Thermalization in high energy heavy ion collisions

Andreas Schäfer (Regensburg),
Berndt Müller (Duke) plus many more

- The problem
- 1. approach: Kolmogorov-Sinai entropy, Husimi transformation and Wehrl entropy
- 2. approach: String theory, AdS/CFT and entanglement entropy
- Conclusions

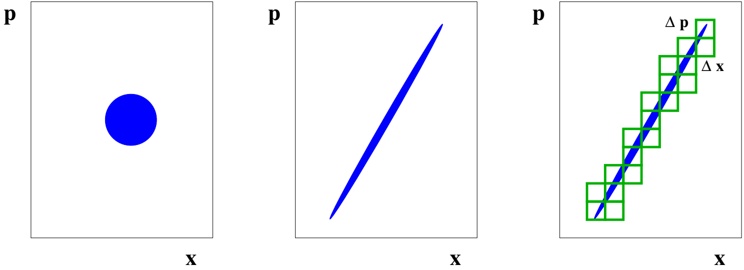
The problem

The theoretical description of high energy heavy ion collisions requires massive entropy production at early stages ($\ll 1$ fm/c)

But QCD is time-reversal invariant. Entropy is typically produced by measurements $\Rightarrow$ Coarse-graining.

How can entropy be produced at all **?**
How can it be produced so fast **?**

Coarse graining



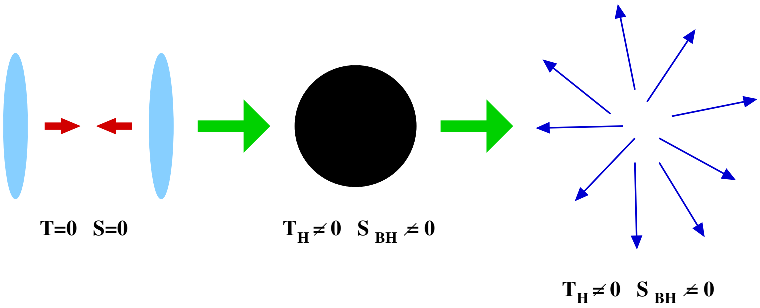
- Under time evolution the phase space volume is conserved
- The finite resolution of any measurement implies an increase in phase space volume

This is a rather generic problem

- still a fundamental problem for the rigorous formulation of quantum mechanics
- black hole physics – the information problem
- decoherence is the main problem for the development of quantum computing
- etc.

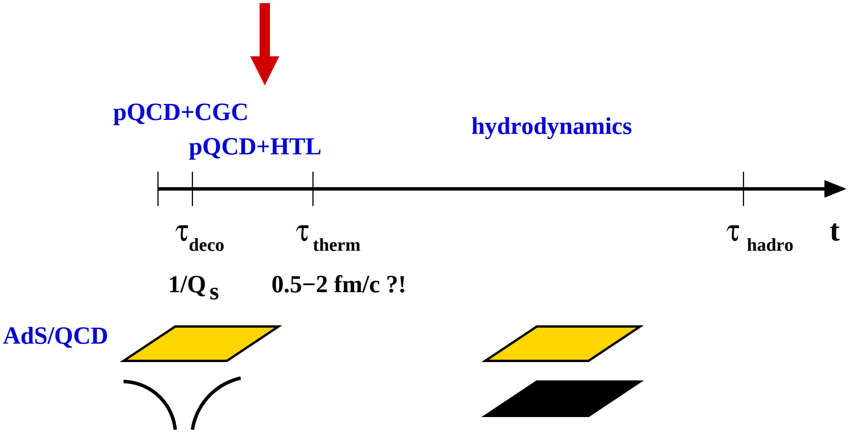
There exists a very large literature, much of which is contradictory
$\Rightarrow$ many papers must be wrong
$\Rightarrow$ what I am going to tell has a fair chance of being wrong in some respects

The information problem of black-hole physics



If the S-Matrix is unitary $S_{inital} = S_{final}$.

Different stages of entropy production in a HIC



Earlier work:

B. Müller, AS; Phys. Rev. C73(2006)054905
R. Fries, B. Müller, AS; Phys. Rev. C79 (2009)034904
pQCD & CGC $\Rightarrow$ entropy production from decoherence is very fast $\tau \sim 1/Q_s \sim 0.2$ fm/c
One gets at most 1/3 of the entropy needed
bottom-up mechanism: thermalization by soft gluon production

R. Fries, B. Müller, AS; Phys. Rev. C78 (2008) 034913
Viscous hydrodynamics can only produce a small amount of entropy

$\Rightarrow$ A large part of the entropy must come from the thermalization phase
Can this happen fast enough **?**

1. approach: Kolmogorov Sinai entropy etc.

