The strong coupling limit of lattice QCD

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PhD thesis of Michael Fromm (ETH → Frankfurt)
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and in progress
with J. Langelage, K. Miura, O. Philipsen and W. Unger

Ph. de Forcrand St.Goar, March 2011 β = 0 LQCD
Scope of lattice QCD simulations

Physics of color singlets

- “One-body” physics: confinement
  - hadron masses
  - form factors, etc.
Physics of color singlets

- “One-body” physics: confinement
  hadron masses
  form factors, etc..

- “Two-body” physics: nuclear interactions
  pioneers: Hatsuda et al, Savage et al

hard-core
+ pion exchange?
QCD phase diagram according to Wikipedia

Here: • “many-body” physics: hadron $\leftrightarrow$ nuclear matter transition
• “two-body”: $T = 0$ nuclear interactions
A different approach to the sign problem

\[ Z = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i (\not{D} + m_i + \mu_i \gamma_0) \psi_i \right) \]

\( \det(\not{D} + m + \mu \gamma_0) \) complex \( \rightarrow \) try integrating over the gauge field first!

- **Problem:** \( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \rightarrow \beta_{\text{gauge}} \text{Tr} U_{\text{Plaquette}}, \) ie. 4-link interaction
- **Solution:** set \( \beta_{\text{gauge}} = \frac{2N_c}{g^2} \) to zero, ie. \( g = \infty \), strong coupling limit
- Then integral over gauge links *factorizes:* \( \sim \int \prod dU \exp(\bar{\psi}_x U_{x,\hat{\mu}} \psi_{x+\hat{\mu}}) \)
  - analytic 1-link integral \( \rightarrow \) only color singlets survive
  - perform Grassmann integration last \( \rightarrow \) hopping of color singlets
  \( \rightarrow \text{hadron worldlines} \)
  - sample gas of worldlines by Monte Carlo

Note: when \( \beta_{\text{gauge}} = 0 \), quarks are *always* confined \( \forall(\mu, T) \), ie. nuclear matter
The price to pay: not continuum QCD

Strong coupling LQCD: why bother?

Asymptotic freedom: $a(\beta_{\text{gauge}}) \propto \exp(-\frac{\beta_{\text{gauge}}}{4N_c b_0})$

ie. $a \to 0$ when $\beta_{\text{gauge}} \equiv \frac{2N_c}{g^2} \to +\infty$. Here $\beta_{\text{gauge}} = 0$ !!

- Lattice “infinitely coarse”
- Physics not universal

Nevertheless:
- Properties similar to QCD: confinement and $\chi_{\text{SB}}$
- Include (perhaps) next term in strong coupling expansion, ie. $\beta_{\text{gauge}} > 0$

When $\beta_{\text{gauge}} = 0$, sign problem is manageable $\to$ complete solution

Valuable insight? understand nuclear interactions
Further motivation

- 25+ years of analytic predictions:
  80’s: Kluberg-Stern et al., Kawamoto-Smit, Damgaard-Kawamoto
  \( \mu_c(T = 0) = 0.66, \ T_c(\mu = 0) = 5/3 \)
  90’s: Petersson et al., 1/g^2 corrections
  00’s: detailed \((\mu, T)\) phase diagram: Nishida, Kawamoto, ...
  now: Ohnishi et al. \( o(\beta) \) & \( o(\beta^2) \), Münster & Philipsen, ...

How accurate is mean-field \((1/d)\) approximation?

