 0$)
- Vacuum
- Nuclear matter
- Crystalline? CFL
- Universe
- ~0.2 GeV

Temperature

Baryon chemical potential
QCD Phase Diagram: Expectation & Facts

- Expectation:
  - Quark-gluon plasma: $\langle \Phi \rangle \sim 1$

- Facts:
  - $\Phi \sim 1$

Graph:
- Temperature: $T$
- Baryon chemical potential: $\mu_B$
- Hadron gas: $\langle q\bar{q} \rangle \neq 0$
- CFL: $\langle q\bar{q} \rangle \neq 0$
- Vacuum
- Universe
- Cross-over
- Nuclear matter
- Neutron stars?
QCD Phase Diagram: Expectation & Facts

- quark-gluon plasma: $\langle \Phi \rangle \sim 1$
- cross-over: $\sim 0.2$ GeV

Universe

- Hadron gas: $\langle \bar{q}q \rangle \neq 0$
- $\bar{q}q$ condensation

Vacuum

- $T_0 = 270$ MeV

- Polyakov loop $\Phi$

- Rapid acquisition of mass is effect of gluon cloud

- $m = 0$ (Chiral limit)
- $m = 30$ MeV
- $m = 70$ MeV

Lattice data from:
- C. D. Roberts, Nucl. Phys. 61, 50 (2008)
- O. Kaczmarek et al. (2002)
- S. Borsanyi et al., JHEP 09, 073 (2010) (BMW)
Outline

1. Symmetries and the Polyakov–Nambu–Jona-Lasinio Model
   - NJL Model
     - spontaneous chiral symmetry breaking
     - chiral condensate $\langle \bar{\psi}\psi \rangle$
   - Polyakov Loop
     - confinement-deconfinement transition
     - Polyakov loop $\langle \Phi \rangle$

   ✦ Nonlocal NJL Model

2. Thermodynamics of the Nonlocal PNJL Model
   - Polyakov Loop and Deconfinement-Confinement Transition
   - Finite Temperatures and Finite Densities

3. Summary
Outline

1. Nambu–Jona-Lasinio Model
   - Schwinger-Dyson Equation

\[ S(p)^{-1} = Z_2(-\phi + m_0) - Z_1 \int \frac{d^4 q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \frac{\lambda_a}{2} \gamma_\mu S(q) \frac{\lambda_a}{2} \Gamma_\nu(q,p) \]

2. Nonlocal NJL Model

2. Thermodynamics of the Nonlocal PNJL Model
   - Polyakov Loop and Deconfinement-Confinement Transition
   - Finite Temperatures and Finite Densities

3. Summary
Schwinger-Dyson Equations
and
Nonlocal Nambu–Jona-Lasinio Model
Schwinger-Dyson Equations

SD Equation for the Full Quark Propagator

\[ S(p)^{-1} = Z_2 \left( -p + m_0 \right) - Z_1 \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p - q) \frac{\lambda_a}{2} \gamma_\mu S(q) \frac{\lambda_a}{2} \Gamma_\nu(q, p) \]
Schwinger-Dyson Equations

SD Equation for the Full Quark Propagator

\[ \frac{1}{S(p)} = \frac{1}{Z_2(-p + m_0)} - Z_1 \int \frac{d^4 q}{(2\pi)^4} g^2 D_{\mu\nu}(p - q) \frac{\lambda_a}{2} \gamma_\mu S(q) \frac{\lambda_a}{2} \Gamma_\nu(q, p) \]

Most General Solution

\[ S(p) = -\frac{1}{A(p^2)\phi - B(p^2)} = -\frac{Z(p^2)}{\phi - M(p^2)} \]

C. D. Roberts, A. G. Williams, S. M. Schmidt

renormalization function

mass function
Schwinger-Dyson Equations

Most General Solution

\[ S(p) = \frac{1}{A(p^2)p - B(p^2)} = \frac{Z(p^2)}{p - M(p^2)} \]

Mass Function


Renormalization Function


Rapid acquisition of mass is the effect of the gluon cloud.