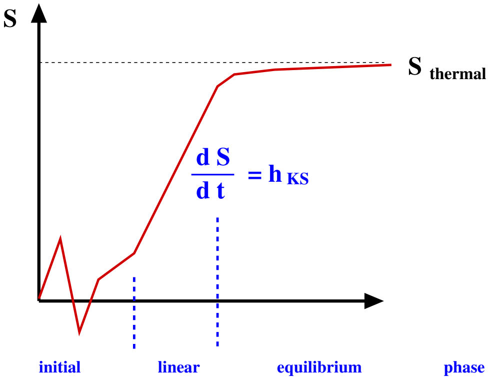
Hamilton Dynamics of classical YM fields

The Lyapunov exponents are determine numerically. The Kolmogorov-Sinai entropy is defined as

$$h_{KS} = \sum_{i,\lambda_i>0} \lambda_i$$

The “Kolmogorov-Sinai entropy” is no entropy, but an entropy growth rate.

a generic (??) picture



In the linear phase: $\frac{dS}{dt} = h_{KS}$

This can be calculated in classical field theory !

Husimi function and Wehrl entropy

with T.Kunihiro, A. Ohnishi, T. Takahashi, A. Yammoto

The Husimi function takes into account that the uncertainty principle implies coarse graining independent of the actual measurement.

It is easiest explained for a simple example: The inverse 1-dim oscillator.

$$\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2$$

Initial state: Gaussian wave packet of width $\sqrt{\hbar/\omega}$:

$$\langle x|\psi_0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\omega x^2/2\hbar}$$

The Wigner function associated with the density matrix $\hat{\rho}$ is defined as

$$W(p, x; t) = \int du\; e^{\frac{i}{\hbar} pu} \langle x - \frac{u}{2} |\; \hat{\rho}(t)\; | x + \frac{u}{2} \rangle$$

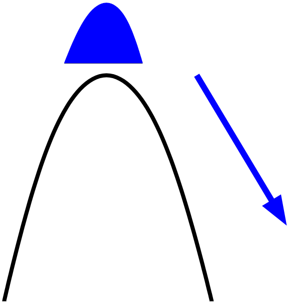
$$\int \frac{dp\, dx}{2\pi\hbar} W(p, x; t) = \mathrm{Tr}\, [\hat{\rho}] = 1$$

$$\int \frac{dp\, dx}{2\pi\hbar} [W(p, x; t)]^2 = \mathrm{Tr}\, [\hat{\rho}^2] \leq 1$$

The Wigner function is constant along the classical path.

$$x = x_0 \cosh \lambda t + \frac{p_0}{\lambda} \sinh \lambda t, \qquad p = \lambda x_0 \sinh \lambda t + p_0 \cosh \lambda t,$$

Thus there is only one positive Lyapunov exponent, namely λ and $h_{KS} = \lambda$



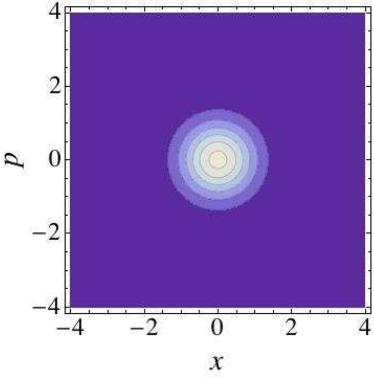
The Husimi function

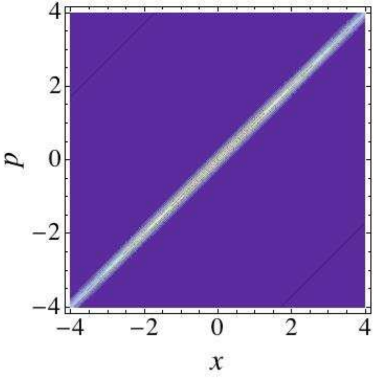
$$H_\Delta(p, x; t) \; = \; \int \frac{dp' \, dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p - p')^2 - \frac{\Delta}{\hbar}(x - x')^2\right) W(p', x'; t)$$

is non-negative, in contrast to the Wigner function. Δ is an arbitrary parameter corresponding to the actual measurement.

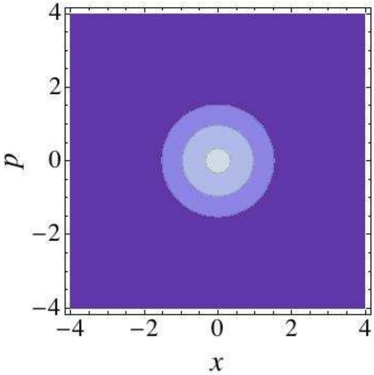
For this simple case the differential equation for W and the integration can be done analytically

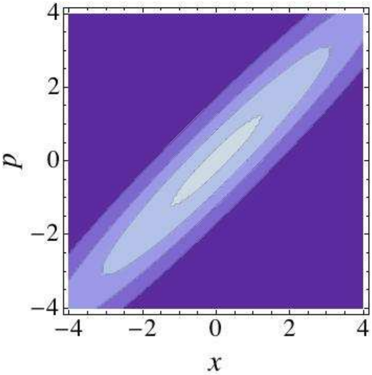
The Wigner function for $t = 0$ and $t = 2/\lambda$





The Husimi function for $t = 0$ and $t = 2/\lambda$ for $\Delta = \lambda$. The phase space volume increases due to smearing.





