

Inhomogeneous chiral symmetry breaking phases



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Michael Buballa

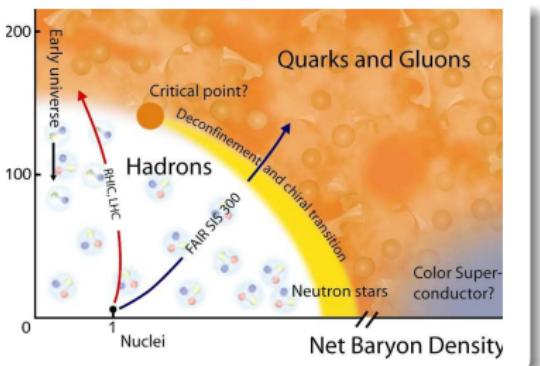
HIC for FAIR Workshop on
Quarks, Gluons, and Hadronic Matter under Extreme Conditions

St. Goar, March 15 - 18, 2011

Motivation

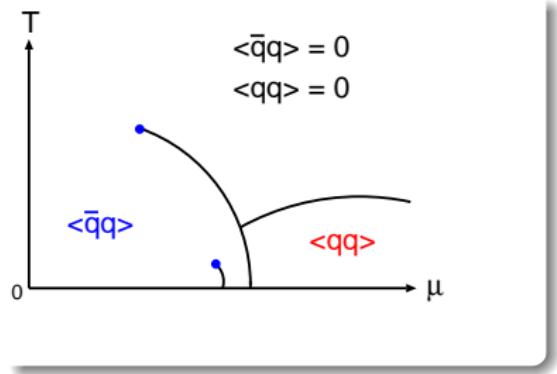


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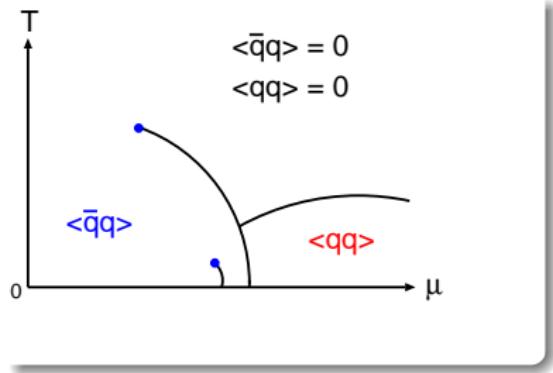
► QCD phase diagram

Motivation



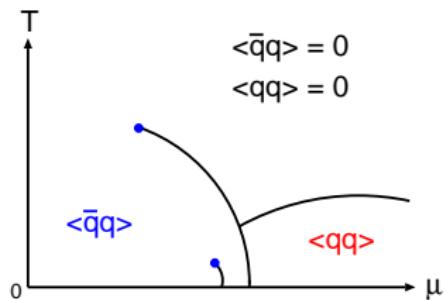
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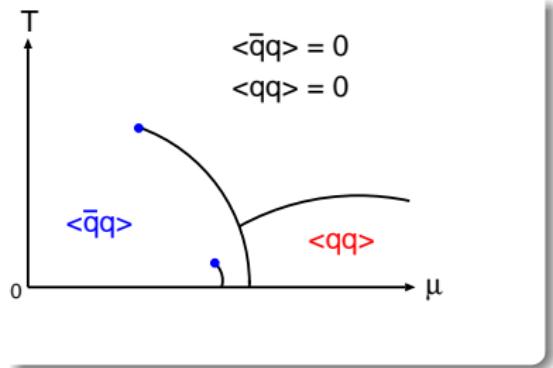
- ▶ QCD phase diagram
- ▶ frequent assumption:
 $\langle \bar{q}q \rangle, \langle qq \rangle$ constant in space

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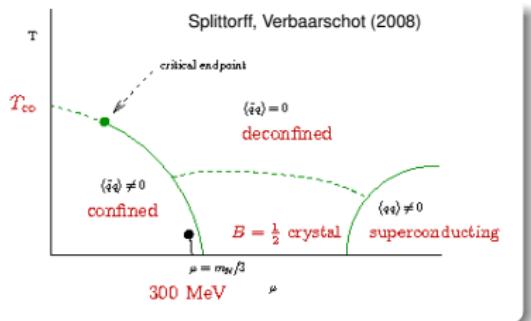


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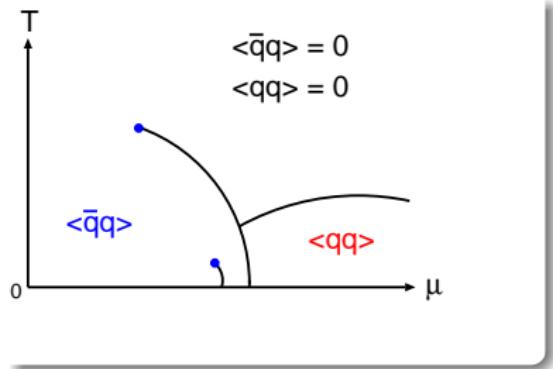
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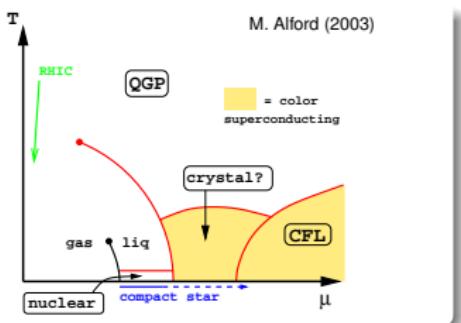
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 - ▶ Skyrme crystal



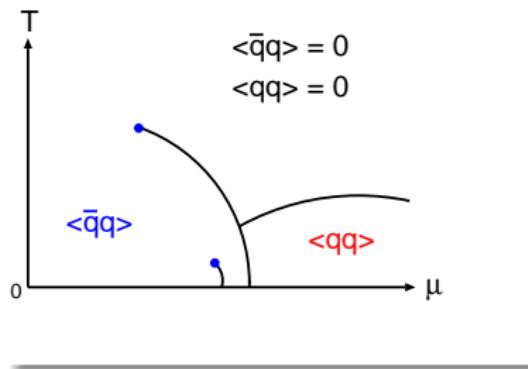
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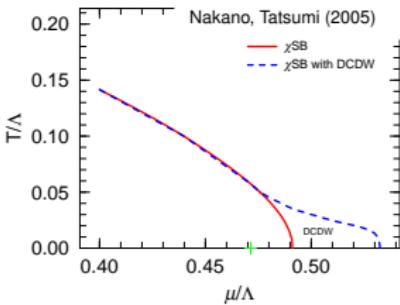
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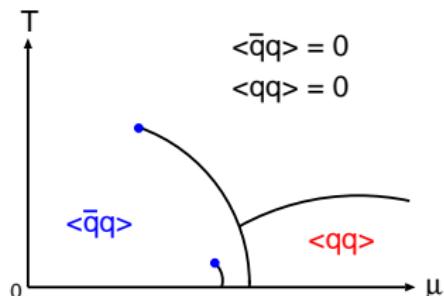
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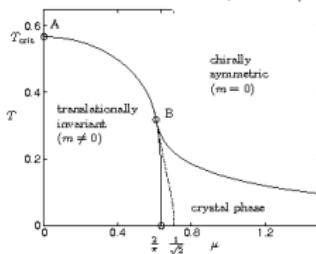


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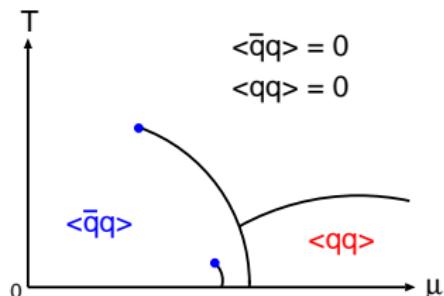


