Polyakov-Loop-Extended NJL Model
meeting
Schwinger-Dyson Equations

Thomas Hell
Kouji Kashiwa, Nino Bratovic, Wolfram Weise

Quarks, Gluons, and Hadronic Matter
under Extreme Conditions
Schlosshotel Rheinfels, St. Goar
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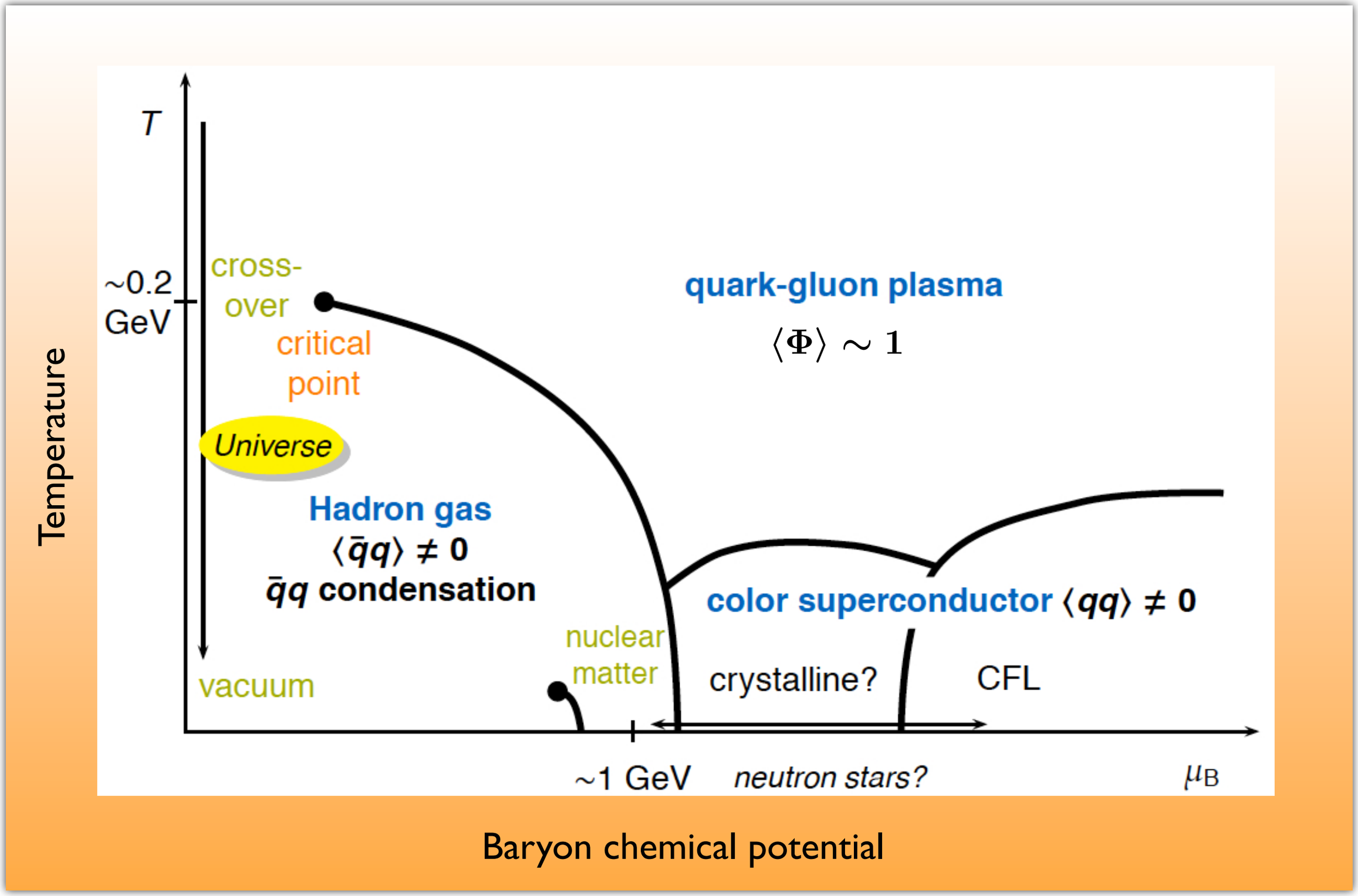
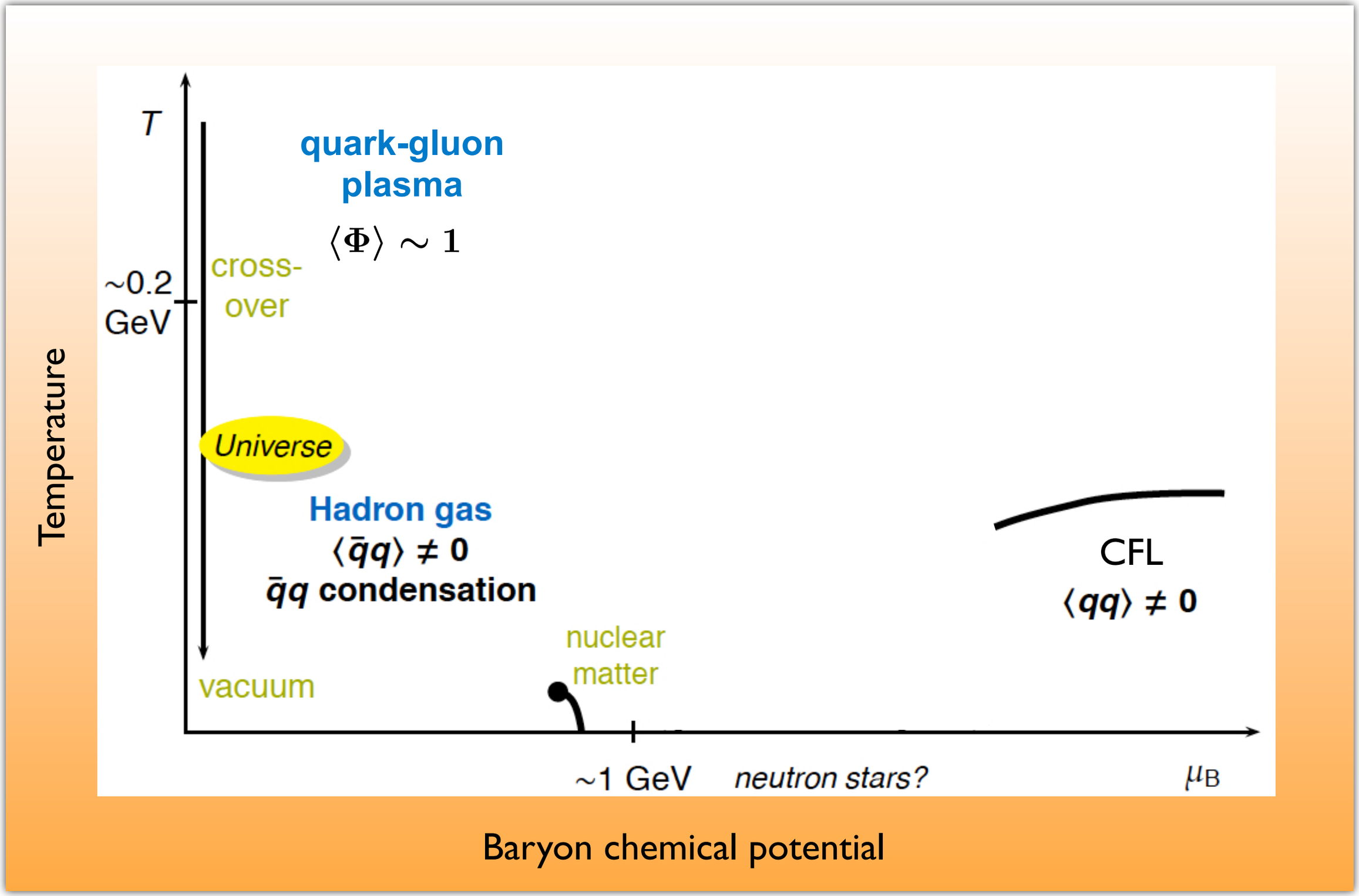
Technische Universität München