- Almost no Monte Carlo crosschecks:
  89: Karsch-Mütter → MDP formalism → \( \mu_c(T = 0) \sim 0.63 \)
  92: Karsch et al. \( T_c(\mu = 0) \approx 1.40 \)
  99: Azcoiti et al., MDP ergodicity ??
  06: PdF-Kim, HMC → hadron spectrum \( \sim 2\% \) of mean-field

Can one trust the details of analytic phase-diagram predictions?
Phase diagram from Nishida (2004, mean field, cf. Fukushima)

- Very similar to conjectured phase diagram of $N_f = 2$ QCD
- But no deconfinement here: high density phase is nuclear matter
- Baryon mass $= M_{\text{proton}} \Rightarrow$ lattice spacing $a \sim 0.6 \text{ fm}$ not universal
Strong coupling $SU(3)$ with staggered quarks

\[ Z = \int \mathcal{D} U \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp\left( -\bar{\psi} (\mathcal{D}(U) + m_q) \psi \right), \text{ no plaquette term (}\beta_{\text{gauge}} = 0\) \]

- One complex colored fermion field per site (no Dirac indices, spinless)
- $\mathcal{D}(U) = \frac{1}{2} \sum_{x,\nu} \eta_{\nu}(x)(U_{\nu}(x) - U_{\nu}^\dagger(x - \hat{\nu}))$, $\eta_{\nu}(x) = (-)^{x_1 + \ldots + x_{\nu-1}}$
- Chemical potential $\mu \rightarrow \exp(\pm a\mu) U_{\pm 4}$
- $\mathcal{D} U = \prod dU$ factorizes $\rightarrow$ integrate over links

\[ \rightarrow \text{ Color singlet degrees of freedom:} \]

- **Meson** $\bar{\psi} \psi$: monomer, $M(x) \in \{0, 1, 2, 3\}$
- **Meson hopping**: dimer, non-oriented $n_{\nu}(x) \in \{0, 1, 2, 3\}$
- **Baryon hopping**: oriented $\bar{B}B_{\nu}(x) \in \{0, 1\} \rightarrow$ self-avoiding loops $C$

Point-like, hard-core baryons in pion bath

No $\pi NN$ vertex
$Z(m_q, \mu) = \sum_{\{M, n_\nu, C\}} \prod_x \frac{m_q^M(x)}{M(x)!} \prod_{x, \nu} \frac{(3 - n_\nu(x))!}{n_\nu(x)!} \prod_{\text{loops}} \rho(C)$

with constraint $(M + \sum_{\nu} n_\nu)(x) = 3 \ \forall x \notin \{C\}$

Constraint: 3 blue symbols or a baryon loop at every site
MDP Monte Carlo

\[ Z(m_q, \mu) = \sum_{\{M, n_\nu, C\}} \prod_x \frac{m_q^M(x)}{M(x)!} \prod_{x, \nu} \frac{(3 - n_\nu(x))!}{n_\nu(x)!} \prod_{\text{loops } C} \rho(C) \]

with constraint \((M + \sum_{\nu} n_\nu)(x) = 3 \ \forall x \notin \{C\}\)

The dense (crystalline) phase: 1 baryon per site; no monomer \(\rightarrow \langle \bar{\psi}\psi \rangle = 0\)
**MDP Monte Carlo**

\[
Z(m_q, \mu) = \sum_{\{M, n_v, C\}} \prod_x \frac{m_q^M(x)}{M(x)!} \prod_{x, v} \frac{(3 - n_v(x))!}{n_v(x)!} \prod_{\text{loops}} \rho(C)
\]

with constraint \((M + \sum_{\pm v} n_v(x)) = 3 \forall x \notin \{C\}\)

Remaining difficulties:

- Baryons are fermions: mild sign problem from \(\rho(C)\) \quad \text{Karsch & Mütter}
  \rightarrow \text{volumes up to } 16^3 \times 4 \forall \mu

- tight-packing constraint \rightarrow \text{local update inefficient, esp. as } m \rightarrow 0
  Solved with worm algorithm \quad \text{(Prokof'ev & Svistunov 1998)}
  Efficient even when \(m_q = 0\)

Local Metropolis, \(4^3 \times 2\) at \(\mu_c\), \(m_q = 0.025\)

Worm, same parameter set
$(\mu, T)$ phase diagram in the chiral limit $m_q = 0$, and for $m_q \neq 0$

- Phase boundary for breaking/restoration of $U(1)$ chiral symmetry
- 2nd order at $\mu = 0$: 3d $O(2)$ universality class
- 1st order at $T = 0$: $\rho_B$ jumps from 0 to 1 baryon per site $\rightarrow$ tricrit. pt. TCP