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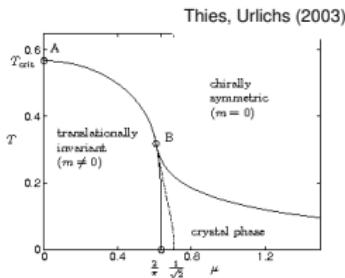
Thies, Urlichs (2003)



Motivation



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 - ▶ Skyrme crystal
 - ▶ crystalline color superconductors
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 - ▶ 1+1 D Gross-Neveu model
- ▶ This talk:
inhomogeneous χ SB in the NJL model



Collaborators



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► based on:

Phys. Rev. D 82 (2010) 054009 [arXiv:1007.1397]

together with



Stefano Carignano
(TU Darmstadt)



Dominik Nickel
(INT Seattle)

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- ▶ NJL model:

$$\mathcal{L} = \bar{\psi}(i\partial - m)\psi + G_S ((\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2)$$

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$$\Rightarrow \quad \mathcal{L} = \bar{\psi} (i\partial - m + 2G_S(\sigma + i\gamma_5\vec{\tau} \cdot \vec{\pi})) \psi - G_S (\sigma^2 + \vec{\pi}^2)$$

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- ▶ mean-field approximation:

$$\sigma(x) \rightarrow \langle \sigma(x) \rangle \equiv S(\vec{x}), \quad \pi_a(x) \rightarrow \langle \pi_a(x) \rangle \equiv P(\vec{x})\delta_{a3}$$

- ▶ $S(\vec{x})$, $P(\vec{x})$ time independent classical fields
- ▶ retain space dependence !

Mean-field model



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- ▶ mean-field Lagrangian: $\mathcal{L}_{MF} = \bar{\psi}(x)\mathcal{S}^{-1}(x)\psi(x) - G_S(S^2(\vec{x}) + P^2(\vec{x}))$

- ▶ inverse dressed propagator:

$$\mathcal{S}^{-1}(x) = i\cancel{\partial} - m + 2G_S(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x}))$$

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► thermodynamic potential:

$$\Omega_{MF}(T, \mu; S, P) = -\frac{T}{V} \text{Tr} \ln \left(\frac{1}{T} (i\partial_0 - \mathcal{H}_{MF} + \mu) \right) + \frac{G_S}{V} \int_V d^3x \left(S^2(\vec{x}) + P^2(\vec{x}) \right)$$

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► mass function: $M(\vec{x}) = m - 2G_S(S(\vec{x}) + iP(\vec{x}))$

► $E_{\lambda} = E_{\lambda}[M(\vec{x})]$ = eigenvalues of \mathcal{H}_{MF}

One dimensional modulations

- ▶ remaining tasks:
 - ▶ calculate eigenvalue spectrum of \mathcal{H}_{MF} for given mass function $M(\vec{x})$
 - ▶ minimize w.r.t. $M(\vec{x})$
- extremely difficult!

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One dimensional modulations

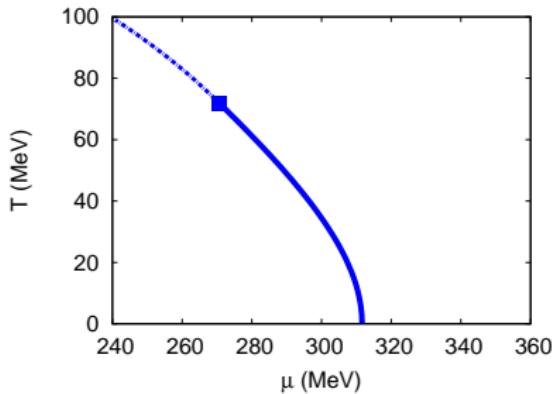
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Phase diagram (chiral limit)

[D. Nickel, PRD (2009)]

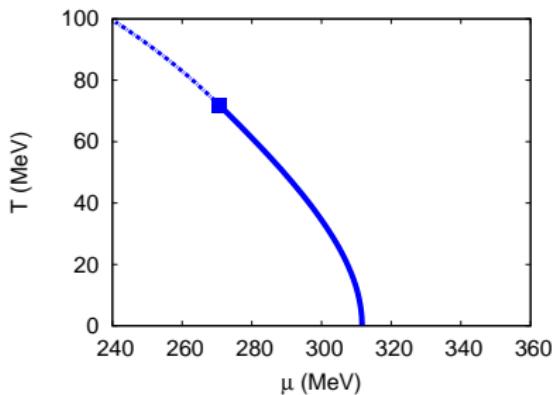
homogeneous phases only



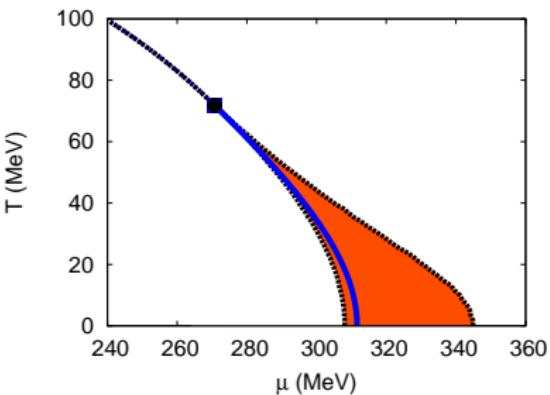
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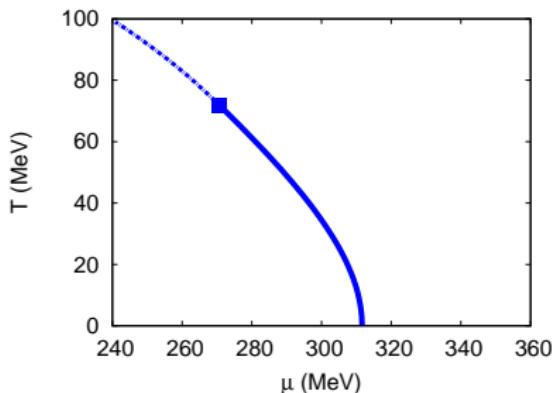
including inhomogeneous phase



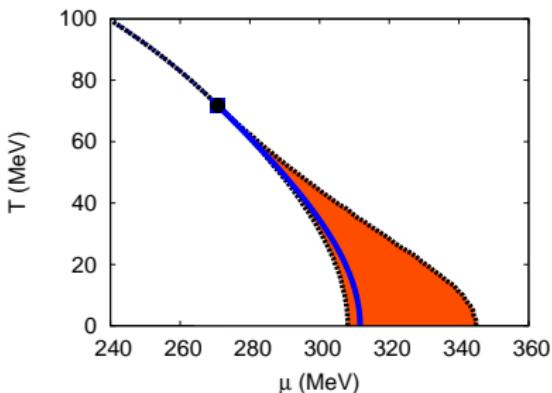
Phase diagram (chiral limit)

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homogeneous phases only



including inhomogeneous phase



- ▶ 1st-order line completely covered by the inhomogeneous phase!
- ▶ all phase boundaries 2nd order
- ▶ critical point coincides with Lifshitz point

Mass functions and density profiles ($T = 0$)

► $M(z) = \sqrt{\nu} \Delta \operatorname{sn}(\Delta z | \nu)$ →
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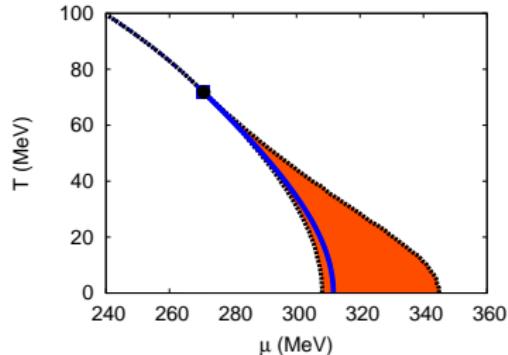
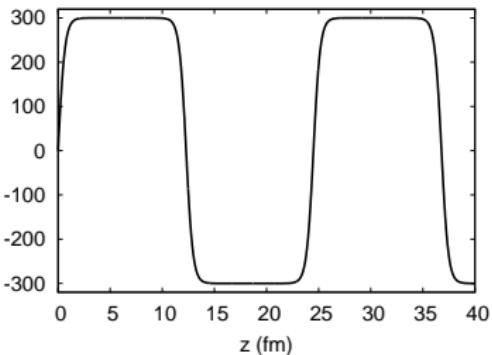
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$M(z)$ ($\mu = 307.5$ MeV)