For the Wehrl entropy

$$S_{H,\Delta}(t) \;=\; -\int \frac{dp\,dx}{2\pi\hbar} H_\Delta(p,x;t) \ln H_\Delta(p,x;t)$$

one finds

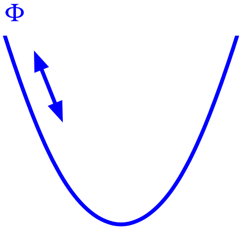
$$\lim_{t\to\infty} \frac{dS_{H,\Delta}}{dt} \;=\; \lambda \;=\; h_{KS}$$

independent of Δ **!**

A simple model: The inflaton field Φ coupled to another scalar field χ

$$\mathcal{L}(\hat{\chi}) = \frac{1}{2}\left(g^{\mu\nu}\frac{\partial\hat{\chi}}{\partial x^\mu}\frac{\partial\hat{\chi}}{\partial x^\nu} - g^2\Phi(t)^2\hat{\chi}^2\right)$$

$$\Phi(t) \approx \Phi_0 \cos(\omega t)$$



mode expansion

$$\hat{\chi}(\mathbf{x},t) = \frac{1}{R(t)} \int \frac{d^3k}{(2\pi)^{3/2}} \left(\hat{X}_k(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{X}_k^\dagger(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \right)$$

$R(t)$ is the cosmological scale factor

$$\frac{\partial^2 \hat{X}_k}{\partial \tau^2} + \left(\bar{\kappa}^2 + \frac{g^2}{2\lambda} \cos(2\omega\tau) \right) \hat{X}_k(\tau) = 0$$

Solutions: Mathieu sine and cosine functions $S(a,q;\omega\tau)$ and $C(a,q;\omega\tau)$ with $a = \bar{\kappa}^2$ and $q = -g^2/4\lambda$. We will henceforth drop the parameters a and q. Asymptotically

$$C(\omega\tau) \; \approx \; e^{\mu\tau} \cos(\omega\tau + \alpha_c(\tau)) \,, \qquad S(\omega\tau) \; \approx \; e^{\mu\tau} \cos(\omega\tau + \alpha_s(\tau))$$

For large times, the constancy of the Wronskian $(C\dot{S} - S\dot{C})/\omega = 1$ implies

$$|\alpha_c(\tau) - \alpha_s(\tau)| \longrightarrow e^{-2\mu\tau}$$

This is similar to coherent light production in a laser: The larger the amplitude the more coherent the light.

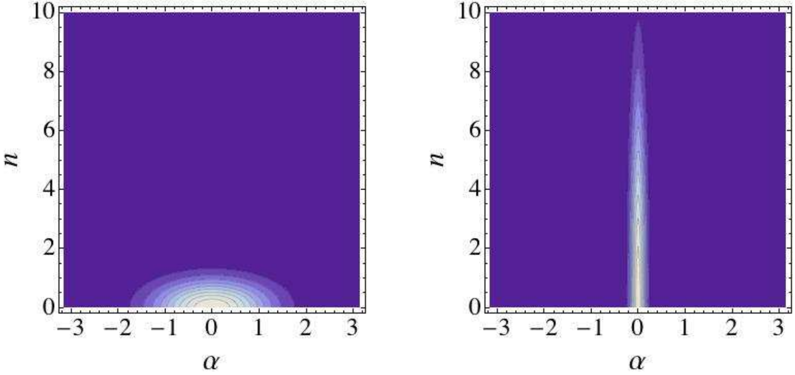
The number - phase uncertainty relation

$$(\Delta N_k)^2(\Delta\alpha_k)^2 = N_k^2(\Delta\alpha_k)^2 \geq \frac{1}{4}$$

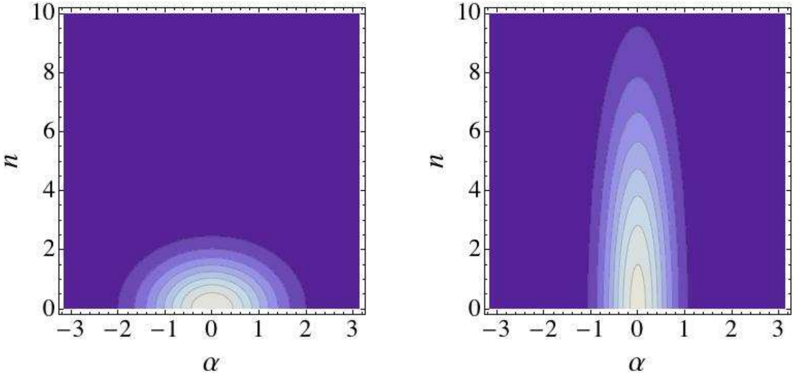
Again the problem is simple enough to be solved analytically

$$H_\Delta(\alpha, n, \tau) \;=\; \int_{-\pi}^{\pi} \frac{d\alpha'}{\pi} \sum_{n'} e^{-\Delta(\alpha-\alpha')^2 - \frac{(n-n')^2}{\Delta}} W(\alpha', n', \tau)$$