Finite-size scaling: $(\mu, T)_{TCP} = (0.33(3), 0.94(7))$ vs $(0.577, 0.866)$ (mean-field)
- Beware of quantitative mean-field predictions for phase diagram
\((\mu, T)\) phase diagram in the chiral limit \(m_q = 0\), and for \(m_q \neq 0\)

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- Beware of quantitative mean-field predictions for phase diagram
- \(m_q \neq 0\): liquid-gas transition \(T_{CEP} \sim 200\text{MeV}\) – traj. of CEP obeys tricrit. scaling
Can compare masses of differently shaped “isotopes”

$E(B = 2) - 2E(B = 1) \sim -0.4$, ie. “deuteron” binding energy ca. 120 MeV

$am(A) \sim a\mu_B^{\text{crit}} A + (36\pi)^{1/3} \sigma a^2 A^{2/3}$, ie. (bulk + surface tension)

Bethe-Weizsäcker parameter-free ($\mu_B^{\text{crit}}$ and $\sigma$ measured separately)

“Magic numbers” with increased stability: $A = 4, 8, 12$ (reduced area)
Nuclear potential: more than hard core

- Nucleons are point-like → no ambiguity with definition of static potential
- Nearest-neighbour attraction \( \sim 120 \) MeV at distance \( \sim 0.5 \) fm: cf. real world
- Baryon worldlines self-avoiding → no direct meson exchange (just hard core)

How do baryons interact at non-zero distance?
How the nucleon got its mass

• Point-like nucleon **distorts pion bath**  cf. Casimir

![Diagram](image)

- Energy = nb. time-like pion lines.
- Constraint: 3 pion lines per site \((m_q = 0)\) → energy density = 3/4 in vacuum.
- No spatial pion lines connecting to site occupied by nucleon → energy increase.

**Steric effect**

- \(am_B \approx 2.88 = (3 - 0.75) + \Delta E_\pi\), ie. "valence"(78%) + "pion cloud"(22%)
So, in fact, nucleon is not point-like

Point-like “bag” of 3 valence quarks $\rightarrow$ macroscopic disturbance in pion vacuum
So, in fact, nucleon is *not* point-like

Point-like “bag” of 3 valence quarks → macroscopic disturbance in pion vacuum

Static baryon prevents monomers = static \((t\)-invariant\) monomer “source”

Linear response \(\propto\) Green’s fct. of lightest \((t\)-invariant\) meson, ie. \(\rho/\omega\)

(pion has factor \((-1)^t\))

\[<n_t(R) - 3/4>\]
So, in fact, nucleon is *not* point-like

Point-like “bag” of 3 valence quarks $\rightarrow$ macroscopic disturbance in pion vacuum

Static baryon prevents monomers $=$ static ($t$-invariant) monomer “source”

Linear response $\propto$ Green’s fct. of lightest $t$-invariant meson, ie. rho/omega

(pion has factor $(-1)^t$)

\[
\langle n_t(R) \rangle - \frac{3}{4} \propto \exp\left(-\frac{m_\rho}{\omega} r\right) \times (-1)^{x+y+z}
\]
Nuclear interaction via pion clouds (thanks W. Weise)

- Here, baryons make self-avoiding loops → no direct meson exchange
- Interaction comes because of pion clouds

The two pion clouds can interpenetrate at \( \approx \) constant energy (2nd order effect)
But each set of valence quarks disturbs pion cloud of other baryon

\[
V_{NN}(R) \approx -2 \times \Delta E_\pi(R) \propto \frac{\exp(-m_\rho/\omega R)}{R} \times (-1)^{x+y+z}
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Meson exchange potential without meson exchange!
Recap: baryons and their interactions at $\beta = 0$

- Baryons are not point-like: \( \text{pion cloud} \sim \exp(-m_\rho/\omega r) \)

- Nuclear potential:
  - Hard-core from Pauli principle
  - Yukawa potential (times \((-1)^r\)) from the two pion clouds