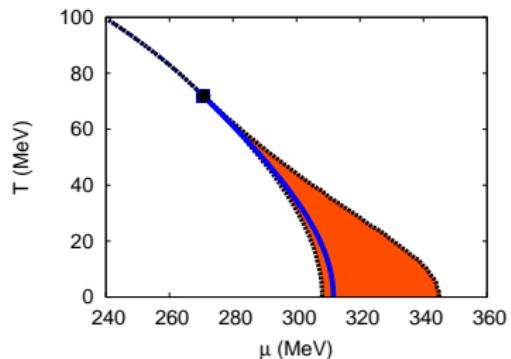
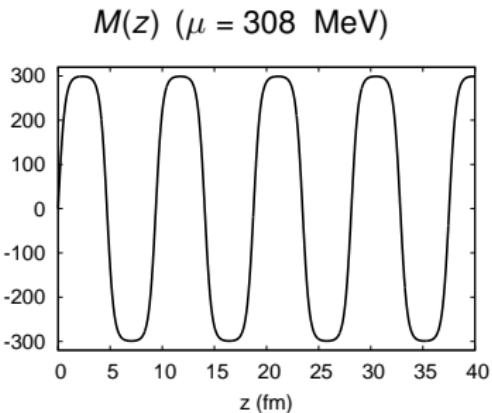


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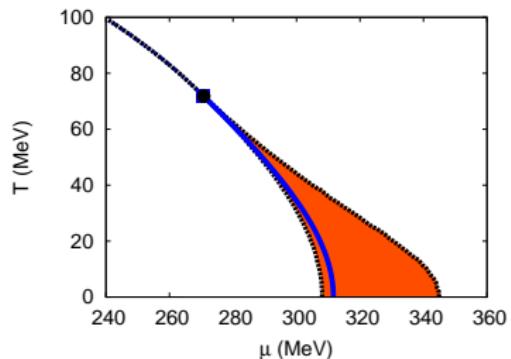
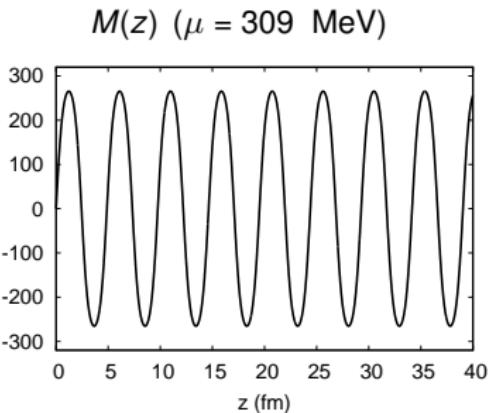


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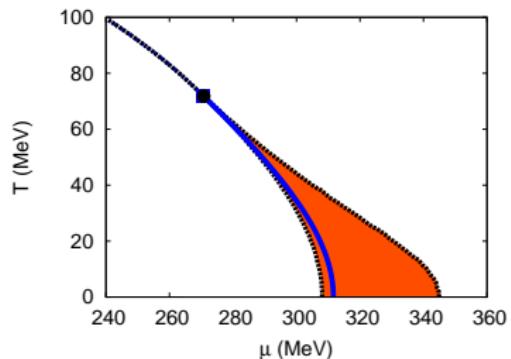
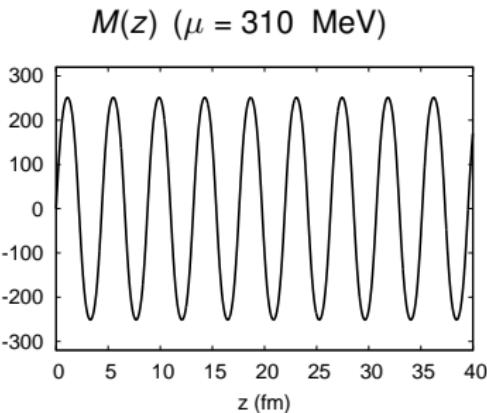


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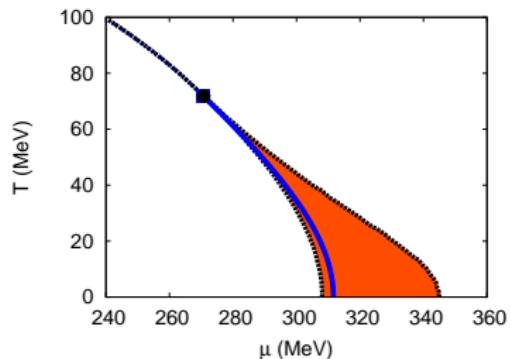
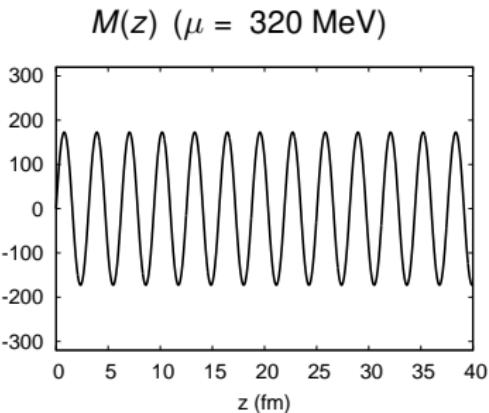


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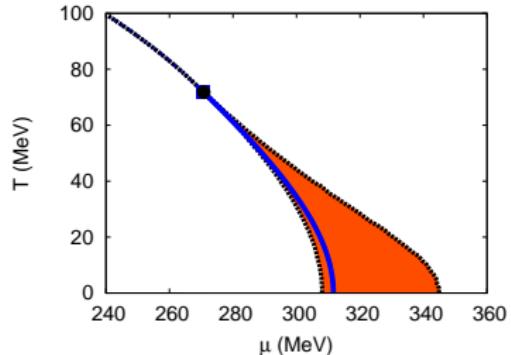
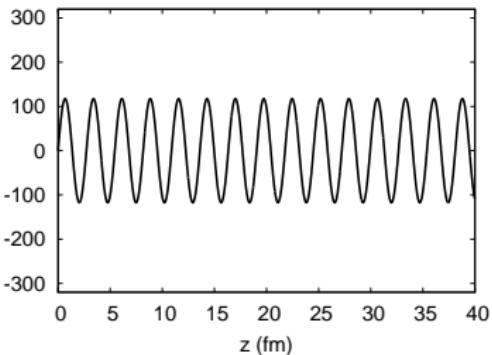
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$M(z)$ ($\mu = 330$ MeV)



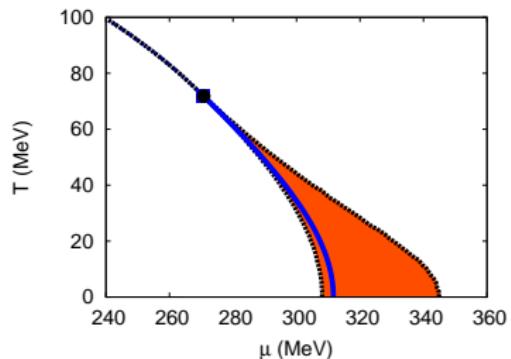
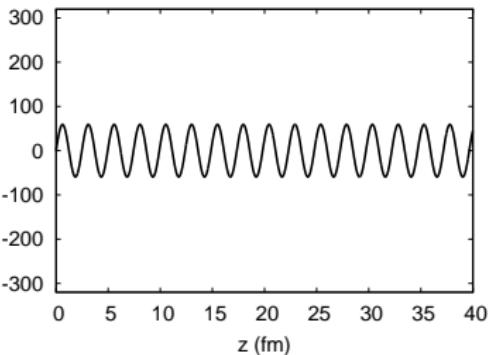
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$M(z)$ ($\mu = 340$ MeV)