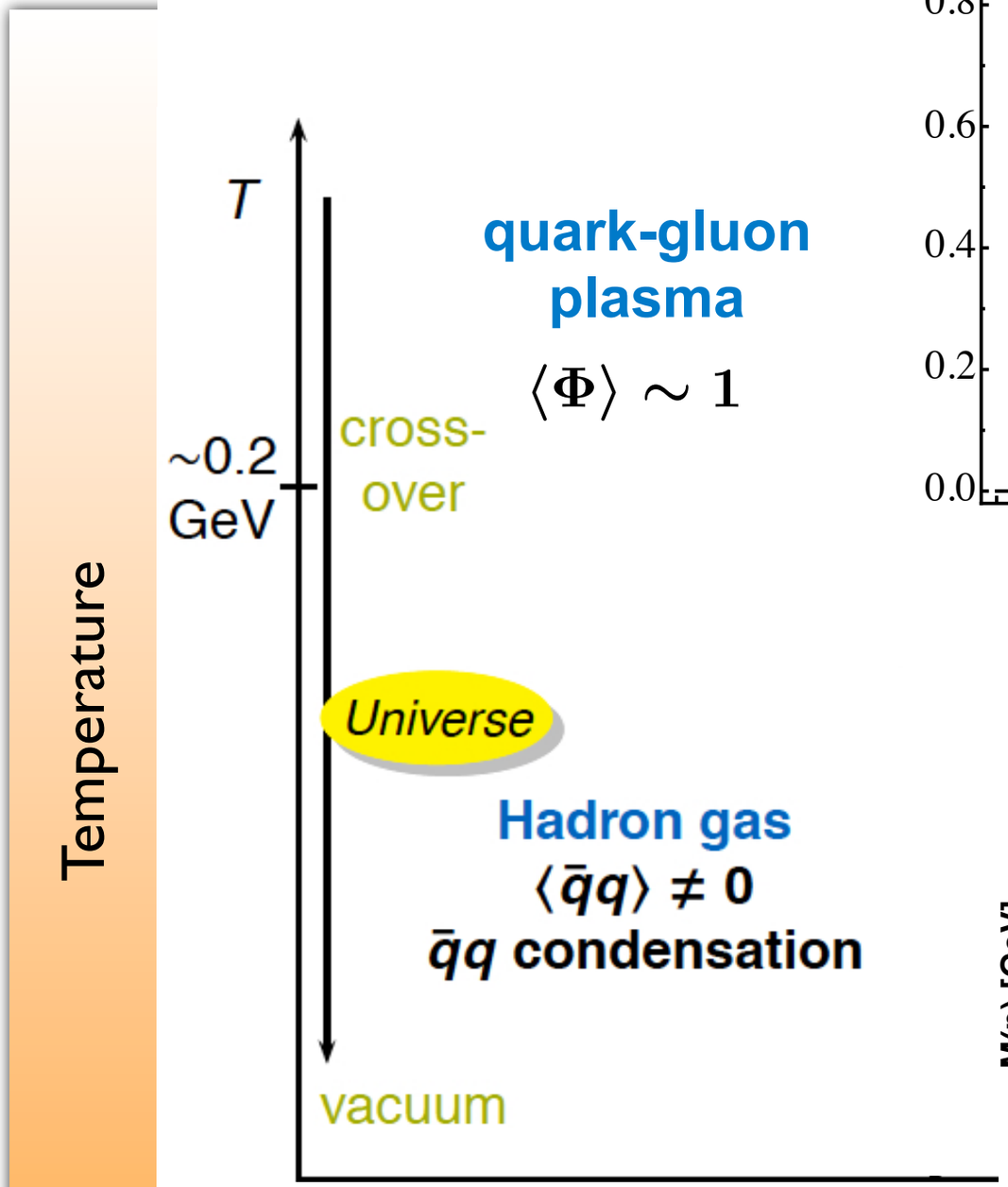
QCD Phase Diagram: Expectation

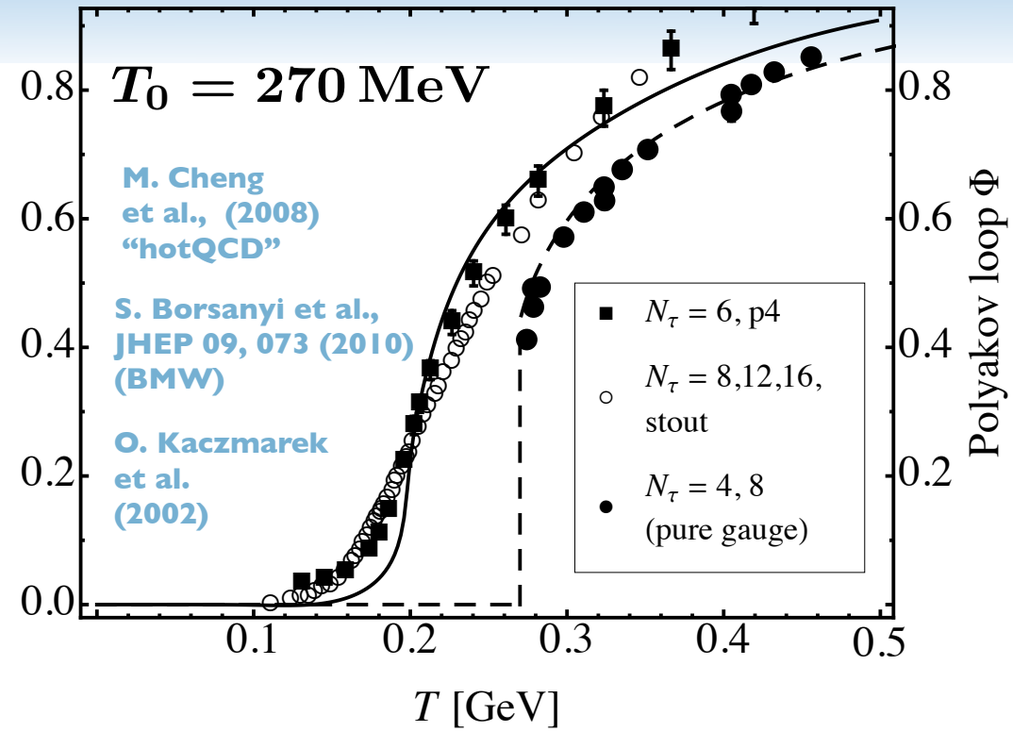


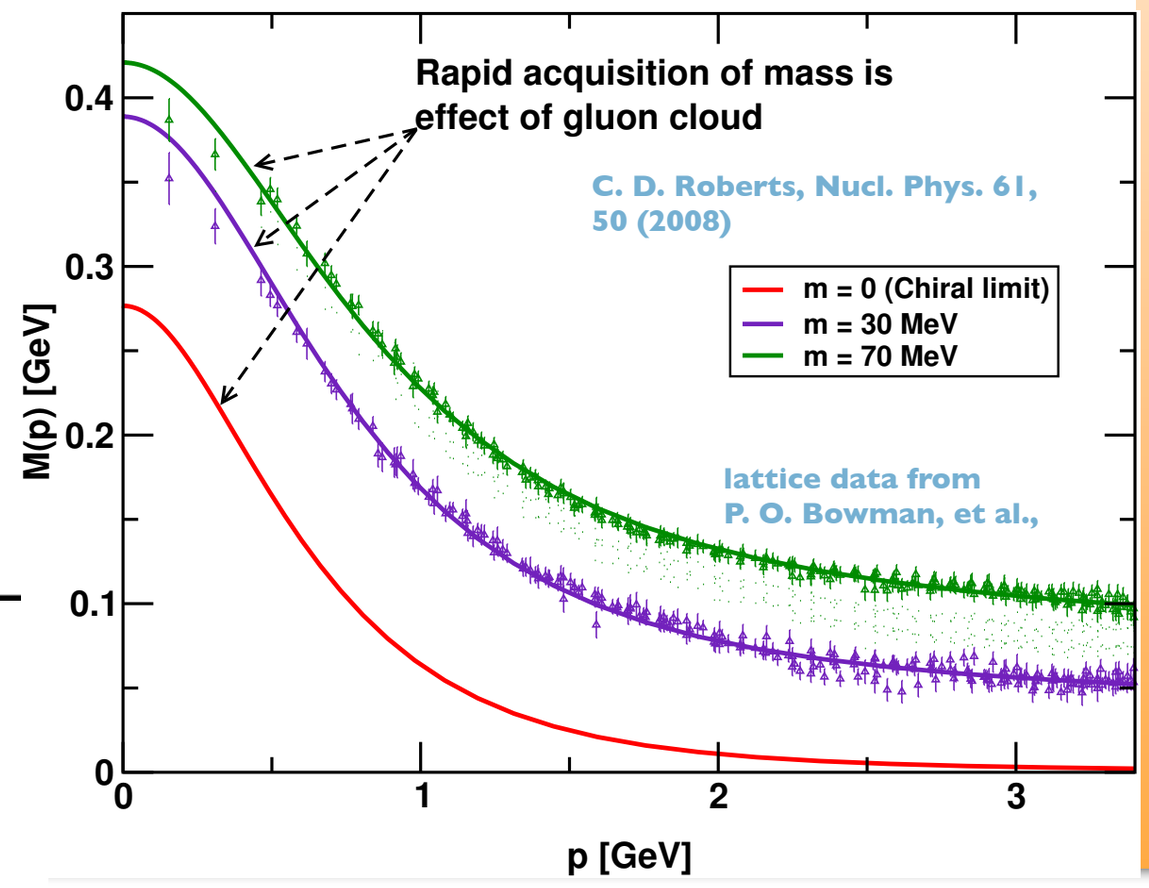
QCD Phase Diagram: Expectation & Facts



QCD Phase Diagram: Expectation & Facts







Outline

❶ Symmetries and the Polyakov–Nambu–Jona-Lasinio Model

NJL Model

spontaneous chiral symmetry breaking

chiral condensate $\langle \bar{\psi}\psi \rangle$

Polyakov Loop

confinement-deconfinement transition

Polyakov loop $\langle \Phi \rangle$

PNJL Model

chiral condensate $\langle \bar{\psi}\psi \rangle$ ⟷ Polyakov loop $\langle \Phi \rangle$

✦ **Nonlocal** NJL Model

❷ Thermodynamics of the Nonlocal PNJL Model

✦ Polyakov Loop and Deconfinement-Confinement Transition

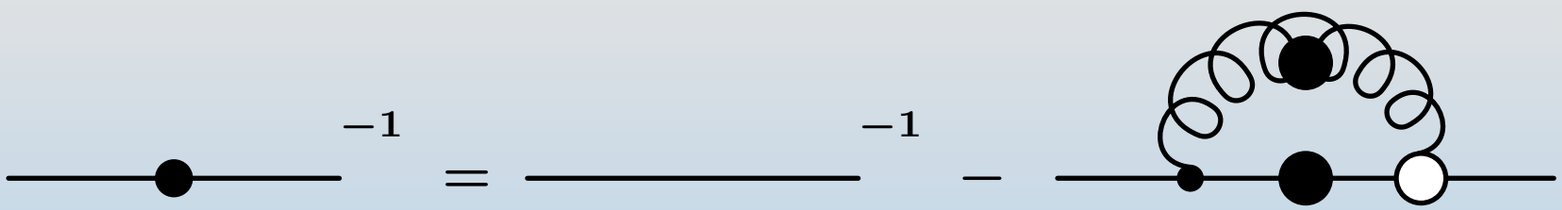
✦ Finite Temperatures and Finite Densities

❸ Summary

Outline

❶ Nambu–Jona-Lasinio Model

✦ Schwinger-Dyson Equation



$$S(p)^{-1} = Z_2(-\not{p} + m_0) - Z_1 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \frac{\lambda_a}{2} \gamma_\mu S(q) \frac{\lambda_a}{2} \Gamma_\nu(q,p)$$

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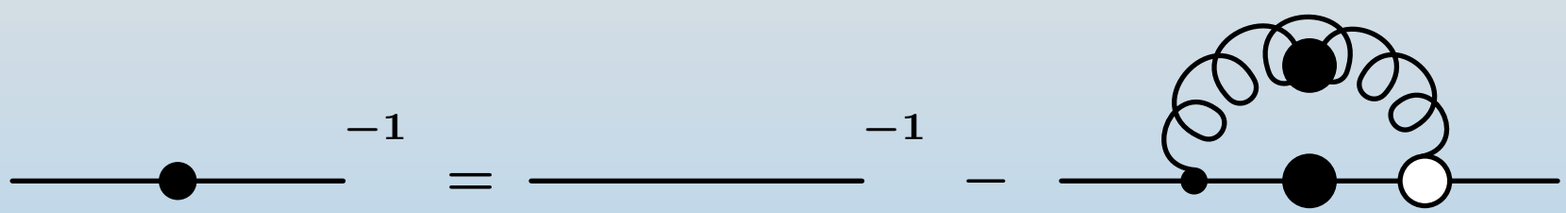
Schwinger-Dyson Equations
and
Nonlocal Nambu–Jona-Lasinio Model