- Precisely like a \textbf{classical hard-sphere fluid}:
  - “Pion cloud” from ripples around tagged sphere
  - Density-density correlation: \( V_{\text{eff}}(r) \equiv -\log(g(r)) \sim \exp(-(m + i\Gamma)r)/r \)

\[ \begin{align*}
\text{Hard spheres } \eta = 0.4, \text{ PY approx } \\
\exp(-(m + i\Gamma)r)/r
\end{align*} \]
Conclusions

**Summary: complete solution of strong coupling limit**
- Phase diagram: take mean-field results with a grain of salt
- [Crude, crystalline] **nuclear matter** from QCD:
  - tabletop simulations of first-principles nuclear physics
- Nucleon: point-like “bag” (→ hard core) + large **pion cloud**
  - cf. “little bag” model (Brown & Rho)
- Yukawa potential without meson exchange

**Outlook**
- Include $o(\beta)$ effects
- Take continuous Euclidean time
- Include second quark species → **isospin**?
First step towards $o(\beta)$: gauge observables at $\beta = 0$

Plaquette vs $m_q$

Polyakov loop vs $T$ ($m_q = 0.1$)
Continuous time: finite-\(T\) transition (\(m_q = 0, \mu = 0\))

Avoid anisotropy \(\gamma \rightarrow\) no error/ambiguity on \(T\)
Relating the strong-coupling and continuum phase diagrams

\[ N_f = 4 \text{ continuum flavors, } m_q = 0 \]

\[ \beta = 0 \]

**Strong coupling (known)**

**Continuum (minimal)**

\[ \beta = \infty \]
Relating the strong-coupling and continuum phase diagrams

\[ N_f = 4 \text{ continuum flavors, } \; m_q = 0 \]

Can differentiate between the two cases
Case \textit{right} favorable for quantitative continuum predictions
Backup: is pion bath essential? Classical hard spheres

\[ g(r) \equiv \langle \rho(0)\rho(r) \rangle \text{ relaxes to } \langle \rho \rangle^2 \text{ with damped oscillations} \quad \rightarrow \quad \text{liquid} \]
Backup: is pion bath essential? Classical hard spheres

“Potential of mean force” $V_{\text{eff}}(r) \equiv -\log(g(r))$ is hard-core + damped oscillatory
Backup: is pion bath essential? Classical hard spheres

"Potential of mean force" $V_{\text{eff}}(r) \equiv -\log(g(r))$ is hard-core + damped oscillatory

Consistent with Yukawa form $\frac{\exp(-mr)}{r} \times \cos(\Gamma r)$
Backup: is pion bath essential? Classical hard spheres

\[ \log(|V_{\text{eff}}(r)|) + \text{Yukawa fit} \]

Perfect fit at large distance

Hard-sphere “potential of mean force” is of Yukawa form

\[ V_{\text{eff}}(r) = \text{Re} \left[ \frac{e^{-(m+i\Gamma)r}}{r} \right] \]
Backup: Karsch & Mütter’s resummation

\[ \varepsilon(C) \exp(+3 \frac{\mu}{T}) \quad \varepsilon(C) \exp(-3 \frac{\mu}{T}) \quad +1 \quad +1 \]

Karsch & Mütter: **Resum into “MDP ensemble”** → sign pb. **eliminated at** \( \mu = 0 \)

\[ 1 + \varepsilon(C) \cosh(3 \frac{\mu}{T}) \quad 1 + \varepsilon(C) \cosh(3 \frac{\mu}{T}) \]
Backup: Sign problem? Monitor $-\frac{1}{V} \log \langle \text{sign} \rangle$

\[ \langle \text{sign} \rangle = \frac{Z}{Z_{||}} \sim \exp \left( -\frac{\nu}{T} \Delta f(\mu^2) \right) \text{ as expected;} \quad \Delta f \sim \mu^2 + O(\mu^4) \]

• Can reach $\sim 16^3 \times 4 \forall \mu$, ie. adequate

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\[ \beta = 0 \text{ LQCD} \]