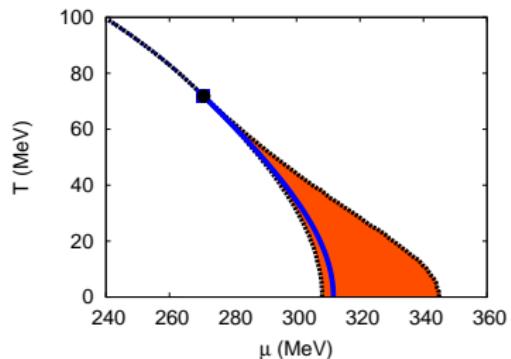
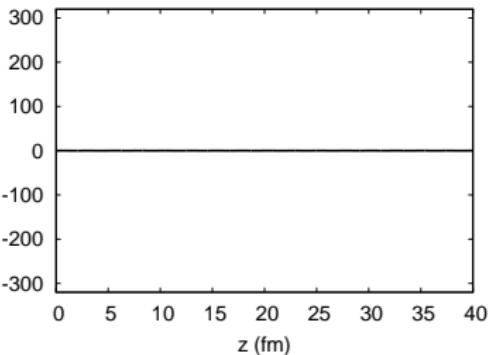
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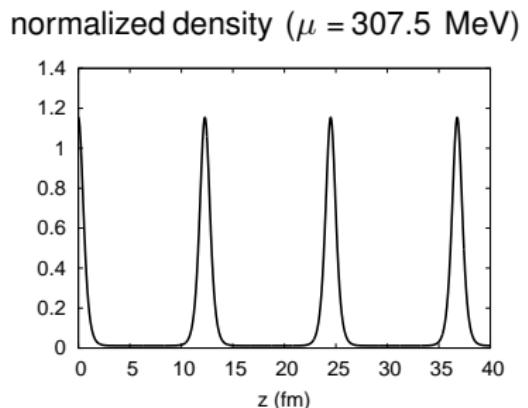
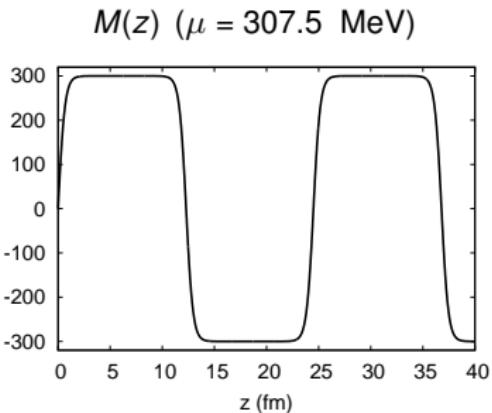


Mass functions and density profiles ($T = 0$)



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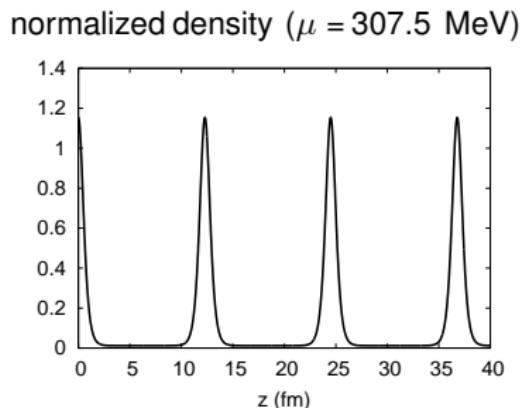
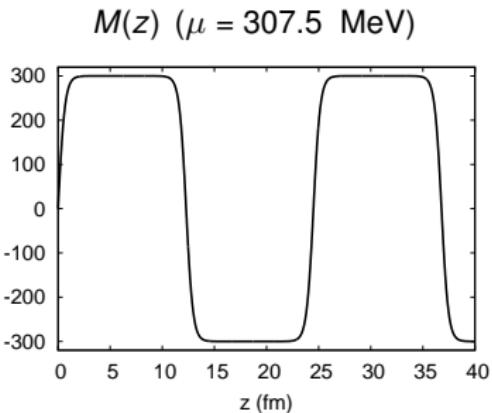


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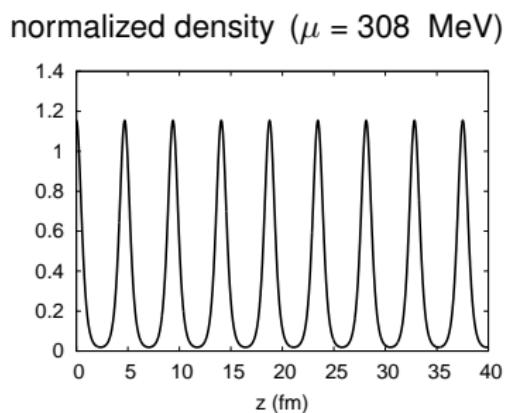
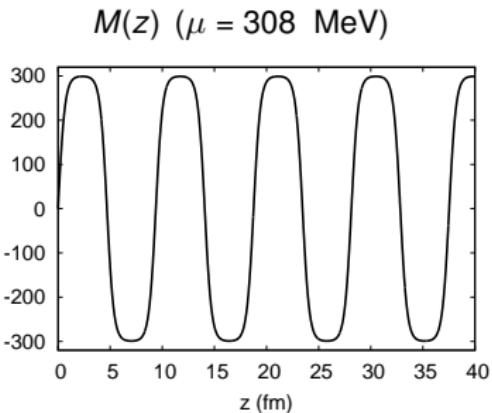


- Quarks reside in the chirally restored regions.

Mass functions and density profiles ($T = 0$)