Schwinger-Dyson Equations

SD Equation for the Full Quark Propagator



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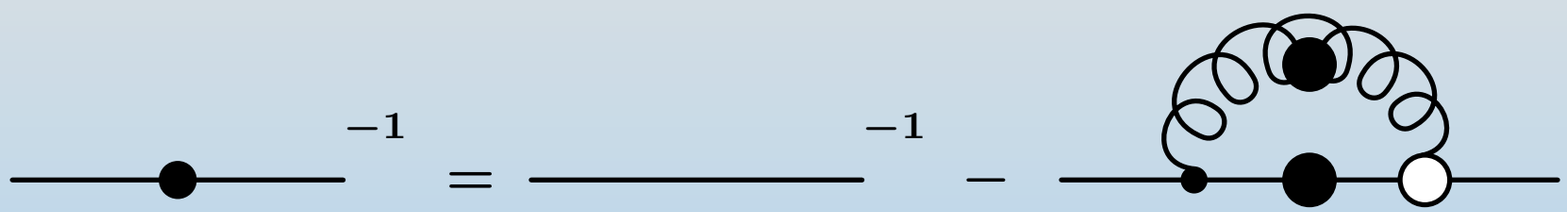
quark-wave-function renormalization

quark-gluon-vertex renormalization

Schwinger-Dyson Equations

SD Equation for the Full Quark Propagator



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Most General Solution

C. D. Roberts, A. G. Williams, S. M. Schmidt

$$S(p) = -\frac{1}{A(p^2)\not{p} - B(p^2)} = -\frac{Z(p^2)}{\not{p} - M(p^2)}$$

renormalization function

mass function

Schwinger-Dyson Equations

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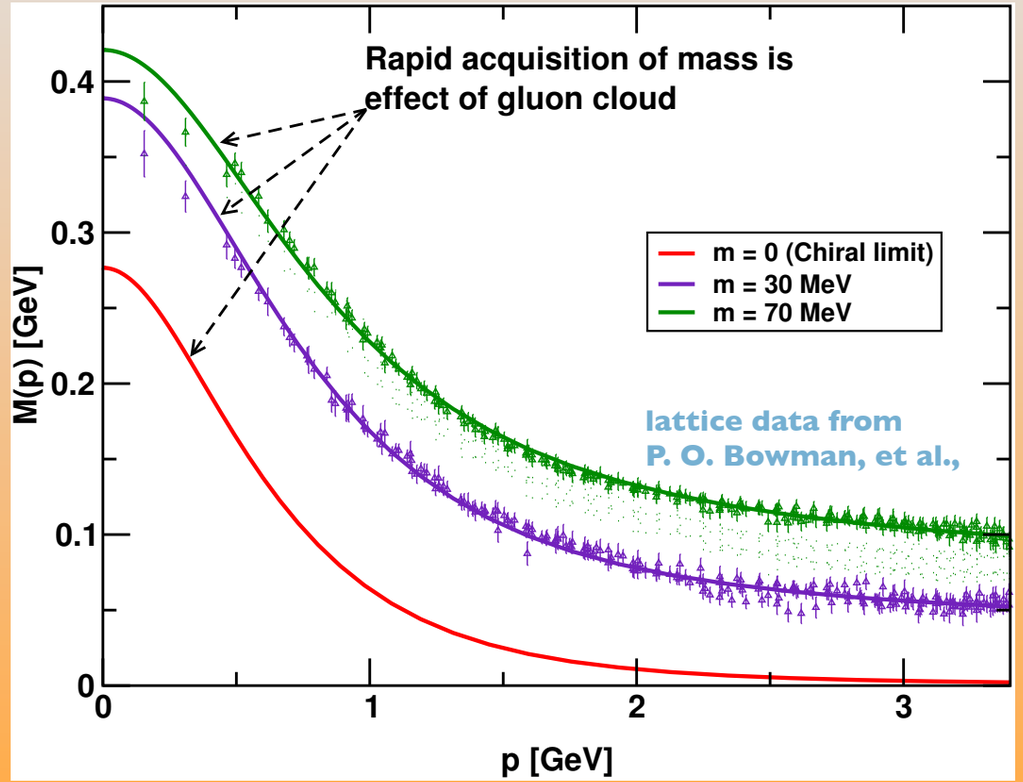
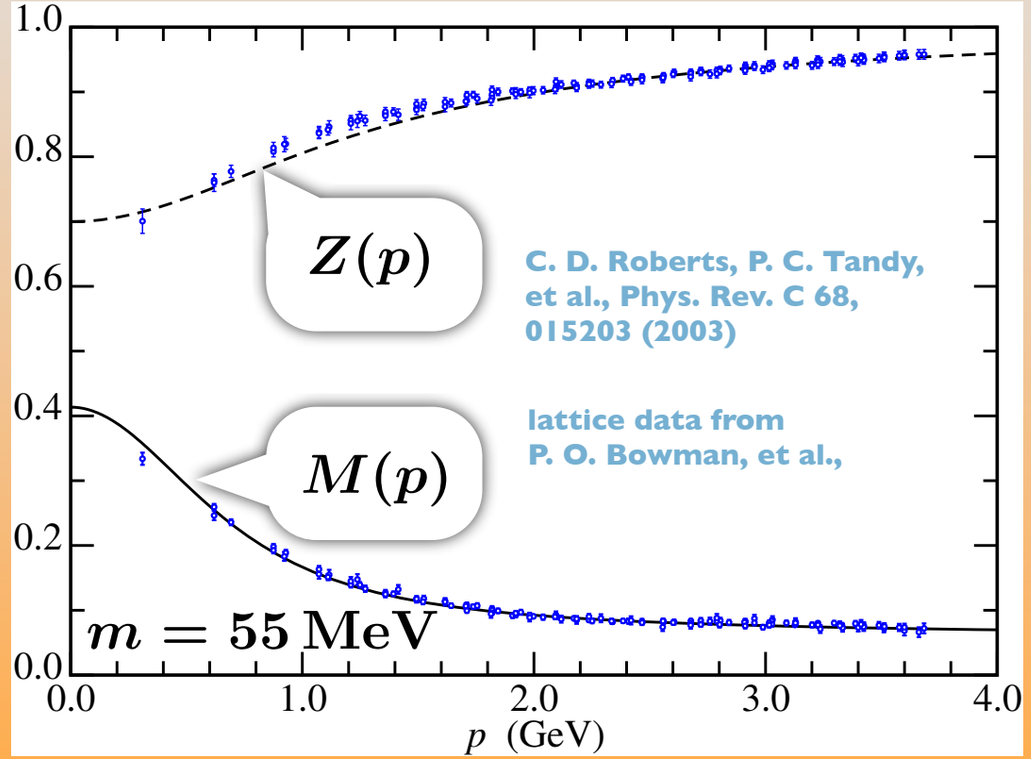
Mass Function

C. D. Roberts, Nucl. Phys. 61, 50 (2008)



Renormalization Function



Three-Flavor Nonlocal NJL Model

Nonlocal Action with Vector-Type Interaction

A. Scarpettini et al., PRD69 (2004)
S. Noguera, N.N. Scoccola, PRD78 (2008)

$$\mathcal{S}_{\mathrm{E}} = \int \mathrm{d}^4x \Big\{ \bar{\psi}(x)\left[-\mathrm{i}\gamma^\mu\partial_\mu + \hat{m}_q\right]\psi(x)$$

$$- \frac{G}{2}\left[j^S_\alpha(x)j^S_\alpha(x) + j^P_\alpha(x)j^P_\alpha(x) + J(x)J(x)\right]$$

$$- \frac{H}{4}\mathcal{A}_{\alpha\beta\gamma}\left[j^S_\alpha(x)j^S_\beta(x)j^S_\gamma(x) - 3j^S_\alpha(x)j^P_\beta(x)j^P_\gamma(x)\right]\Big\}$$

$\hat{m}_q = \mathrm{diag}(m_u, m_d, m_s)$

chirally invariant

KM-'t Hooft determinant: axial-U(1) breaking

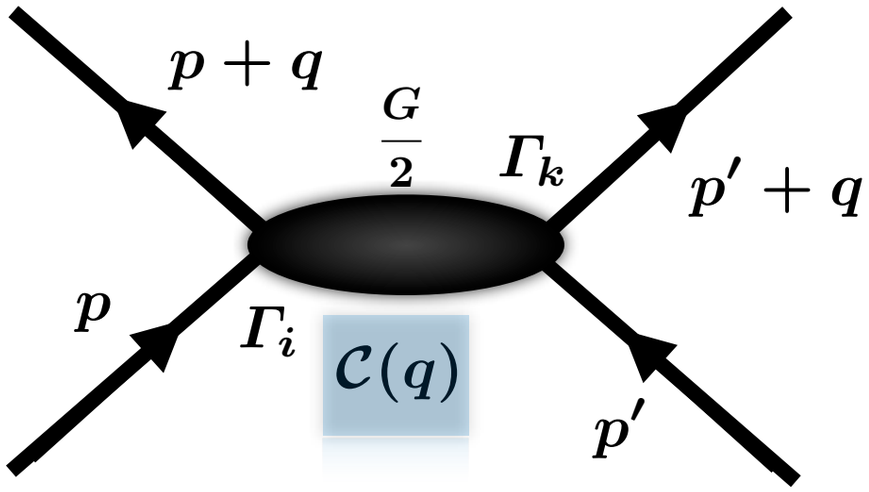
Currents

$$j^S_\alpha(x) = \int \mathrm{d}^4z\, \mathcal{C}(z)\, \bar{\psi}\left(x+\frac{z}{2}\right)\lambda_\alpha\,\psi\left(x-\frac{z}{2}\right)$$

$$j^P_\alpha(x) = \int \mathrm{d}^4z\, \mathcal{C}(z)\, \bar{\psi}\left(x+\frac{z}{2}\right)\mathrm{i}\gamma_5\lambda_\alpha\,\psi\left(x-\frac{z}{2}\right)$$

$$J(x) = \int \mathrm{d}^4z\, \mathcal{F}(z)\, \bar{\psi}\left(x+\frac{z}{2}\right)\frac{\mathrm{i}\overleftrightarrow{\partial\!\!\!/}}{2\kappa}\psi\left(x-\frac{z}{2}\right)$$

$$\bar{\psi}(x')\overleftrightarrow{\partial\!\!\!/}\psi(x) := \bar{\psi}(x')\,(\partial\!\!\!/\psi)(x) - (\partial\!\!\!/\bar{\psi})(x')\,\psi(x)$$