The Wigner function for $\tau = 0.5/\mu$ and $\tau = 1/\mu$



The Husimi function



The Wehrl entropy is again given by the Lyapunov exponent for an individual mode

$$
\begin{aligned}
S_{\mathrm{H},\Delta}(\tau) &= -\int \frac{d\alpha}{2\pi} \sum_n H_\Delta \ln H_\Delta \\
&\stackrel{\tau\to\infty}{\longrightarrow} 2\mu\tau + \mathrm{const.}
\end{aligned}
$$

and by the Kolmogorov-Sinaï entropy for all modes.

Numerical studies: Tobias Weindler

$\Rightarrow$ For typical chaotic inflation parameters thermalization occures very fast, e.g., in one oscillation period.

Relation to Other Definitions of Entropy

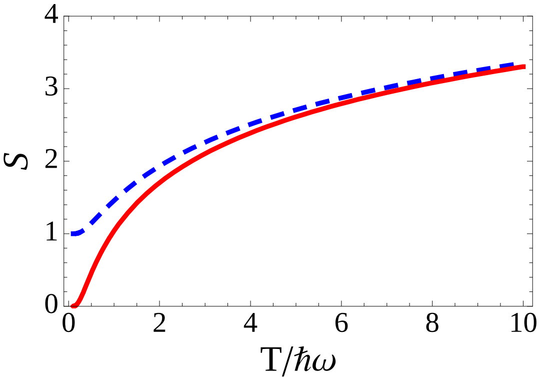
one harmonic oscillator; occupation probability of eigenstate $|n\rangle$:

$$
\begin{aligned}
w_n &= e^{-n\beta\hbar\omega}/\mathcal{Z}_\beta \\
\mathcal{Z}_\beta &= \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} = (1 - e^{-\beta\hbar\omega})^{-1}
\end{aligned}
$$

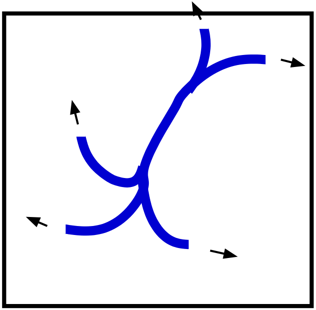
von Neumann entropy

$$
\begin{aligned}
S_{\mathrm{vN}} &\equiv -\sum_{n=0}^{\infty} w_n \ln w_n = \frac{\beta\hbar\omega}{e^{\beta\hbar\omega} - 1} - \ln(1 - e^{-\beta\hbar\omega}) \\
&= -\bar{n} \ln \bar{n} + (\bar{n} + 1) \ln(\bar{n} + 1)
\end{aligned}
$$

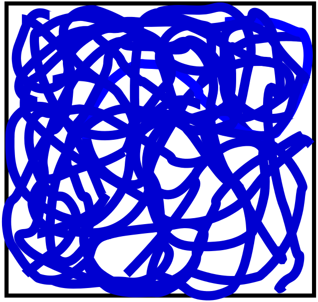
S_{vN} (solid) versus S_H (dashed)



The general idea



Linear phase



Late time phase

The crucial point: the linear slope can be calculated in classical gauge theory

A quantitative study: 1008.1156, Kunihiro, Müller, Ohnishi, AS, Takahashi, Yamamoto

$$
\begin{aligned}
H &= \frac{1}{2}\sum_{x,a,i} E_i^a(x)^2 + \frac{1}{4}\sum_{x,a,i,j} F_{ij}^a(x)^2 \\
F_{ij}^a(x) &= \partial_i A_j^a(x) - \partial_j A_i^a(x) + \sum_{b,c} f^{abc} A_i^b(x) A_j^c(x)
\end{aligned}
$$