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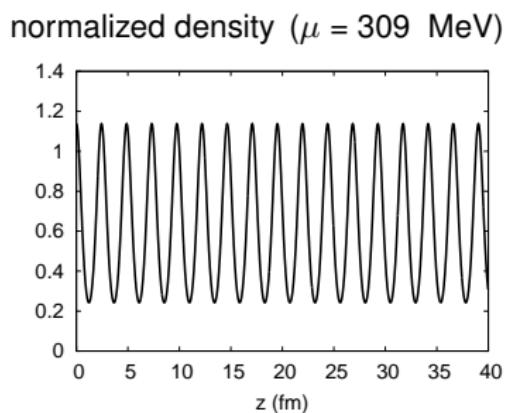
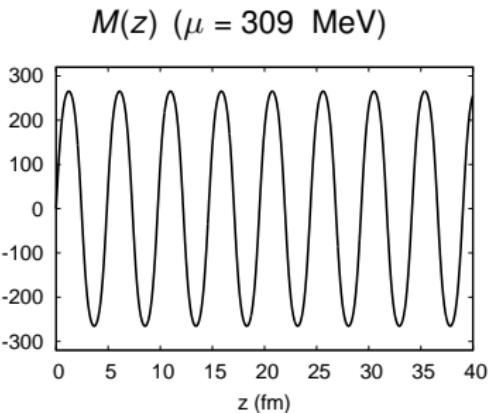
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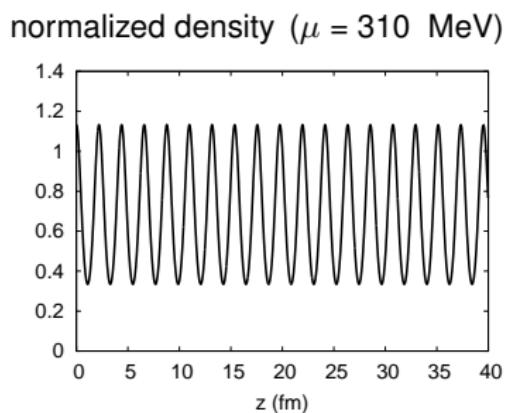
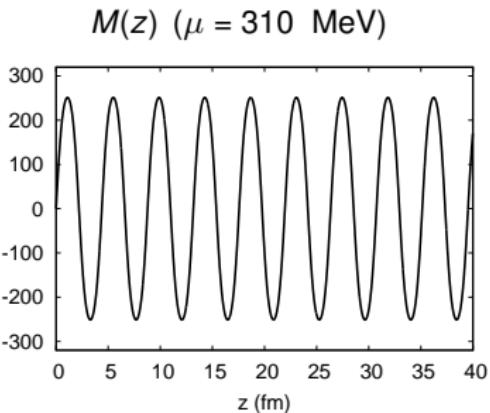
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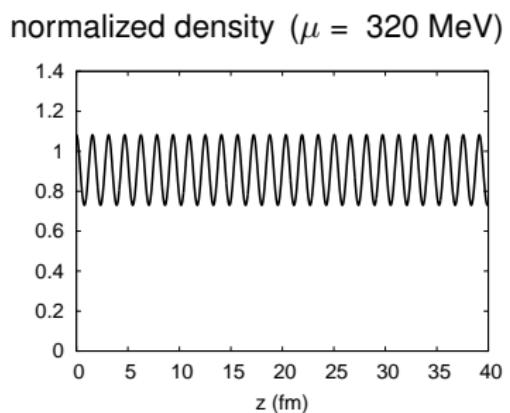
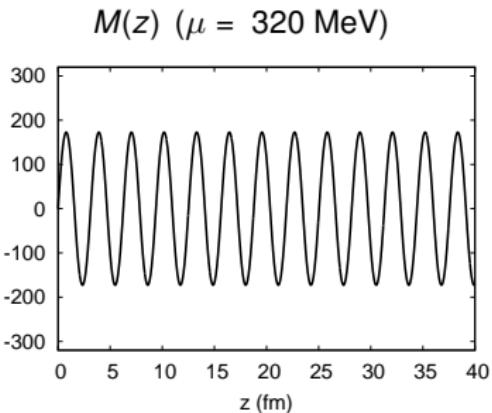
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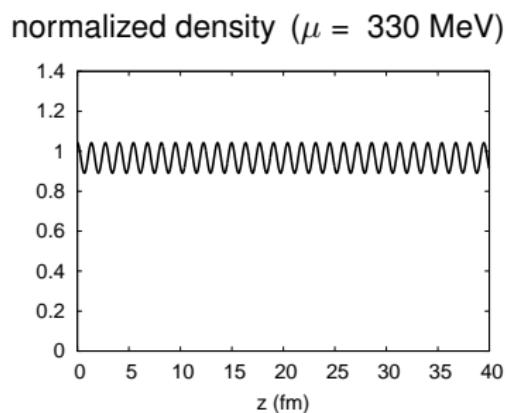
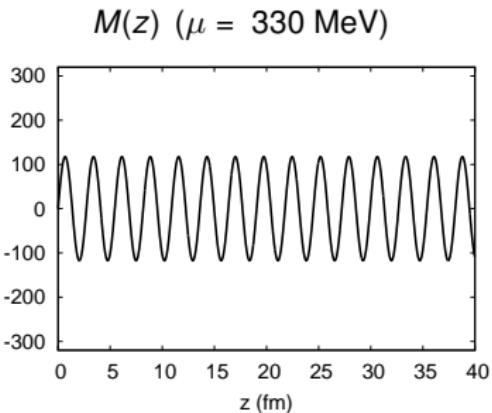
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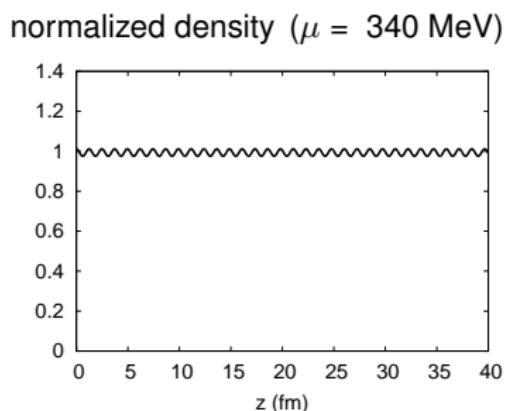
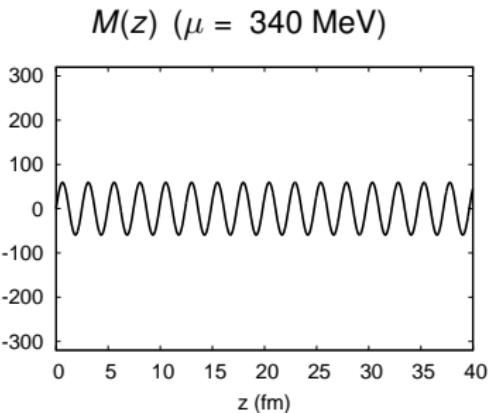


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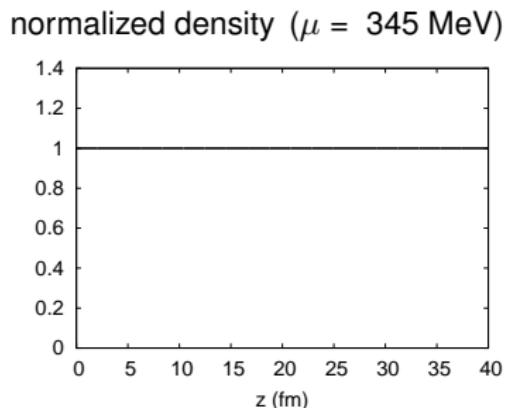
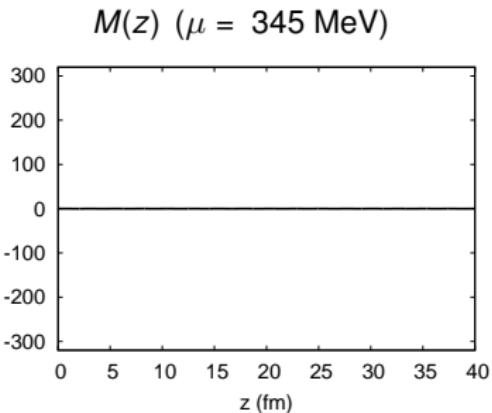


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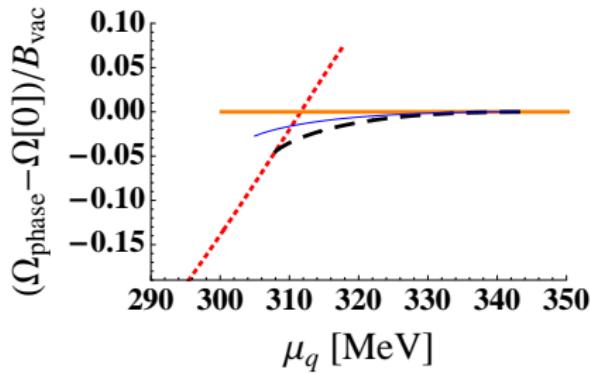
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Free energy difference

[D. Nickel, PRD (2009)]



- ▶ homogeneous chirally broken
- ▶ solitons
- ▶ chiral density wave:

$$M_{CDW}(z) = \Delta e^{iqz}$$

- ▶ soliton phase favored, when it exists
- ▶ $\delta\Omega_{\text{soliton}} \approx 2\delta\Omega_{CDW} \Rightarrow \text{CDW never favored}$