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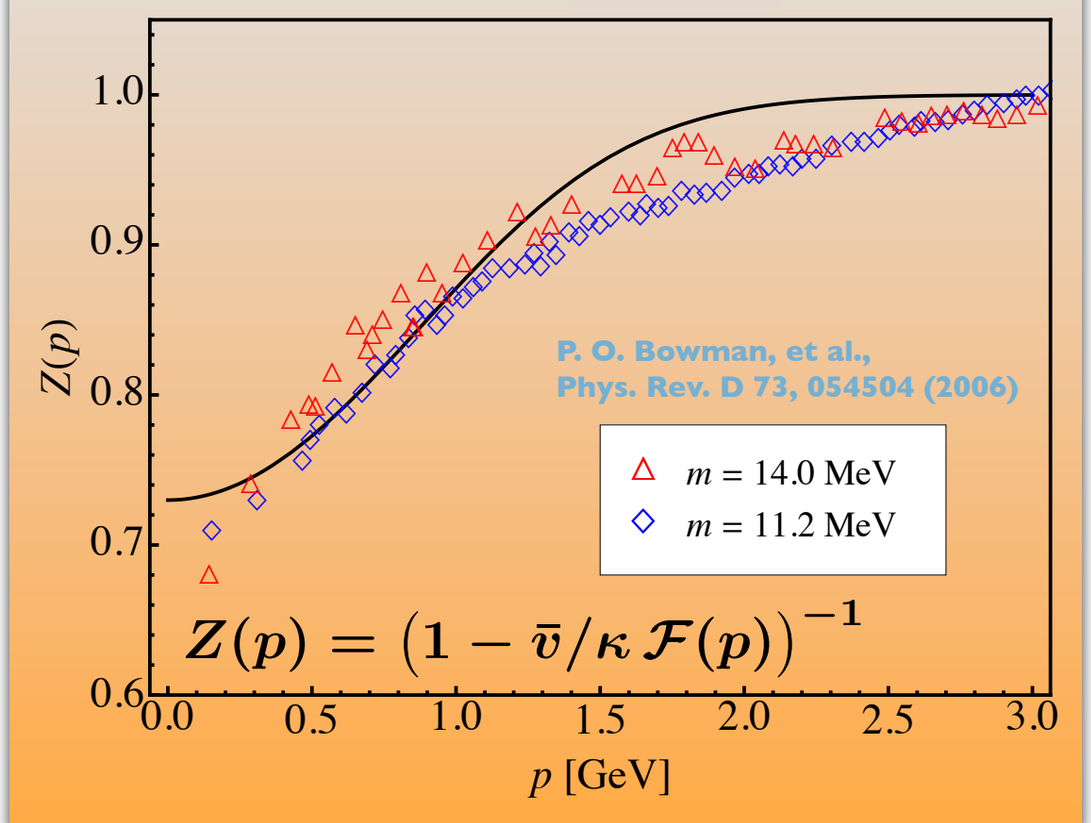
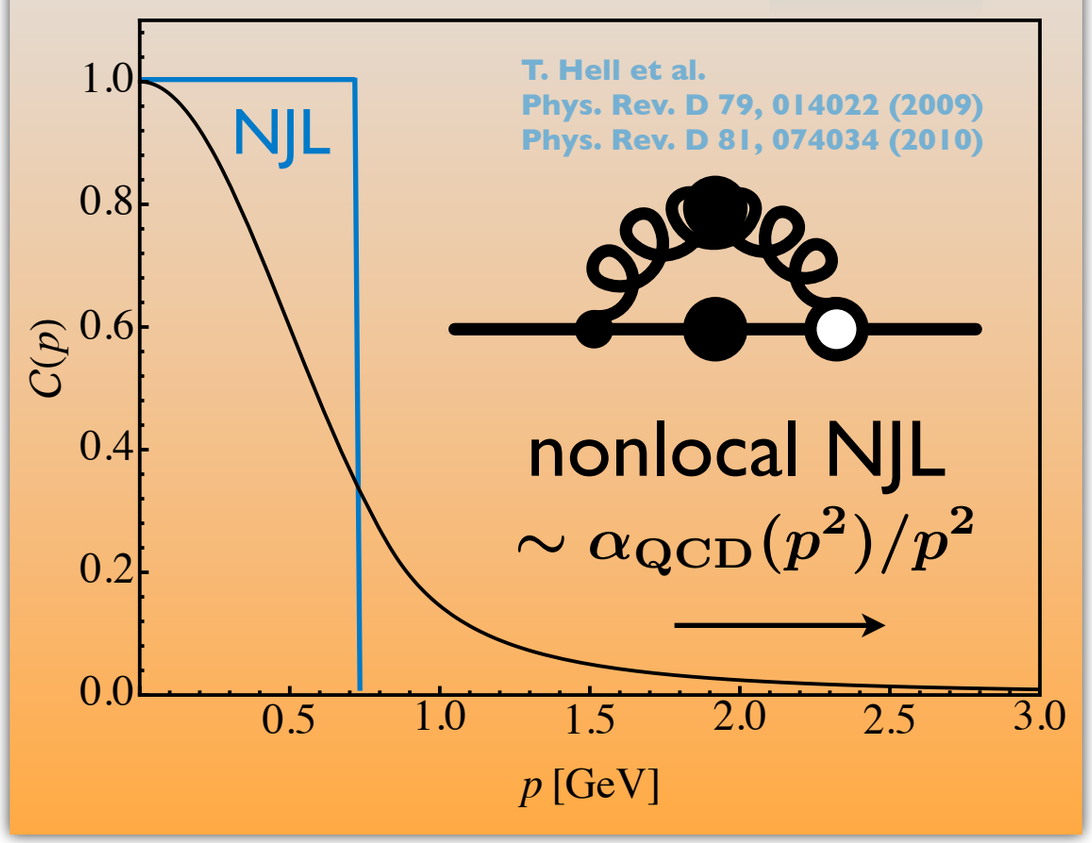
WFR Function $Z(p)$



Correlation Function $\mathcal{C}(p)$



Nonlocal NJL Model

Parameters and Results

$m_u = m_d$	m_s	G	H	κ
2.9 MeV	69.3 MeV	1.63 fm^2	$-0.77\ \mathrm{fm}^5$	5.28 GeV

$\langle\bar{u}u\rangle^{1/3} = \langle\bar{d}d\rangle^{1/3}$	$\langle\bar{s}s\rangle^{1/3}$	$M_u(0) = M_d(0)$	$M_s(0)$	$\bar{v}$
-0.290 GeV	-0.309 GeV	0.369 GeV	0.583 GeV	-1.95 GeV

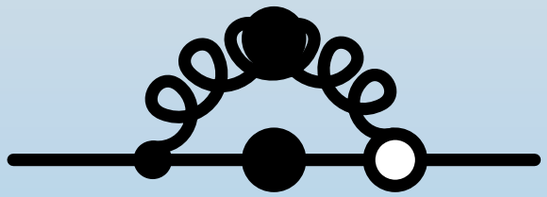
m_π	m_K	m_η	$m_{\eta'}$	f_π	f_K	$\theta_{\eta'}$
142 MeV	503 MeV	553 MeV	957 MeV	84.4 MeV	100.3 MeV	$-29.9°$

Gap Equations in Mean-Field Approximation

SPA

$$\bar{\sigma}_i = -G\bar{S}_i - \frac{H}{4}\varepsilon_{ijk}\varepsilon_{ijk}\bar{S}_j\bar{S}_k$$

$$\bar{S}_i = -8N_\mathrm{c}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathcal{C}(p)\,\frac{Z(p)M_i(p)}{p^2+M_i^2(p)}$$



$$\bar{v} = -\frac{4N_\mathrm{c}G}{\kappa}\sum_{i\in\{u,d,s\}}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathcal{F}(p)\frac{p^2\,Z(p)}{p^2+M_i^2(p)}$$

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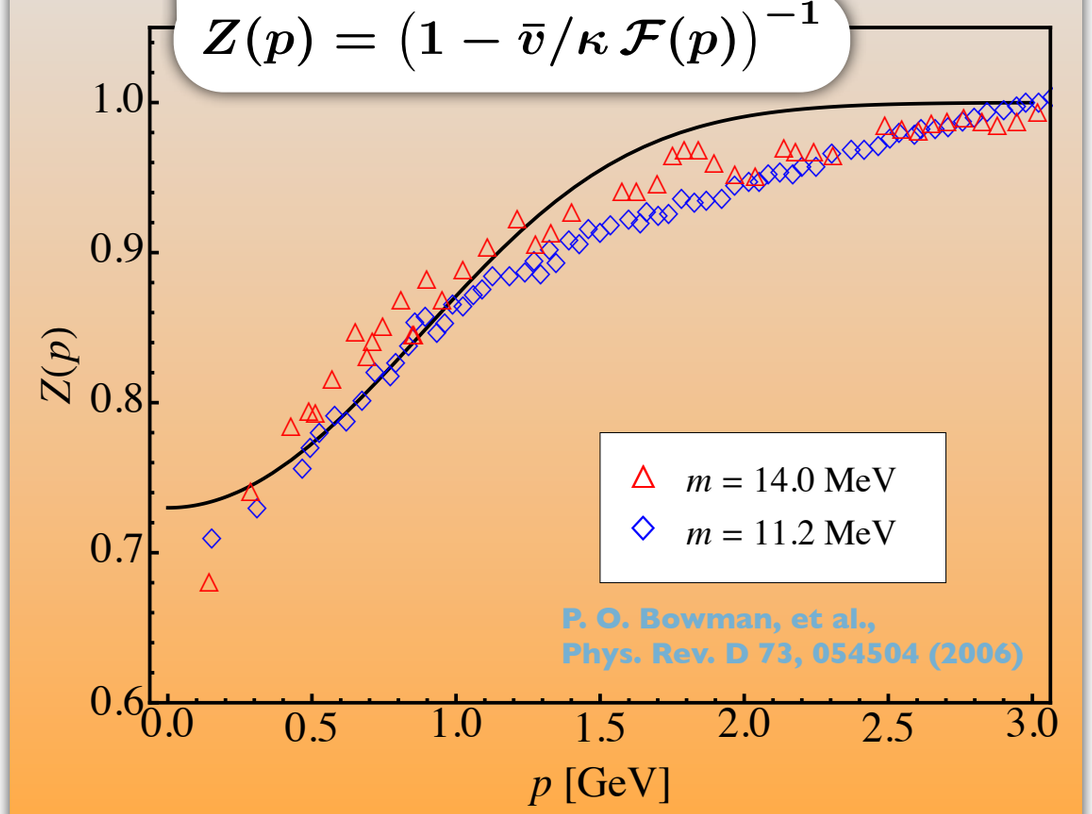
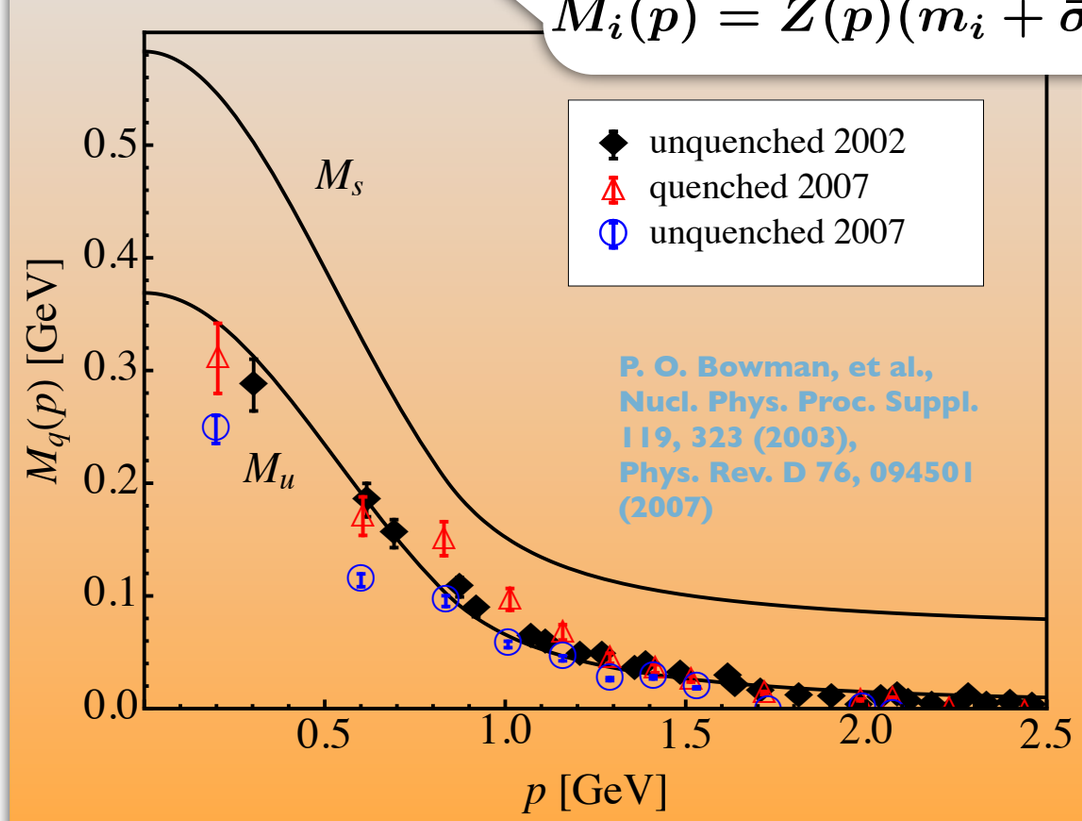
WFR Function

$$Z(p) = \left(1 - \bar{v}/\kappa\,\mathcal{F}(p)\right)^{-1}$$



Mass Functions

$$M_i(p) = Z(p)(m_i + \bar{\sigma}_i\,\mathcal{C}(p))$$



The Nonlocal Polyakov-Loop-Extended NJL Model

Polyakov Loop and Confinement

Polyakov Loop

$$\langle \Phi(\vec{x}\,)\rangle = \frac{1}{N_c}\langle \mathrm{tr_c}\,[L(\vec{x}\,)]\rangle$$

$$L(\vec{x}\,) = \mathcal{P}\exp\left\{ \mathrm{i}\int_0^\beta \mathrm{d}\tau\, A_4(\tau,\vec{x}\,)\right\}\,, \qquad \beta = 1/T$$

$$\langle \Phi\rangle = \mathrm{e}^{-\beta\mathscr{F}_q}$$

L. D. McLerran and B. Svetitsky
Phys. Lett. B 98, 195 (1981)
Phys. Rev. D 24, 450 (1981)

Polyakov loop serves as an order parameter for confinement-deconfinement phase transition