Lattice data from P. O. Bowman, et al.,
Three-Flavor Nonlocal NJL Model

Nonlocal Action with Vector-Type Interaction

\[
S_E = \int d^4x \left\{ \bar{\psi}(x) \left[ -i \gamma^\mu \partial_\mu + \hat{m}_q \right] \psi(x) \right.
\]
\[
- \frac{G}{2} \left[ j_\alpha^S(x) j_\alpha^S(x) + j_\alpha^P(x) j_\alpha^P(x) + J(x) J(x) \right]
\]
\[
- \frac{H}{4} A_{\alpha\beta\gamma} \left[ j_\alpha^S(x) j_\beta^S(x) j_\gamma^S(x) - 3 j_\alpha^S(x) j_\beta^P(x) j_\gamma^P(x) \right]
\]

\[
\hat{m}_q = \text{diag}(m_u, m_d, m_s)
\]

Currents

\[
j_\alpha^S(x) = \int d^4z \, C(z) \bar{\psi} \left( x + \frac{z}{2} \right) \lambda_\alpha \psi \left( x - \frac{z}{2} \right)
\]
\[
j_\alpha^P(x) = \int d^4z \, C(z) \bar{\psi} \left( x + \frac{z}{2} \right) i\gamma_5 \lambda_\alpha \psi \left( x - \frac{z}{2} \right)
\]
\[
J(x) = \int d^4z \, F(z) \bar{\psi} \left( x + \frac{z}{2} \right) \frac{i\partial}{2\kappa} \psi \left( x - \frac{z}{2} \right)
\]

KM-'t Hooft determinant: axial-U(1) breaking

\[
\bar{\psi}(x') \partial_x \psi(x) := \bar{\psi}(x') (\partial_x \psi)(x) - (\partial_x \bar{\psi})(x') \psi(x)
\]
Three-Flavor Nonlocal NJL Model

\[ S_E = \int d^4x \left\{ \bar{\psi}(x) \left[ -i\gamma^\mu \partial_\mu + \hat{m}_q \right] \psi(x) \right. \]
\[ \left. - \frac{G}{2} \left[ j^S_\alpha(x) j^S_\alpha(x) + j^P_\alpha(x) j^P_\alpha(x) + J(x) J(x) \right] \right. \]
\[ \left. - \frac{H}{4} \mathcal{A}_{\alpha\beta\gamma} \left[ j^S_\alpha(x) j^S_\beta(x) j^S_\gamma(x) - 3 j^S_\alpha(x) j^P_\beta(x) j^P_\gamma(x) \right] \right\} \]

WFR Function \( Z(p) \)

\[ Z(p) = \left( 1 - \bar{v}/\kappa F(p) \right)^{-1} \]

Correlation Function \( C(p) \)

\[ C(p) \sim \alpha_{\text{QCD}} (p^2)/p^2 \]

A. Scarpettini et al., PRD69 (2004)
S. Noguera, N.N. Scoccola, PRD78 (2008)


T. Hell et al.
Nonlocal NJL Model

Parameters and Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_u = m_d$</td>
<td>2.9 MeV</td>
</tr>
<tr>
<td>$m_s$</td>
<td>69.3 MeV</td>
</tr>
<tr>
<td>$G$</td>
<td>1.63 fm$^2$</td>
</tr>
<tr>
<td>$H$</td>
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<tr>
<td>$\kappa$</td>
<td>5.28 GeV</td>
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<tbody>
<tr>
<td>$\langle \bar{u}u \rangle^{1/3}$</td>
<td>$-0.290$ GeV</td>
</tr>
<tr>
<td>$\langle \bar{d}d \rangle^{1/3}$</td>
<td>$-0.309$ GeV</td>
</tr>
<tr>
<td>$\langle \bar{s}s \rangle^{1/3}$</td>
<td>0.369 GeV</td>
</tr>
<tr>
<td>$M_u(0) = M_d(0)$</td>
<td>0.583 GeV</td>
</tr>
<tr>
<td>$M_s(0)$</td>
<td>$-1.95$ GeV</td>
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<td>$\bar{v}$</td>
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<thead>
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<tr>
<td>$m_\pi$</td>
<td>142 MeV</td>
</tr>
<tr>
<td>$m_K$</td>
<td>503 MeV</td>
</tr>
<tr>
<td>$m_\eta$</td>
<td>553 MeV</td>
</tr>
<tr>
<td>$m_{\eta'}$</td>
<td>957 MeV</td>
</tr>
<tr>
<td>$f_\pi$</td>
<td>84.4 MeV</td>
</tr>
<tr>
<td>$f_K$</td>
<td>100.3 MeV</td>
</tr>
<tr>
<td>$\theta_{\eta'}$</td>
<td>$-29.9^\circ$</td>
</tr>
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Gap Equations in Mean-Field Approximation

\[
\bar{\sigma}_i = -G \bar{S}_i - \frac{H}{4} \varepsilon_{ijk} \varepsilon_{ijk} \bar{S}_j \bar{S}_k
\]

\[
\bar{S}_i = -8N_c \int \frac{d^4p}{(2\pi)^4} C(p) \frac{Z(p)M_i(p)}{p^2 + M_i^2(p)}
\]

\[
\bar{v} = -\frac{4N_cG}{\kappa} \sum_{i \in \{u,d,s\}} \int \frac{d^4p}{(2\pi)^4} F(p) \frac{p^2 Z(p)}{p^2 + M_i^2(p)}
\]
Nonlocal NJL Model