Including vector interactions

- ▶ additional vector term: $\mathcal{L}_V = -G_V(\bar{\psi}\gamma^\mu\psi)^2$

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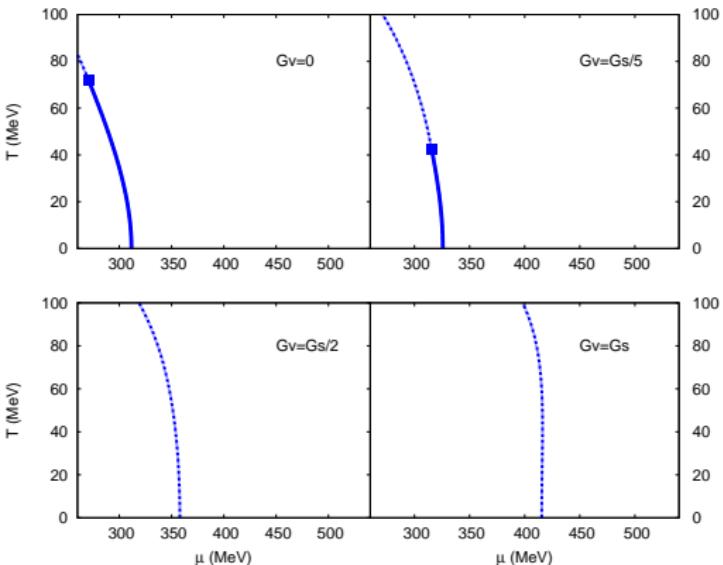
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Phase diagram

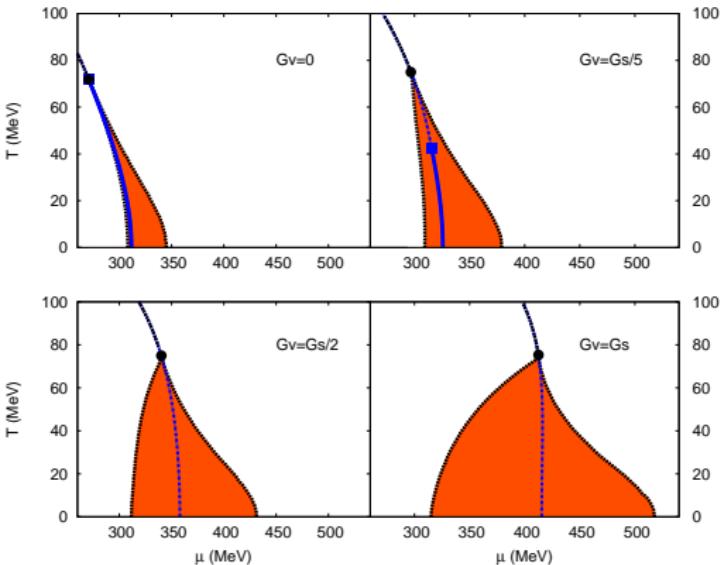


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- ▶ homogeneous phases: strong G_V -dependence of the critical point

Phase diagram

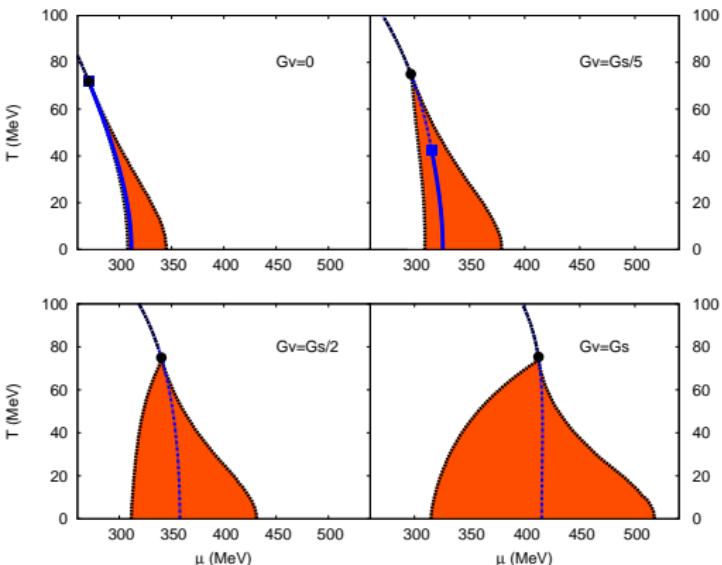


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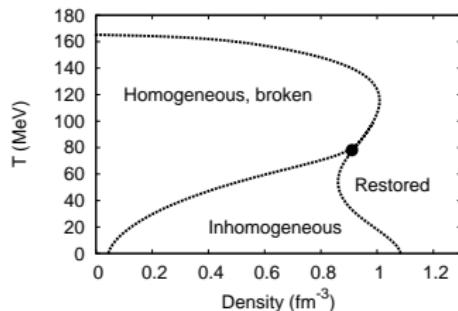
Phase diagram



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$T\langle n \rangle$ phase diagram:

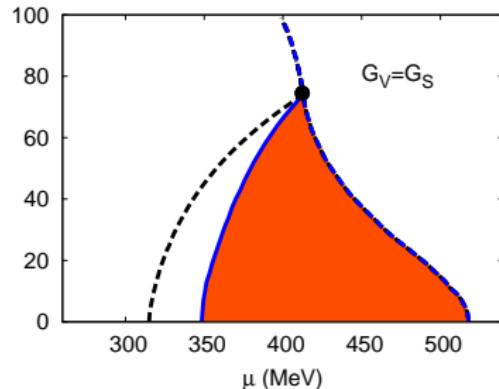
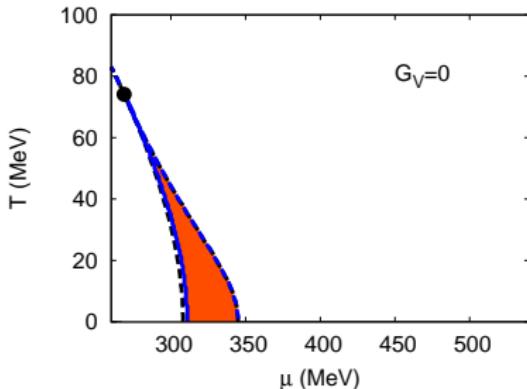


► independent of G_V !

- homogeneous phases: strong G_V -dependence of the critical point
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Chiral density wave

- ▶ How much can we trust the approximation $\tilde{\mu} = \mu - 2G_V\bar{n}$?
- ▶ Chiral density wave: $M(z) = \Delta e^{iqz} \Rightarrow n(z) = \text{const.}$



- ▶ CDW → restored and Lifshitz point agree with soliton solution
- ▶ chirally broken → CDW: 1st order and at higher μ
- ▶ exact phase boundary somewhere in between

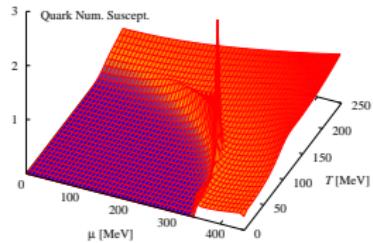
Susceptibilities



- ▶ signature of the critical point:
divergent susceptibilities
- ▶ e.g., quark number susceptibility:

$$\chi_{nn} = -\frac{\partial^2 \Omega}{\partial \mu^2} = \frac{\partial n}{\partial \mu}$$

homogeneous phases only:



[K. Fukushima, PRD (2008)]

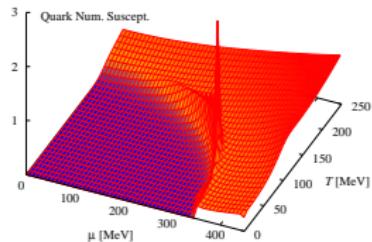
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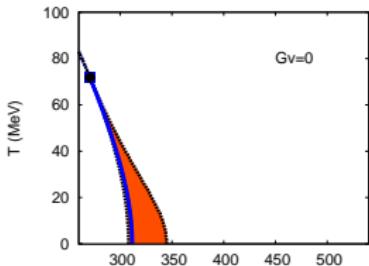
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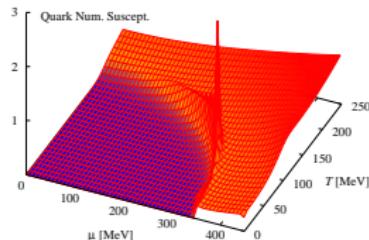
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- ▶ $G_V = 0$:
CP = Lifshitz point
→ no qualitative change