Polyakov Loop and Confinement

Polyakov-Fukushima Effective Potential

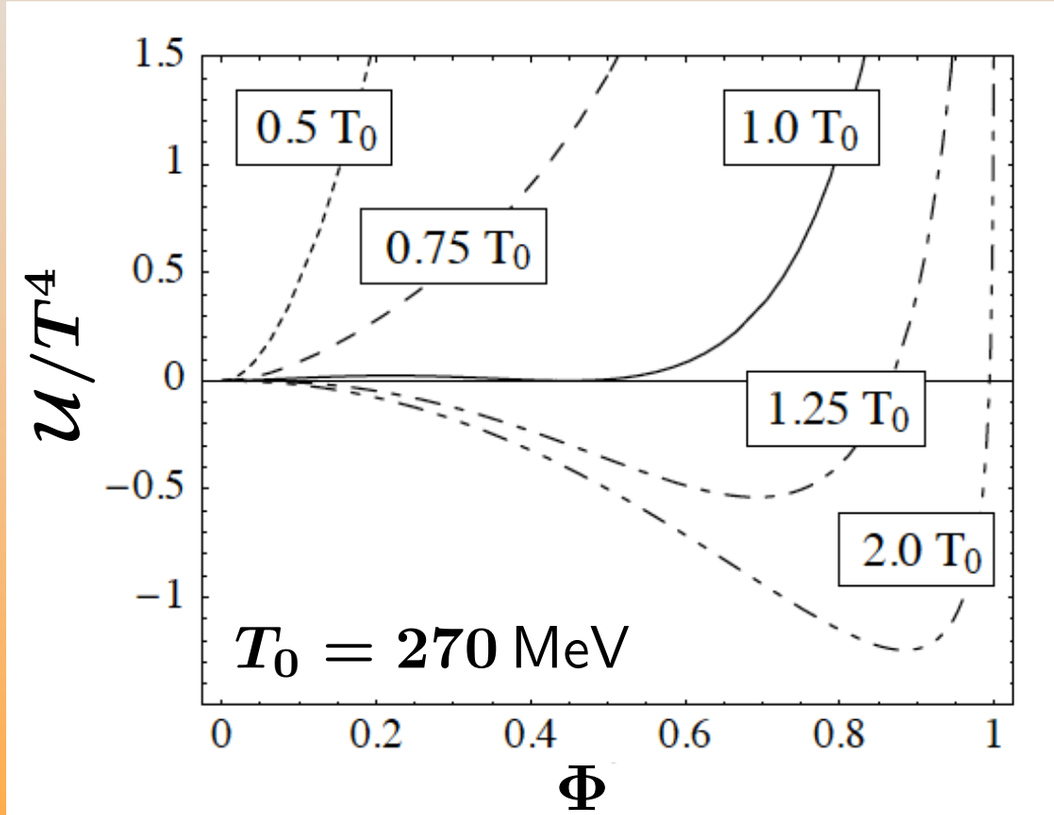
K. Fukushima, Phys. Lett. B 591, 277 (2004)
S. Rößner et al., Phys. Rev. D 73, 014019 (2006)

$$\frac{\mathcal{U}}{T^4} = -\frac{1}{2} b_2(T)\Phi^*\Phi + b_4(T)\ln\left[1 - 6\Phi^*\Phi + 4\left(\Phi^{*3} + \Phi^3\right) - 3(\Phi^*\Phi)^2\right]$$

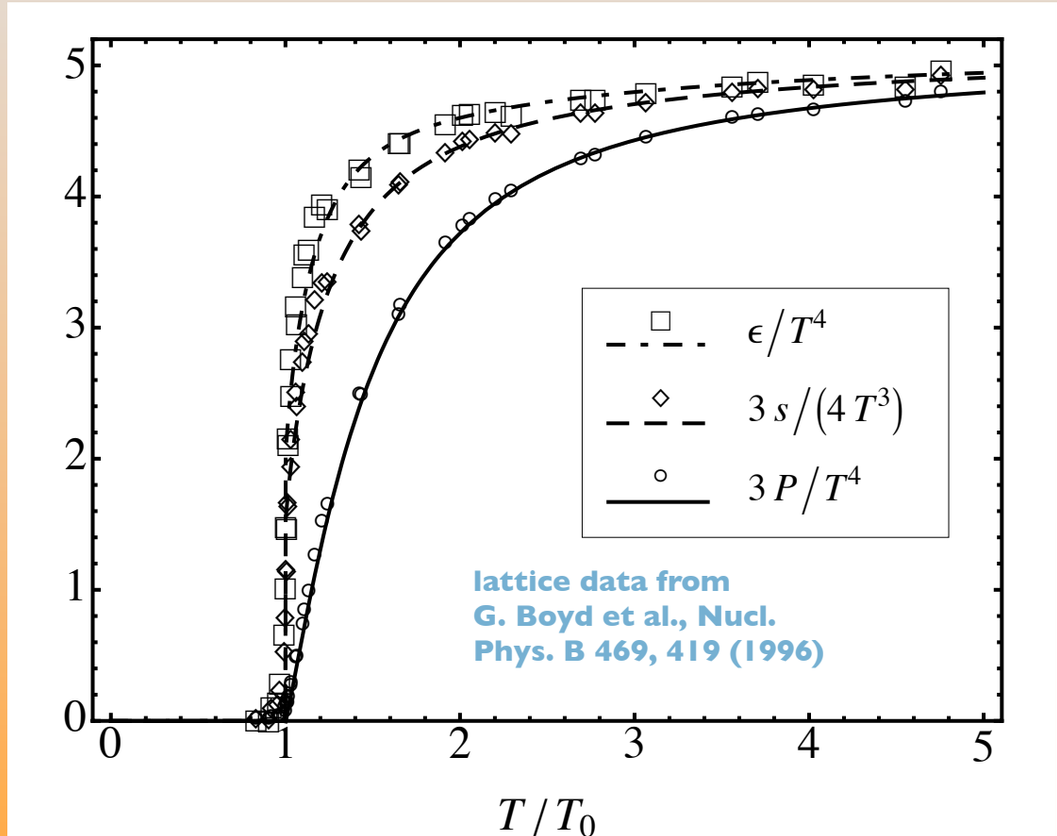
$Z(3)$ invariant

with $$\Phi = \frac{1}{3}\mathrm{tr}_c\left[\exp\left(\mathrm{i}\left(\beta A_4^3\lambda_3 + \beta A_4^8\lambda_8\right)\right)\right]$$

Effective Potential



Pure-Gauge Lattice Results



Nonlocal PNJL Model

Partition Function

$$Z = \int \mathscr{D}\varphi_a \int \mathscr{D}A \mathrm{e}^{-\mathcal{S}_{\mathrm{E}}^{\mathrm{bos}}(\varphi_a, A)}$$

Thermodynamic Potential in Mean-Field Approximation

$$\Omega = -2T \sum_{f\in\{u,d,s\}} \sum_{i=0,\pm} \sum_{n\in\mathbb{Z}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \ln\left[\frac{{\omega^i_{f,n}}^2 + \vec{p}^{\,2} + M_f^2(\omega^i_{f,n}, \vec{p})}{Z^2(\omega^i_{f,n}, \vec{p})}\right]$$
$$- \frac{1}{2}\left\{\sum_{f\in\{u,d,s\}} \left(\bar{\sigma}_f \bar{S}_f + \frac{G}{2}\bar{S}_f\bar{S}_f\right) + \frac{H}{2}\bar{S}_u\bar{S}_d\bar{S}_s\right\} + \frac{\bar{v}^2}{2G} + \mathcal{U}(\Phi, \Phi^*, T)$$

- $M_f(\omega^i_{f,n}, \vec{p}) = Z(\omega^i_{f,n}, \vec{p})\left(m_f + \bar{\sigma}_f\, \mathcal{C}(\omega^i_{f,n}, \vec{p})\right)$, $\quad Z(\omega^i_{f,n}, \vec{p}) = \left(1 - \frac{\bar{v}}{\kappa}\mathcal{F}(\omega^i_{f,n}, \vec{p})\right)^{-1}$
- Matsubara frequencies $\omega_n = (2n+1)\pi T, n \in \mathbb{Z}$

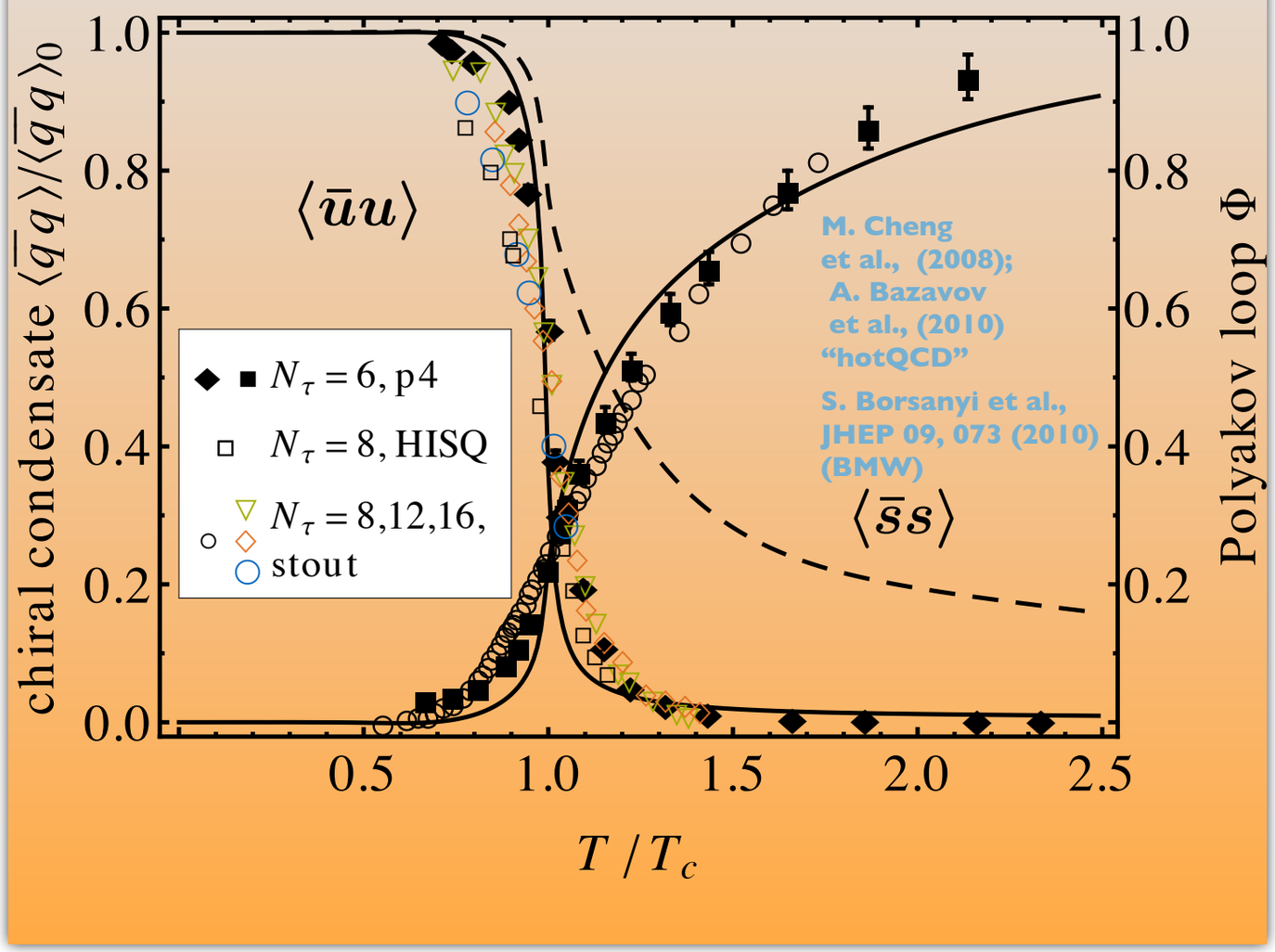
  $\omega^{\pm}_{f,n} = \omega_n - \mathrm{i}\mu_f \pm A_4^3/2 - A_4^8/(2\sqrt{3}), \quad \omega^0_{f,n} = \omega_n - \mathrm{i}\mu_f + A_4^8/\sqrt{3}$