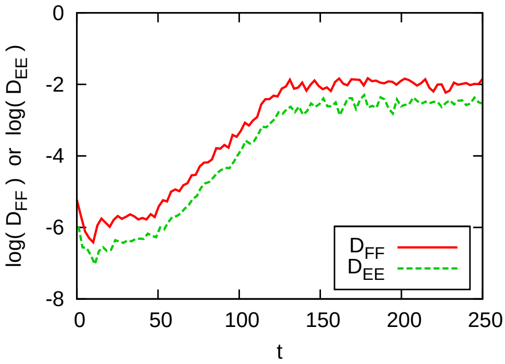
$$\delta\dot{X}(t) = \mathcal{H}\delta X(t)$$

$$
\begin{aligned}
\dot{A}_i^a(x) &= E_i^a(x) \\
\dot{E}_i^a(x) &= \sum_j \partial_j F_{ji}^a(x) + \sum_{b,c,j} f^{abc} A_j^b(x) F_{ji}^c(x)
\end{aligned}
$$

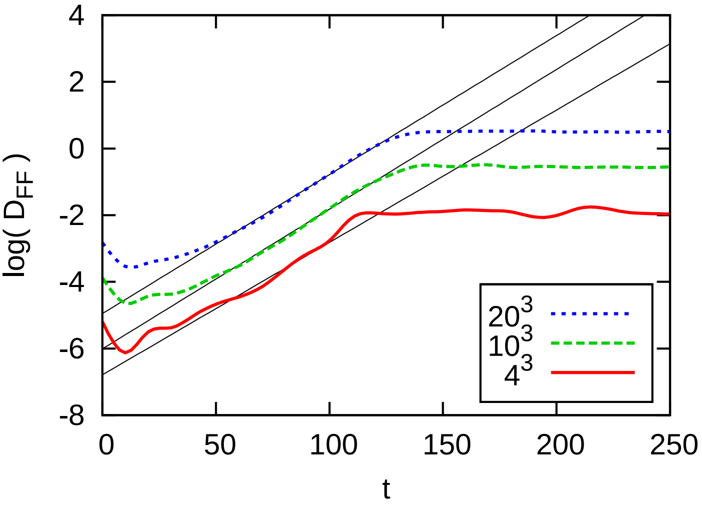
Different distance measures give the same result.

$$D_{EE} = \sqrt{\sum_x \{\sum_{a,i} E_i^a(x)^2 - \sum_{a,i} E'^a_i(x)^2\}^2}$$

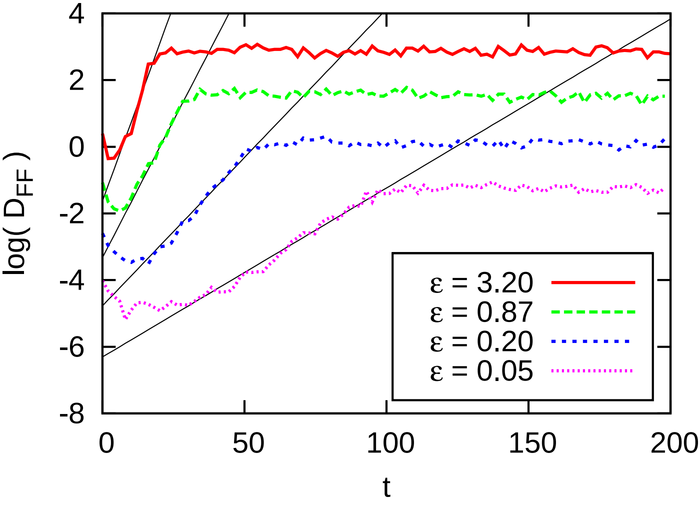
$$D_{FF} = \sqrt{\sum_x \{\sum_{a,i,j} F_{ij}^a(x)^2 - \sum_{a,i,j} F'^a_{ij}(x)^2\}^2}$$



Time evolution in SU(2) simulation on 4^3, 10^3, and 20^3 lattices with the same energy density



Time evolution in SU(3) simulation on a 4^3 lattice



Classical YM theories are UV divergent
$\Rightarrow$ The lattice constant has the physical meaning of a screening length

Scale setting can be done by different means:
equation the energy densities on the lattice

$$\begin{aligned} \epsilon_{cl}(T) &= \frac{\epsilon^{\mathrm{L}}}{a^4 g^2} \\ \epsilon^{\mathrm{L}} &= 2(N_c^2-1)\frac{1}{L^3}\sum_{\mathbf{k}} |\mathbf{k}| \frac{T^{\mathrm{L}}}{|\mathbf{k}|} \\ \epsilon(T) &= 2(N_c^2-1)\int \frac{d^3k}{(2\pi)^3}\frac{|\mathbf{k}|}{e^{|\mathbf{k}|/T}-1} \end{aligned}$$