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### Mass Functions

$M_i(p) = Z(p)(m_i + \bar{\sigma}_i C(p))$

- $M_u(p)$
- $M_d(p)$
- $M_s(p)$

### WFR Function

$Z(p) = (1 - \bar{\nu}/\kappa F(p))^{-1}$

- $m = 14.0$ MeV
- $m = 11.2$ MeV

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The Nonlocal Polyakov-Loop-Extended NJL Model
Polyakov Loop and Confinement

<table>
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| \[
\langle \Phi(\vec{x}) \rangle = \frac{1}{N_c} \langle \text{tr}_c [L(\vec{x})] \rangle 
\]
| \[
L(\vec{x}) = \mathcal{P} \exp \left\{ i \int_0^\beta d\tau A_4(\tau, \vec{x}) \right\}, \quad \beta = 1/T
\] |

| \[
\langle \Phi \rangle = e^{-\beta \mathcal{F}_q}
\] |

Polyakov loop serves as an order parameter for confinement-deconfinement phase transition.
Polyakov Loop and Confinement

Polyakov-Fukushima Effective Potential

\[
\frac{U}{T^4} = -\frac{1}{2} b_2(T) \Phi^* \Phi + b_4(T) \ln \left[ 1 - 6 \Phi^* \Phi + 4 \left( \Phi^3 + \Phi^3 \right) - 3 (\Phi^* \Phi)^2 \right]
\]

with

\[
\Phi = \frac{1}{3} \text{tr}_c \left[ \exp \left( i \left( \beta A_3^3 \lambda_3 + \beta A_8^8 \lambda_8 \right) \right) \right]
\]

Z(3) invariant

Effective Potential

Pure-Gauge Lattice Results

\[
T_0 = 270 \text{ MeV}
\]

lattice data from
Nonlocal PNJL Model

Partition Function

\[ Z = \int \mathcal{D}\varphi_a \int \mathcal{D}A e^{-S_{\text{E bos}}^\text{bos}(\varphi_a, A)} \]

Thermodynamic Potential in Mean-Field Approximation

\[ \Omega = -2T \sum_{f \in \{u,d,s\}} \sum_{i=0,\pm} \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \ln \left[ \frac{\omega_{f,n}^2 + \bar{p}^2 + M_f^2(\omega_{f,n}^i, \bar{p})}{Z^2(\omega_{f,n}^i, \bar{p})} \right] \]

\[ - \frac{1}{2} \left\{ \sum_{f \in \{u,d,s\}} \left( \bar{\sigma}_f \bar{S}_f + \frac{G}{2} \bar{S}_f \bar{S}_f \right) + \frac{H}{2} \bar{S}_u \bar{S}_d \bar{S}_s \right\} + \frac{\bar{v}^2}{2G} + U(\Phi, \Phi^*, T) \]

- \( M_f(\omega_{f,n}^i, \bar{p}) = Z(\omega_{f,n}^i, \bar{p}) \left( m_f + \bar{\sigma}_f C(\omega_{f,n}^i, \bar{p}) \right) \), \( Z(\omega_{f,n}^i, \bar{p}) = \left( 1 - \frac{\bar{v}}{\kappa} F(\omega_{f,n}^i, \bar{p}) \right)^{-1} \)

- Matsubara frequencies \( \omega_n = (2n + 1)\pi T, n \in \mathbb{Z} \)

\[ \omega_{\pm}^{\pm} = \omega_n - i\mu_f \pm A_4^3/2 - A_4^8/(2\sqrt{3}), \quad \omega_0^{\pm} = \omega_n - i\mu_f + A_4^8/\sqrt{3} \]
Condensates and Polyakov Loop

Gap Equations

\[
\frac{\partial \Omega}{\partial \bar{\sigma}_u} = \frac{\partial \Omega}{\partial \bar{\sigma}_s} = \frac{\partial \Omega}{\partial \bar{v}} = \frac{\partial \Omega}{\partial A_4^3} = \frac{\partial \Omega}{\partial A_4^8} = 0
\]