Condensates and Polyakov Loop

Gap Equations

$$\frac{\partial\Omega}{\partial\bar{\sigma}_u} = \frac{\partial\Omega}{\partial\bar{\sigma}_s} = \frac{\partial\Omega}{\partial\bar{v}} = \frac{\partial\Omega}{\partial A_4^3} = \frac{\partial\Omega}{\partial A_4^8} = 0$$

Normalized Temperature



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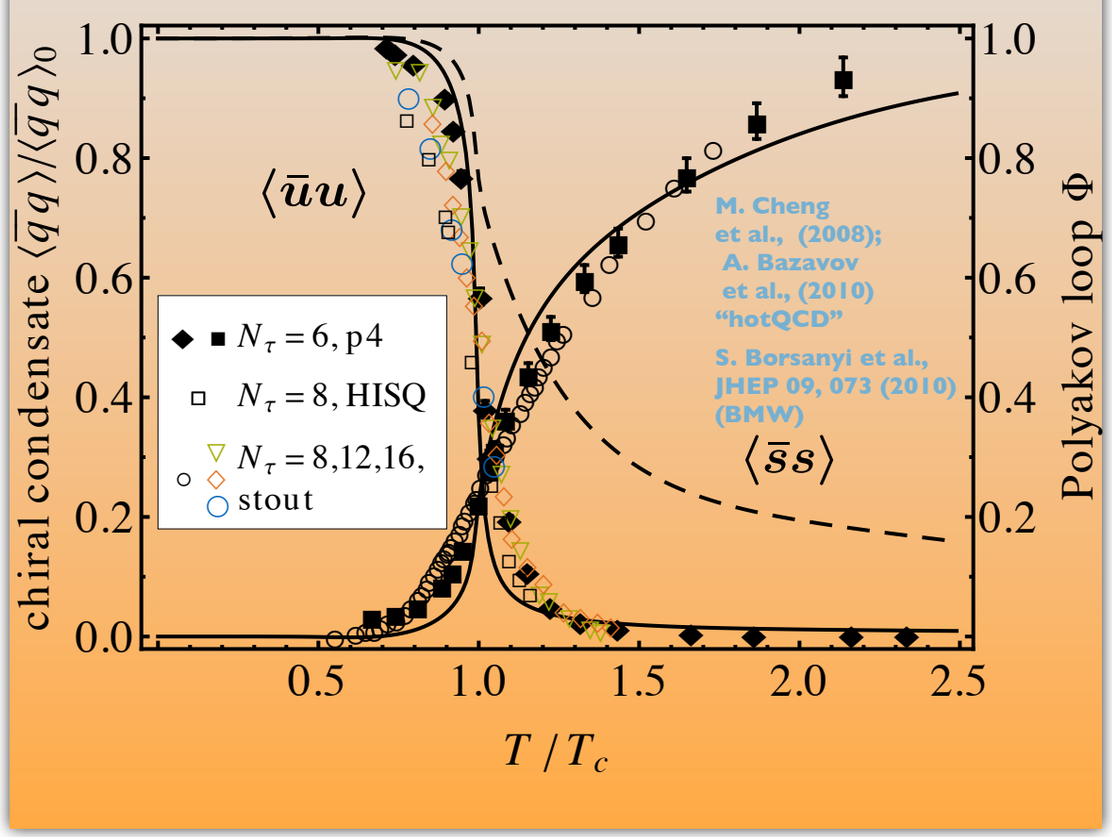
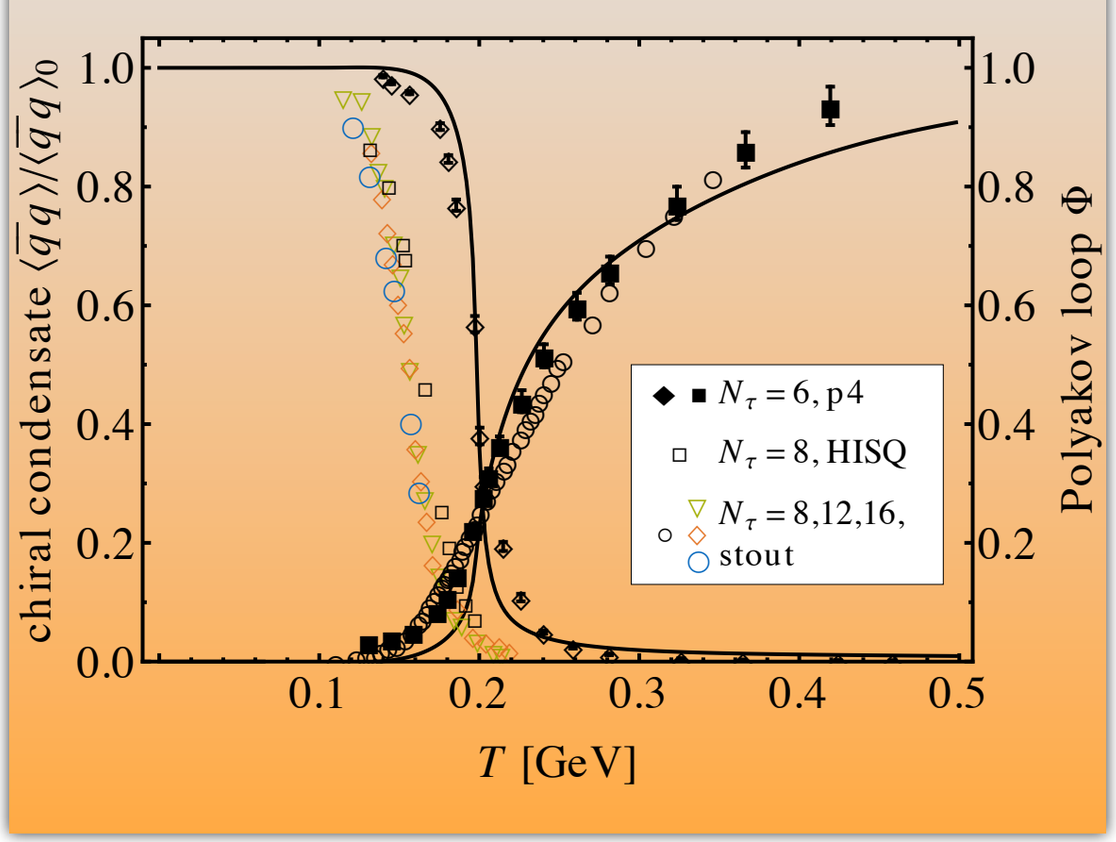


Absolute Temperature



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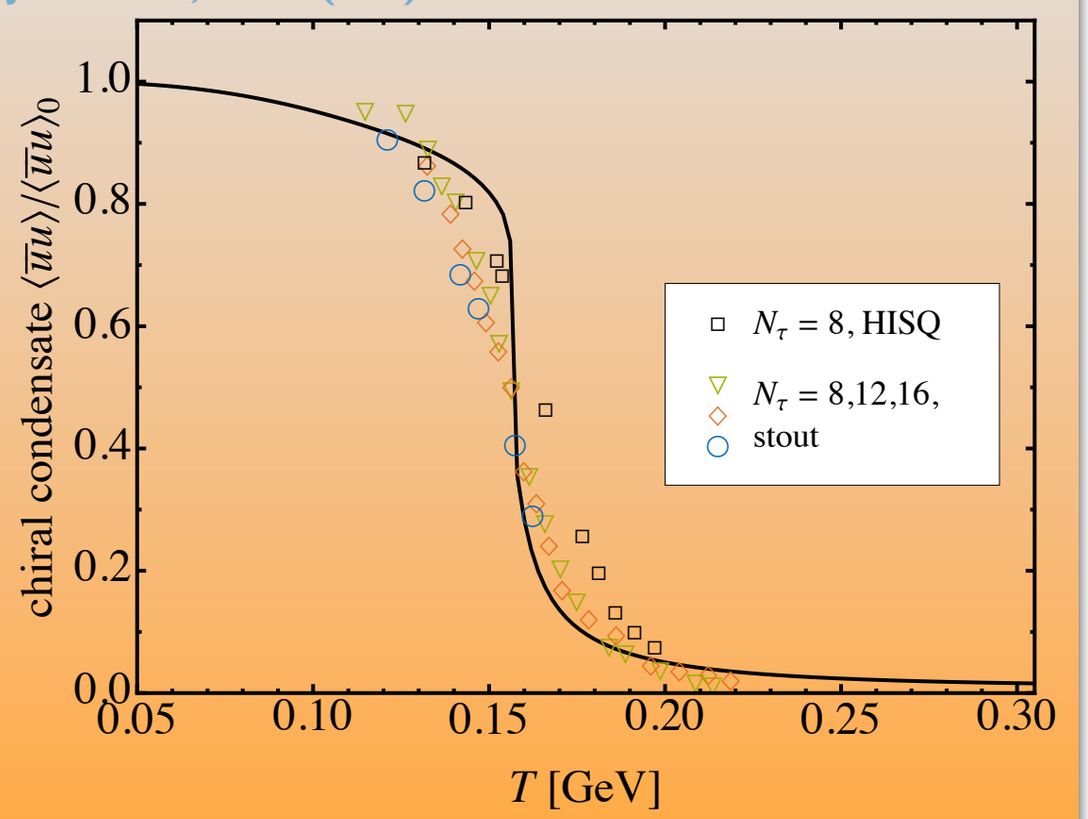
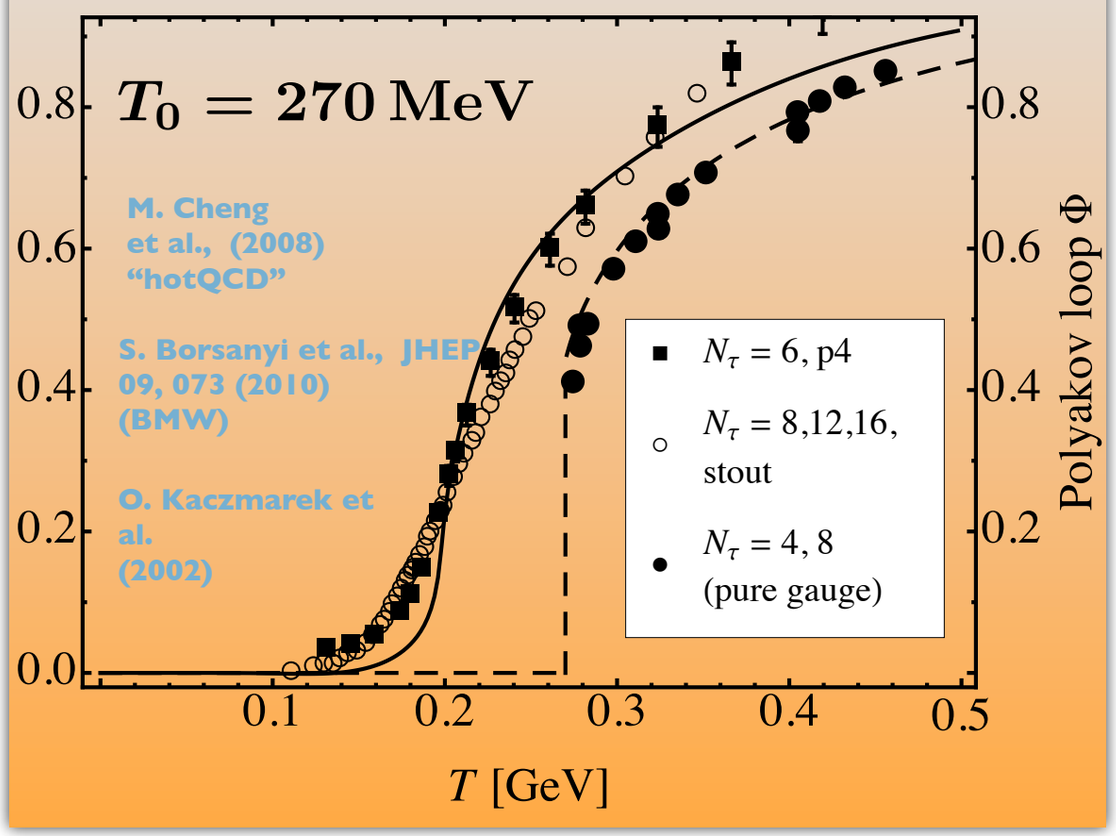
$T_0 = 187\,\mathrm{MeV}$, with pions