or choosing the damping length of SU(N) gauge theories

All approaches give

$$a = \frac{\theta}{T} \sim \epsilon^{-1/4}$$

with θ of order unity

The thermalization time is estimated from

$$\tau_{\rm eq} = \frac{\Delta s}{s_{KS}} + \tau_{delay}$$

The delay time is found to be of the order

$$\frac{1}{6(N_c^2-1)L^3} e^{\lambda_{\rm max}\tau_{\rm delay}} \approx 1$$

numerically we find

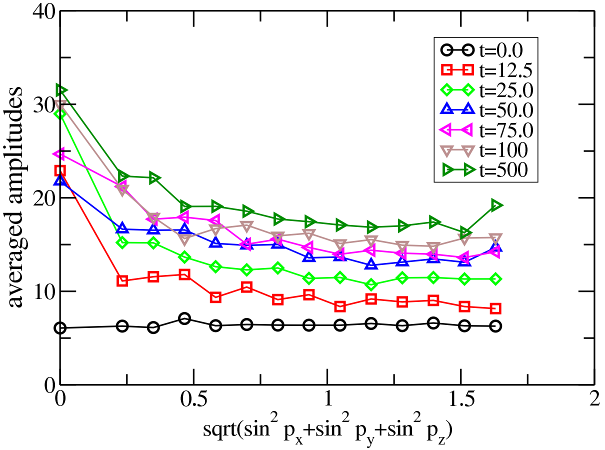
$$\tau_{\mathrm{eq}} \approx 2\ \mathrm{fm/c}$$

with substantial theoretical uncertainties

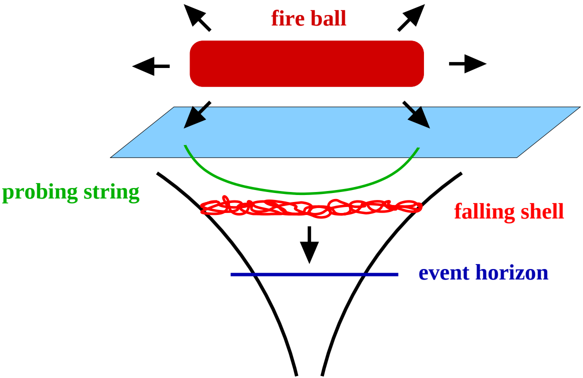
a value below 1 fm/c is very unlikely

This is rather a bottom-up mechanism

Fourier spectrum: the driving modes are infrared



Idea: Probe black brane formation with a string or membrane.

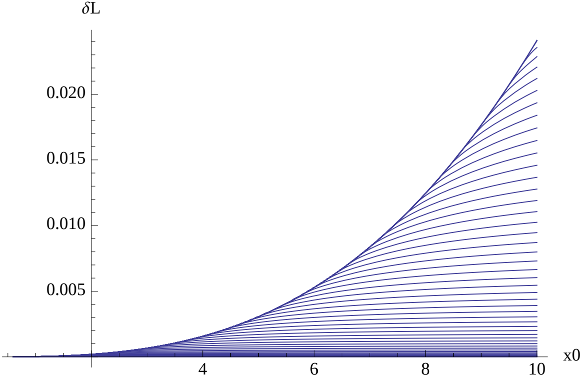


de Boer, Craps, Keski-Vakkuri, Bernamonti, Staessens, Balasubramanian, Shigemori, Copland
results from 1012.4753 and 1103.2683

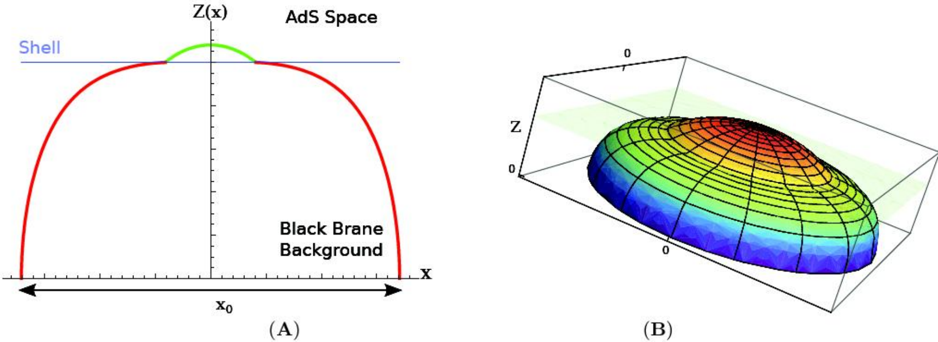
Disclaimer: This is a very active field, there are many highly relevant papers [Lin & Shuryak, Calabrese & Cardy, Albash & Johnson, Abajo-Arrastia & Aparicio & Lopez, ...]. I will not cite them for lack of time.