Normalized Temperature

\[\langle \bar{u}u \rangle \]

\[\langle \bar{s}s \rangle \]

\[N_\tau = 6, p4\]

\[N_\tau = 8, \text{HISQ}\]

\[N_\tau = 8, 12, 16, \text{stout}\]

S. Borsanyi et al., JHEP 09, 073 (2010) (BMW)
Condensates and Polyakov Loop

Gap Equations

\[
\frac{\partial \Omega}{\partial \bar{\sigma}_u} = \frac{\partial \Omega}{\partial \bar{\sigma}_s} = \frac{\partial \Omega}{\partial \bar{v}} = \frac{\partial \Omega}{\partial A^3_4} = \frac{\partial \Omega}{\partial A^8_4} = 0
\]

Normalized Temperature

Absolute Temperature

\( \langle \bar{u}u \rangle \) and \( \langle \bar{s}s \rangle \)

M. Cheng et al., (2008); A. Bazavov et al., (2010) "hotQCD"

S.~Borsanyi et al., JHEP 09, 073 (2010) (BMW)

\( N_T = 6, p4 \)

\( N_T = 8, \text{HISQ} \)

\( N_T = 8,12,16, \) stout

\( T / T_c \)

\( T \) [GeV]
Condensates and Polyakov Loop

Gap Equations

\[ \frac{\partial \Omega}{\partial \bar{\sigma}_u} = \frac{\partial \Omega}{\partial \bar{\sigma}_s} = \frac{\partial \Omega}{\partial \bar{v}} = \frac{\partial \Omega}{\partial A^3_4} = \frac{\partial \Omega}{\partial A^8_4} = 0 \]

\[ T_0 = 187 \text{ MeV}, \text{ with pions} \]


\[ T_0 = 270 \text{ MeV}, \text{ Polyakov loop} \]

S. Borsanyi et al., JHEP 09, 073 (2010) (BMW)
O. Kaczmarek et al., (2002)
QCD Phase Diagram

Phase Diagram for 2 + 1 Flavors

\[ \mu_u = \mu_d = \mu_s \equiv \mu \]
QCD Phase Diagram

Phase Diagram for 2 + 1 Flavors

Critical point

\[ \mu_{\text{crit}} = 135 \text{ MeV} \]
\[ T_{\text{crit}} = 188 \text{ MeV} \]

Polyakov Loop

Chiral crossover

Chiral 1st order

Nuclear territory
QCD Phase Diagram

Phase Diagram for 2 + 1 Flavors

\[ \mu_u = \mu_d = \mu_s \equiv \mu \]

**critical point**

\[ \mu_{\text{crit}} = 135 \text{ MeV} \]
\[ T_{\text{crit}} = 188 \text{ MeV} \]

\[ T_c(\mu) / T_c(\mu = 0) = 1 - \kappa_\chi \left( \frac{\mu}{T} \right)^2 \]
\[ + \mathcal{O} \left( \left( \frac{\mu}{T} \right)^4 \right) \]

**Polyakov Loop**

**chiral crossover**

**chiral 1^{st} order**

O. Kaczmarek et al., PRD83 (2011)

\[ \kappa_\chi = 0.11 \pm 0.02 \]
\[ (\kappa_\chi, \text{lat} = 0.059(2)(4)) \]
QCD Phase Diagram

Inclusion of back-reaction effects of the matter sector

2 + 1 Flavors

\[ T/T_c \]

Polyakov Loop
chiral crossover
chiral 1\textsuperscript{st} order

2 Flavors

\[ T \text{ [MeV]} \]

\[ \mu \text{ [MeV]} \]

\[ \chi \text{ crossover} \]
\[ \Phi \text{ crossover} \]
\[ \Phi \text{ crossover} \]
CEP
\[ \chi \text{ first order} \]
Summary and Outlook
Summary and Outlook

Nonlocal NJL Model
- spontaneous chiral symmetry breaking
- chiral condensate $\langle \bar{\psi} \psi \rangle$

Polyakov Loop
- confinement-deconfinement transition
  Polyakov loop $\langle \Phi \rangle$
Summary and Outlook

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**Nonlocal PNJL Model**

- nonlocal PNJL model $\leftrightarrow$ SD approach
- momentum-dependent quark mass
- Thermodynamics of strong interaction
- entanglement of chiral and deconfinement transition
- schematic “sketch” of QCD phase diagram