B.-J. Schäfer, J. M. Pawlowski, J. Wambach, PRD76 (2007)

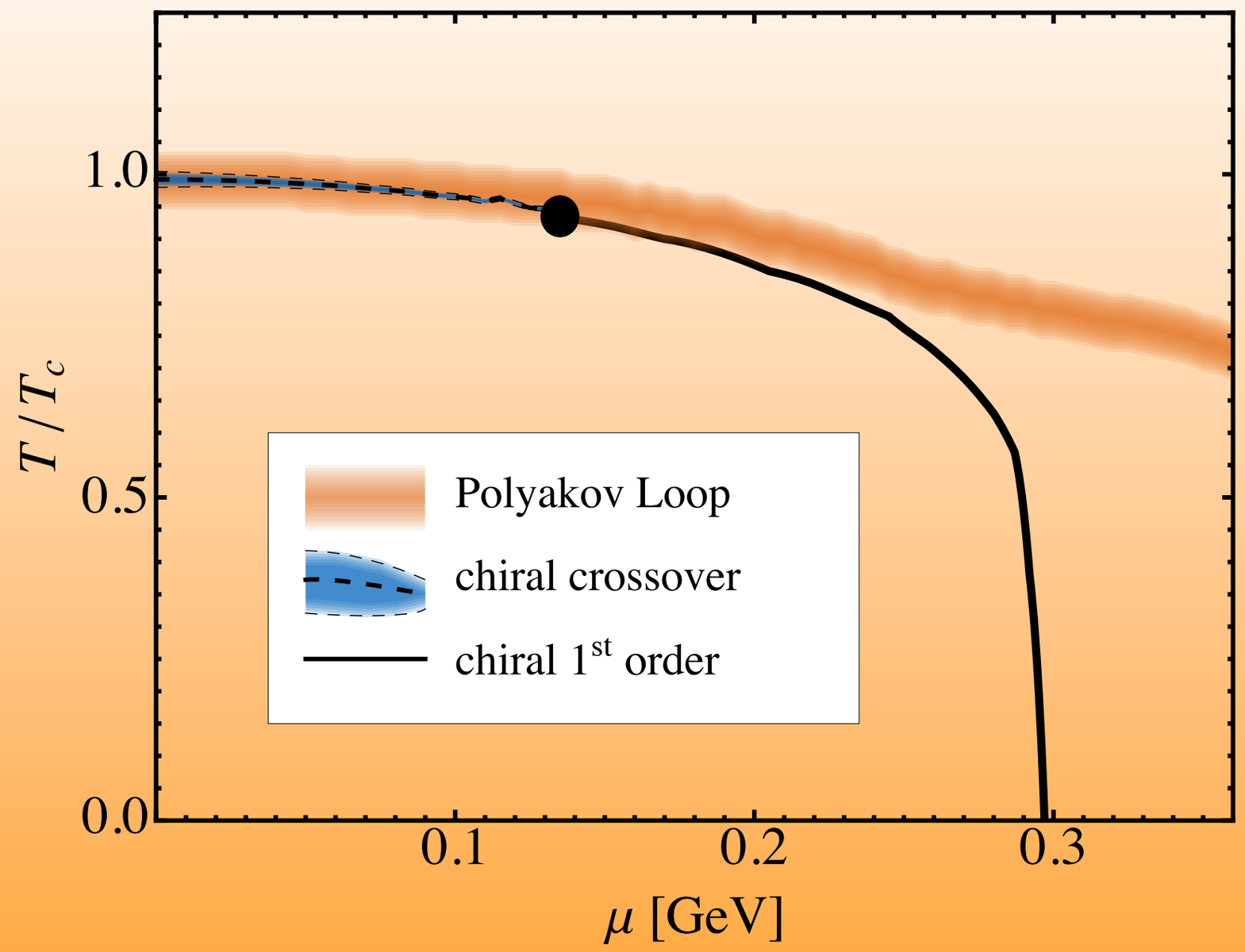


$T_0 = 270\,\mathrm{MeV}$, Polyakov loop



QCD Phase Diagram

Phase Diagram for 2 + 1 Flavors $\mu_u = \mu_d = \mu_s \equiv \mu$



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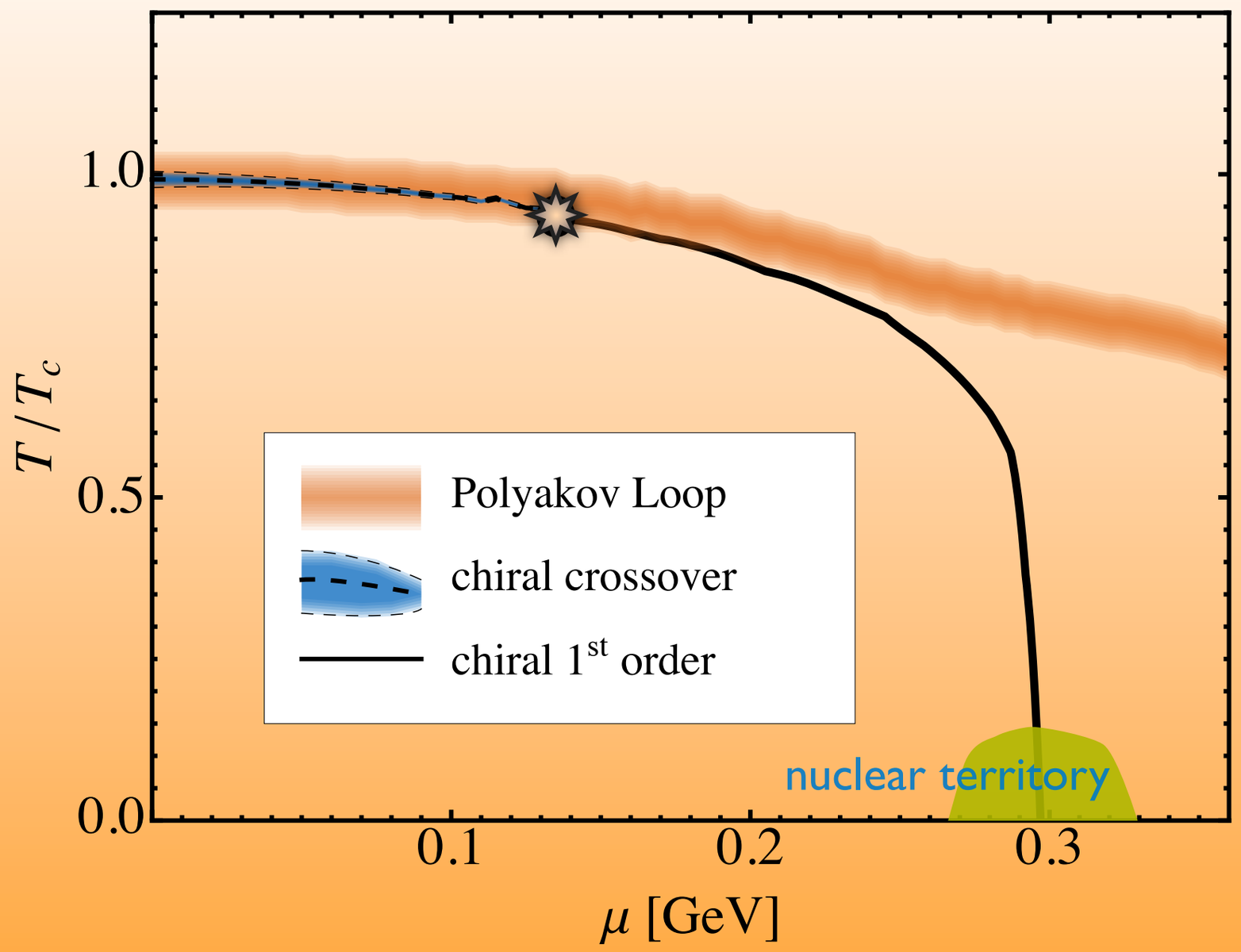