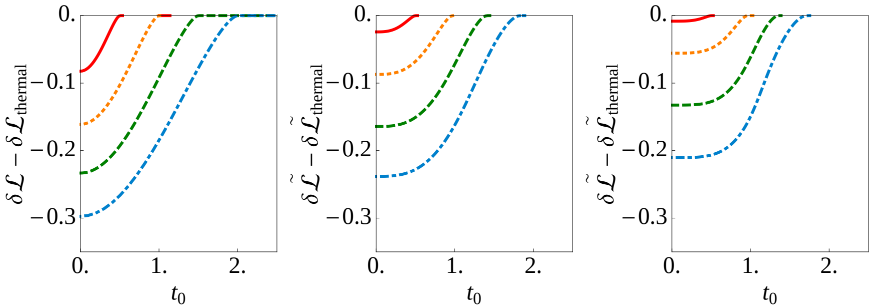
The change in geodetic length is sensitive to equal time correlators of high dimension gluonic operators.

$$\frac{\langle \mathcal{O}(t_{shell}, x)\mathcal{O}(t_{shell}, 0)\rangle_{\rm shell}}{\langle \mathcal{O}(t_{shell}, x)\mathcal{O}(t_{shell}, 0)\rangle_{\rm AdS}} \approx e^{-\Delta \delta L(t_{shell}, x)}$$

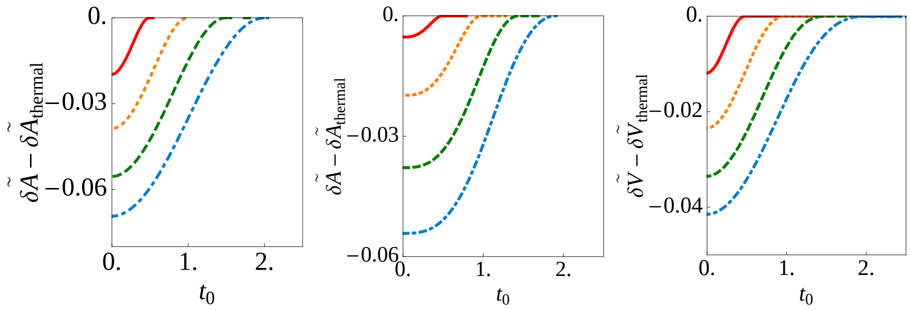


We solved analytically and numerically different cases:
$AdS_3 \sim CFT(1+1)$, $AdS_4 \sim CFT(1+2)$, $AdS_5 \sim CFT(1+3)$
and analysed how the length of the geodesic/the area of the surface approaches its thermal value, as a function of ℓ and t_0.





$\delta\tilde{\mathcal{L}} - \delta\tilde{\mathcal{L}}_{\text{thermal}}$ ($\tilde{\mathcal{L}} \equiv \mathcal{L}/\ell$) for $d = 2, 3, 4$ (left,right, middle) and $\ell = 1, 2, 3, 4$ (top to bottom curve).



$\delta\tilde{\mathcal{A}} - \delta\tilde{\mathcal{A}}_{\rm thermal}$ ($\tilde{\mathcal{A}} \equiv \mathcal{A}/\pi R^2$ and $\delta\tilde{V} - \delta\tilde{V}_{\rm thermal}$ ($\tilde{V} \equiv V/(4\pi R^3/3)$) as a function of t_0 for radii $R = 0.5, 1, 1.5, 2$ (top to bottom) in $d = 3$ (left panel) and $d = 4$ (middle and right panel) CFT

Observations

- Thermalization is approached as fast as compatible with causality.
  Note: spead of light (gluons), not speed of sound (density fluctuations)
  For heavy ion collisions this implies
  $\tau \sim 1/(2Q_s) \sim 0.1 fm/c$
- Short distances thermalize first, top-down rather than bottom-up thermalization
  Unavoidable in the AdS dual theory. A fundamental difference between strong and weak coupling ???
- Finite time till complete thermalization
  Probably an artefact of the treatment of UV divergencies

Entanglement entropy, a candidate for thermal entropy ?
A quantum system X is composed of two subsystems A and B.
Definition of *entanglement entropy*:

$$S(A) = -\mathrm{Tr}_A[\rho_A \ln \rho_A]$$

Symmetry $A \Leftrightarrow B$ imposes $S(A) \sim$ *surface* $A =\sim$ *surface* B.
toy model: one-dimensional CFT

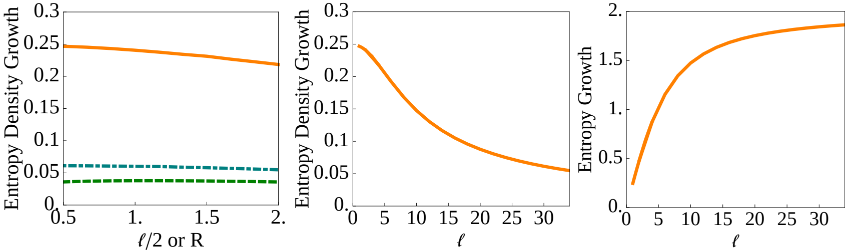
$$S(\ell) = \frac{c}{3} \ln(\ell/a),$$

c central charge, a short-distance cut-off.
Generalization for Finite T:

$$S(\ell) = \frac{c}{3} \ln\left(\frac{1}{\pi a T} \sinh(\pi \ell T)\right),$$

$T\ell \ll 1 \Rightarrow S(\ell)$, $T\ell \gg 1 \Rightarrow S_{\mathrm{th}} = (c/3)\pi T\ell$
the thermal entropy could be equal to the volume part of the entanglement entropy