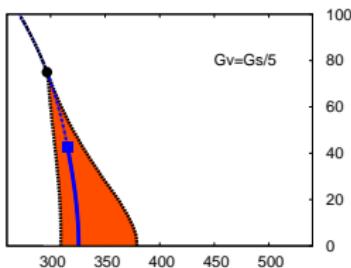
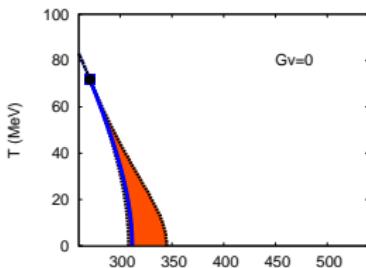
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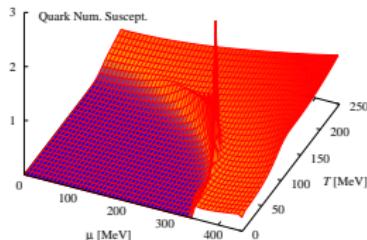
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no CP → no divergence

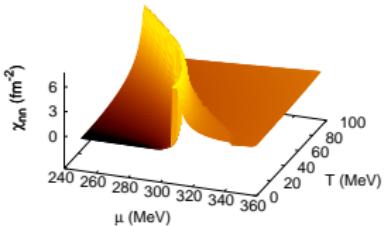
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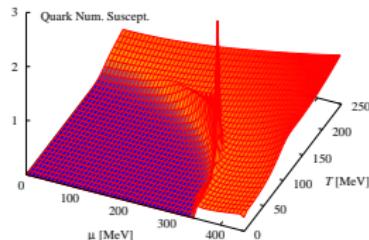
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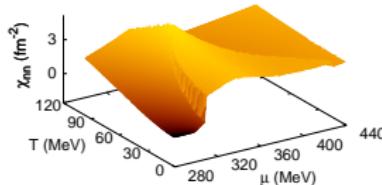
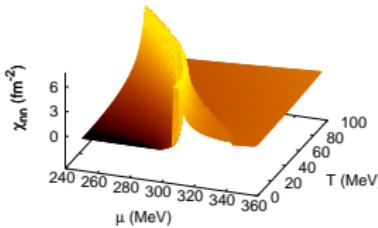
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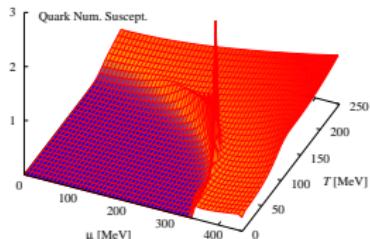
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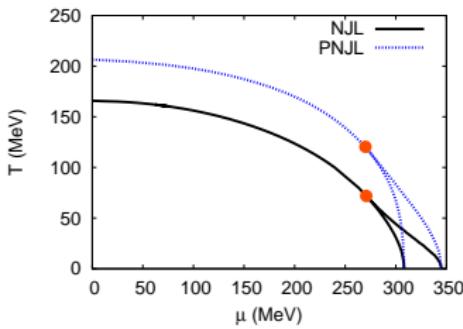
Including Polyakov-loop dynamics



- ▶ PNJL model: $\mathcal{L} = \bar{\psi}(i\cancel{D} - m)\psi + G_S ((\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2) + U(\ell, \bar{\ell})$
 - ▶ covariant derivative: $D_\mu = \partial_\mu + iA_0\delta_{\mu 0}$,
 - ▶ Polyakov loop: $L(\vec{x}) = \mathcal{P} \exp[i \int_0^{1/T} d\tau A_4(\tau, \vec{x})]$, $A_4(\tau, \vec{x}) = iA_0(t = -i\tau, \vec{x})$
 - ▶ expectation values: $\ell = \frac{1}{N_c} \langle \text{Tr}_c L \rangle$, $\bar{\ell} = \frac{1}{N_c} \langle \text{Tr}_c L^\dagger \rangle$
- ▶ assumption:
 $\ell, \bar{\ell}$ space-time independent, even in inhomogeneous phases
- ▶ main effect:
 $T \ln \left(1 + e^{-\frac{E-\mu}{T}} \right) \rightarrow T \ln \left(1 + e^{-3\frac{E-\mu}{T}} + 3\ell e^{-\frac{E-\mu}{T}} + 3\bar{\ell} e^{-2\frac{E-\mu}{T}} \right)$
→ suppression of thermally excited quarks at small $\ell, \bar{\ell}$

Results

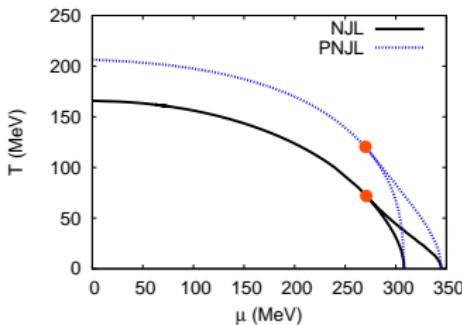
NJL vs. PNJL



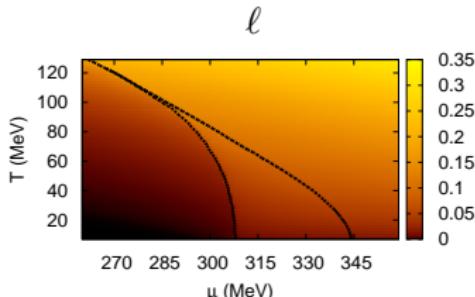
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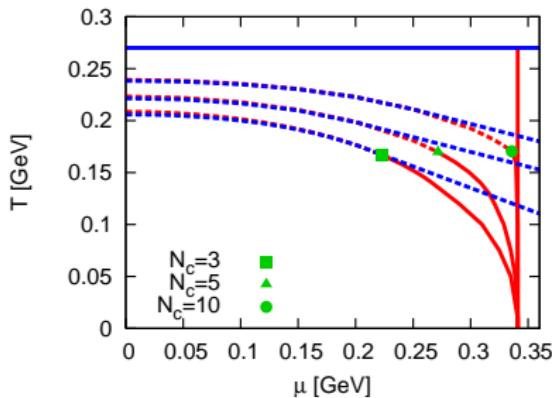
- ▶ Polyakov-loop expectation value:
 - ▶ inhomogeneous regime:
 $\ell \lesssim 0.15$, $\bar{\ell} \lesssim 0.2$
 - ▶ effects of neglecting spatial variations of $\ell, \bar{\ell}$ presumably small

Varying N_c



- ▶ homogeneous phases only

[McLerran, Redlich, Sasaki, NPA (2009)]



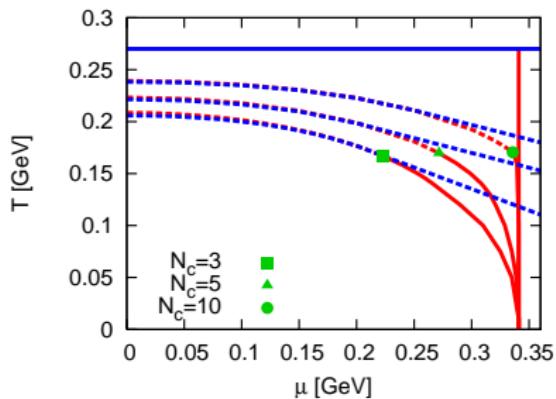
Varying N_c



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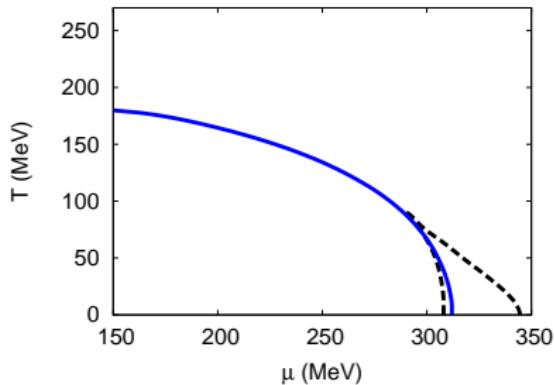
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[McLerran, Redlich, Sasaki, NPA (2009)]