critical point

$\mu_{\rm crit} = 135\,{\rm MeV}$

$T_{\rm crit} = 188\,{\rm MeV}$

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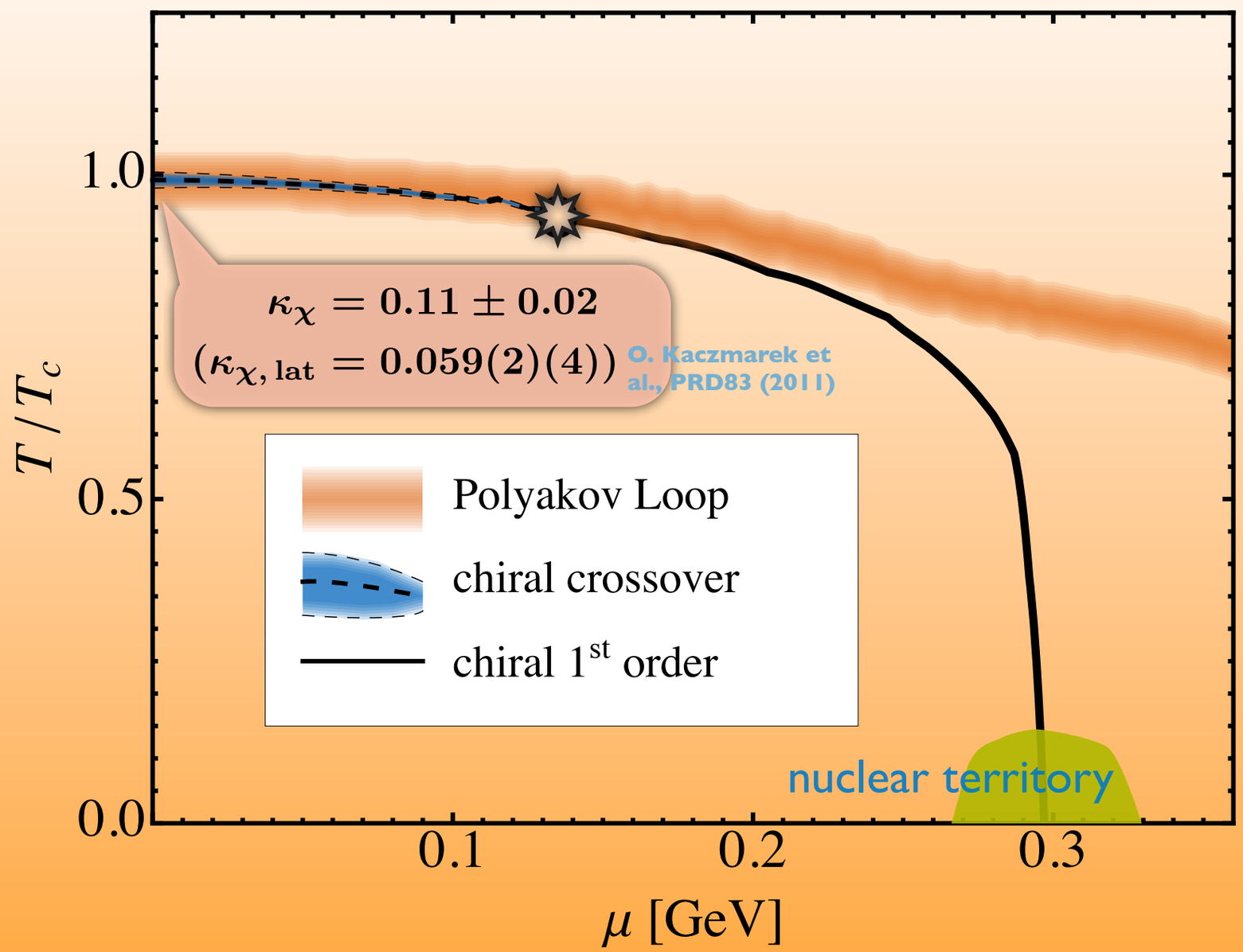


critical point

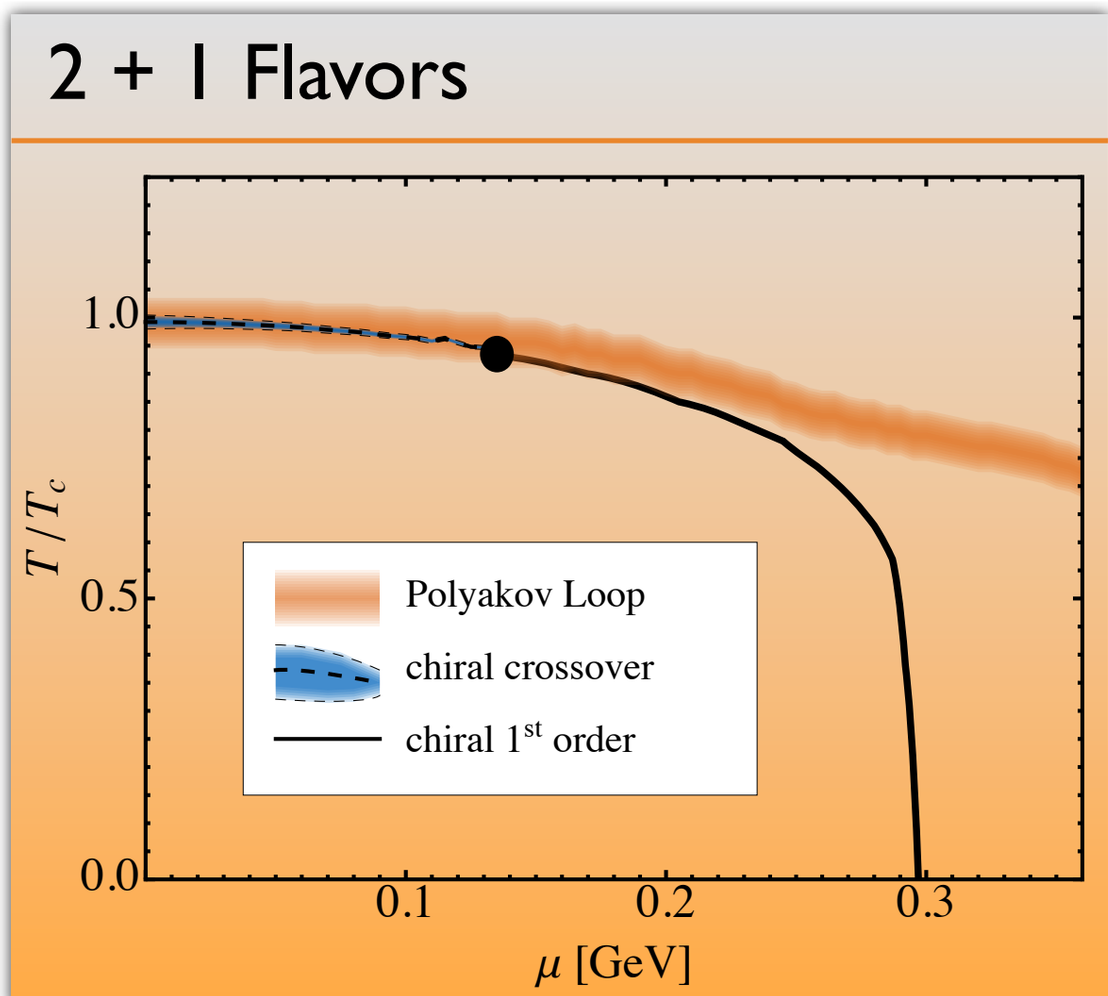
$\mu_{\mathrm{crit}} = 135\,\mathrm{MeV}$

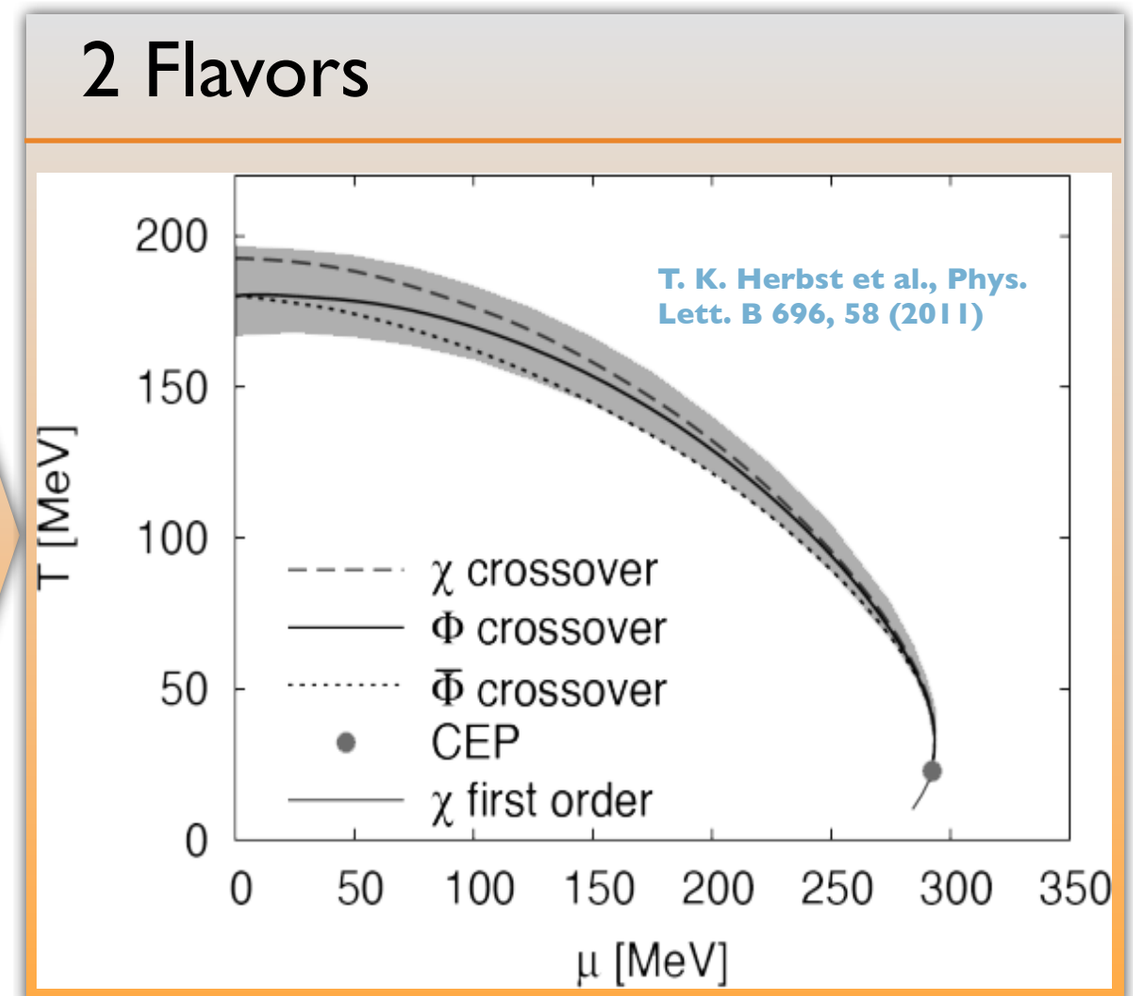
$T_{\mathrm{crit}} = 188\,\mathrm{MeV}$

$$\frac{T_c(\mu)}{T_c(\mu=0)} = 1 - \kappa_\chi \left(\frac{\mu}{T}\right)^2 + \mathcal{O}\left(\left(\frac{\mu}{T}\right)^4\right)$$

QCD Phase Diagram

Inclusion of back-reaction effects
of the matter sector





Summary and Outlook

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Nonlocal PNJL Model

meeting Schwinger-Dyson Calculations

- nonlocal PNJL model $\Longleftrightarrow$ SD approach
  - → momentum-dependent quark mass
- Thermodynamics of strong interaction
  - → entanglement of chiral and deconfinement transition
  - → schematic "sketch" of QCD phase diagram