But: The entanglement entropy does not show the expected behaviour

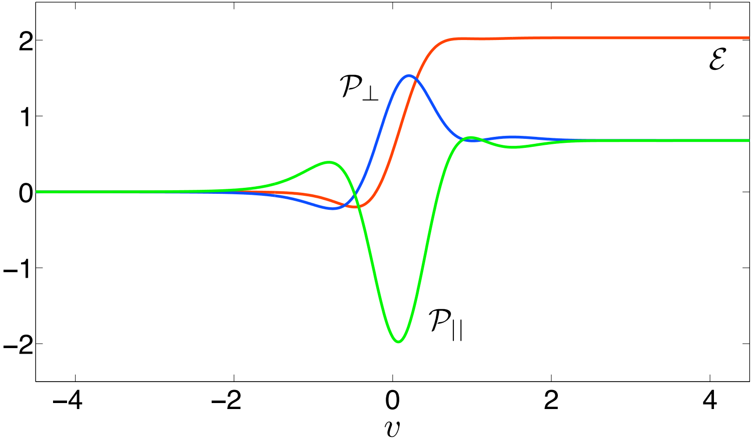


Maximal growth rate of entanglement entropy and entanglement entropy *density* vs. radius of entangled region; $d = 2$ (orange), $d = 3$ (blue), $d = 4$ (green)

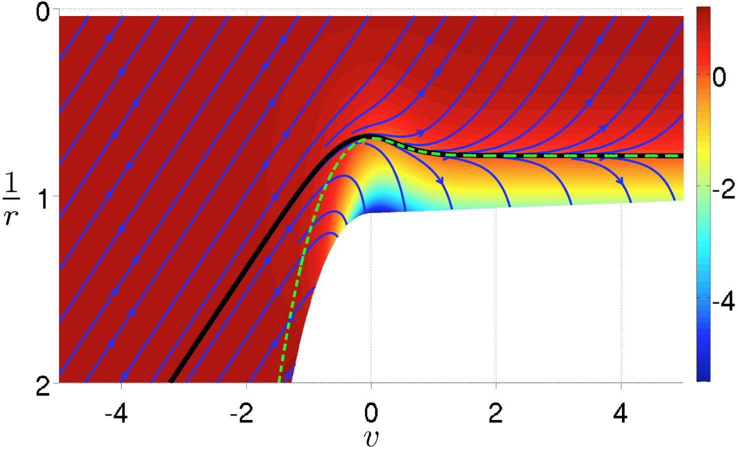
There are other candidates

Chesler and Yaffe 0812.2053

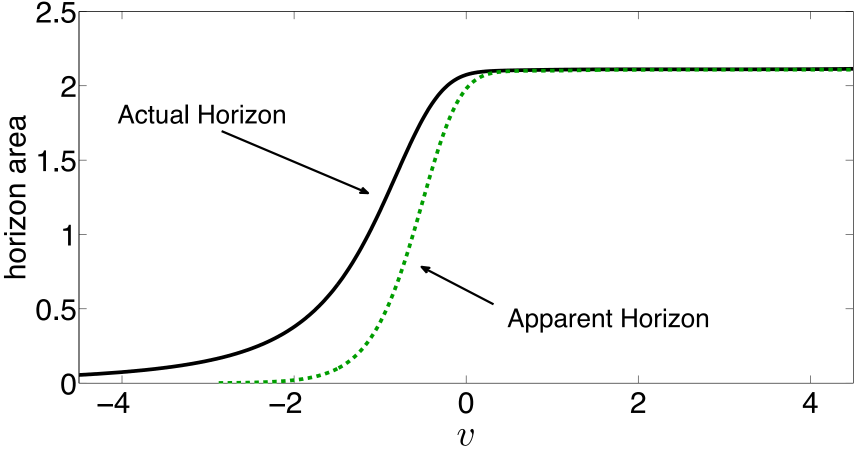
Thermalization of energy deposited in Minowski space



geodesics and horizon formation



horizon area $\stackrel{?}{=}$ entropy as function of time



But the growth rate of the apparent horizon depends on choice of coordinates.

Conclusions

- Understanding entropy production during thermalization in HICs is a problem of fundamental importance. HICs allow to study this process experimentally.
- Thermalization via non-linear dynamics and coarse graining with $\hbar$ is closer to the bottom-up scenario and needs $\tau \approx 2\mathrm{fm/c}$.
- Thermalization for strong coupling as described by AdS/CFT is top-down and very fast $\tau \approx 0.1\mathrm{fm/c}$. The time evolution is different.
- Time evolution might be a key to identify physical candidates for non-equilibrium definitions of entropy.