- ▶ including inhomogeneous phases

$$N_c = 3$$



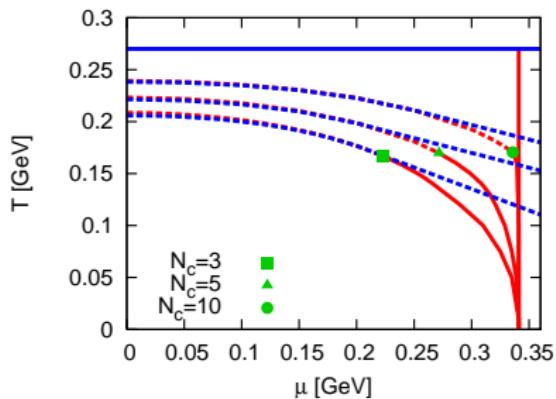
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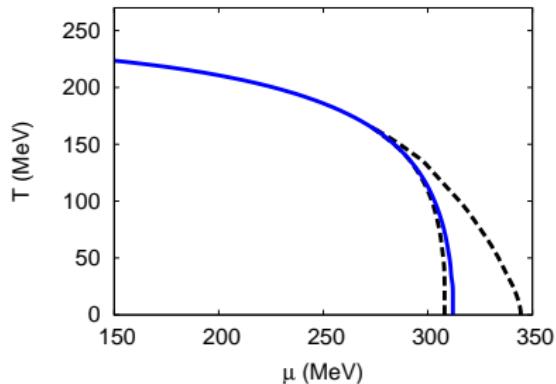
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$$N_c = 10$$



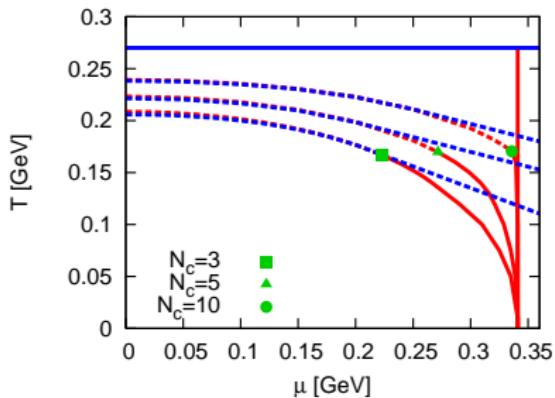
Varying N_c



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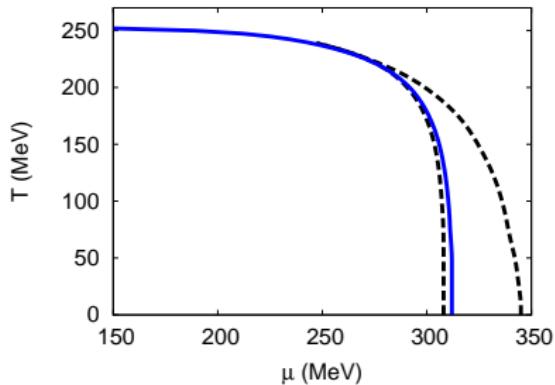
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$$N_c = 50$$



Conclusions



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- ▶ Inhomogeneous phases must be considered!

Conclusions



- ▶ Inhomogeneous phases must be considered!
- ▶ NJL model with one-dimensional modulations of $\langle \bar{q}q \rangle$:
 - ▶ 1st-order line and critical point covered by an inhomogeneous region
 - ▶ inhomogeneous phase rather stable w.r.t. vector interactions
 - ▶ number susceptibility always finite (for $G_V > 0$)
 - ▶ usual effect of the Polyakov loop

Conclusions



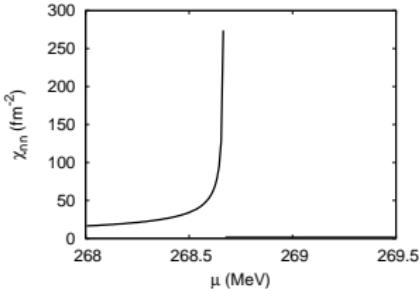
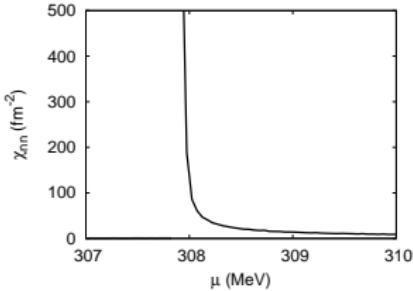
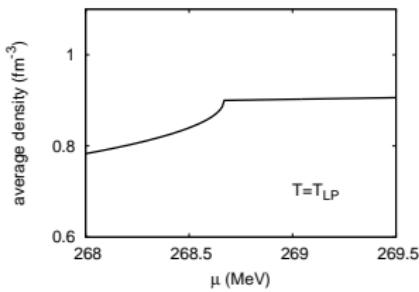
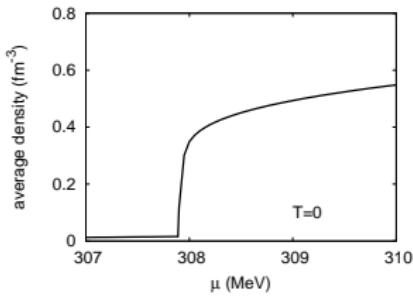
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- ▶ Inhomogeneous phases must be considered!
- ▶ NJL model with one-dimensional modulations of $\langle \bar{q}q \rangle$:
 - ▶ 1st-order line and critical point covered by an inhomogeneous region
 - ▶ inhomogeneous phase rather stable w.r.t. vector interactions
 - ▶ number susceptibility always finite (for $G_V > 0$)
 - ▶ usual effect of the Polyakov loop
- ▶ outlook:
 - ▶ include strange quarks
 - ▶ include color superconductivity
 - ▶ relax approximations (constant density, constant Polyakov loop)
 - ▶ higher dimensional modulations

Susceptibilities



- ▶ densities and quark number susceptibilities for $G_V = 0$:



- ▶ $T = T_{CP}, \mu < \mu_c :$

$$\chi_{nn} \propto \frac{1}{\sqrt{\mu_c - \mu}}$$

- ▶ $T = 0, \mu > \mu_{cr} :$

$$\chi_{nn} \propto \frac{1}{(\mu - \mu_{cr}) \log^2(\mu - \mu_{cr})}$$

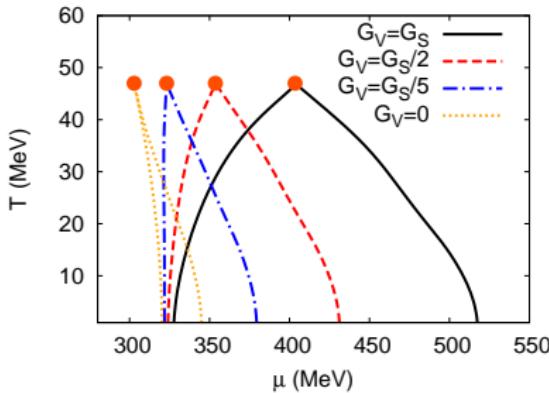
- ▶ $G_V > 0:$

$$\delta \chi_{nn}|_{T=0, \mu=\mu_{cr}} \approx \frac{1}{2G_V}$$

Finite current quark masses



- ▶ phase diagrams for $m = 5$ MeV:



- ▶ same qualitative behavior