The Schwinger function in the Nambu-Jona-Lasinio model

Bachelor-Thesis von Bardiya Bahrampour

Februar 2011

TECHNISCHE UNIVERSITÄT DARMSTADT

Institut für Kernphysik
Theoretische Kernphysik
NHQ

The Schwinger function in the Nambu-Jona-Lasinio model

Vorgelegte Bachelor-Thesis von Bardiya Bahrampour

1. Gutachten: Dr. Buballa
2. Gutachten: Prof. Dr. Fischer

Tag der Einreichung:

Abstract

According to arguments made by Gribov, if the backreaction of pions, the Goldstone bosons of chiral symmetry breaking, is taken into account, the analytic properties of the quark propagator change in a way that positivity of the Schwinger function might be violated. If this is the case, this violation can be interpreted as an explanation of confinement in QCD then. For QCD this has been investigated in [1] but the positivity violation due to pion loops did not occur. Although the NJL-model is has no confinement in Hartree approximation, it might be interesting to investigate if positivity is violated in this model if a pion loop is taken into account. This investigation is done here. We will see at the end of this work that Gribov's confinement scenario is not realized in the NJL model, which agrees with the fact that the NJL model is expected to have no confinement mechanism.

Contents

1 Introduction

Quantum chromodynamics is the quantum field theory of strong interactions. In this gauge theory the interaction between quarks is mediated by its gauge bosons, the gluons. There are 6 different types of quarks (up, down, strange, charm, bottom, top). Similarly to QED, where the electric charge is the reason of a coupling to the photon field, quarks have also a charge called color-charge and it comes in three different types, called red, green, and blue. In general the gluon field couples to all color-charged particles. In contrast to QED the gauge bosons of QCD carry color-charges themselves. The fact that gluons themselves are colored and consequently couple to other gluons makes the theory much more complicated than QED, since much more interaction terms are possible in pertubation theory.

Another feature of QCD that arises from the self coupling of gluons is the fact that the coupling constant becomes weak for small distances or alternatively large momenta (asymptotic freedom), whereas it becomes strong for large distances. In the case of strong coupling, calculations in QCD become very complicated again. One important feature that is part of the strong coupling case in the strong interactions is confinement, the fact that quarks never appear free in nature, but are always confined in colorless hadrons. In QCD this phenomenom arises due to the fact that the potential of the interaction between quarks grows linearly. So the quarks cannot be seperated from each other until enough energy is put into the system to break the interaction tubes by creation of quark-antiquark pairs and the quarks are confined in new groups then. However the full mechanism of QCD color-confinement is still not known.

Gribov had the idea that confinement might be explained in Eucdlidean quantum field theory by positivity violation of the Schwinger function ([2], [3], [4]). In order to obtain the probabilistic interpretation of quantum theory, the axioms of Euclidean quantum field theory (sec. 2.1) demand "reflection positivity" for the Euclidean Green's functions (Schwinger functions). Gribov supposed that confinement might arise by taking the Goldstone bosons of chiral symmetry breaking (pions) into account. The idea is that a pion loop in the quark propagator might change its structure such that reflection positivity is violated. If this happens, then the propagating quark does not describe physical degrees of freedom. So a single quark is unable to propagate, e.g. to a measurement device. This scenario can be interpreted as confinement then, since free quarks cannot be observed. With this argumentation, violation of reflection positivity is a sufficient condition for confinement, but it does not follow that this condition is necessary. For QCD this has been investigated, e.g. in [1], but no violation of positivity due to pion loops occured. The NJL model of quantum chromodynamics (ch. 3) is expected to have no confinement, but it shares with QCD, besides some other symmetries, the feature of chiral symmetry and its breaking. Since in Gribov's argumentation the Goldstone boson of chiral symmetry breaking, which also appears in the NJL-model, is essential for the Gribov confinement scenario, it might hold for this model. In this work we calculate the quark propagtor numerically, taking a pion loop into account, to investigate if Gribov's scenario is realized, i.e. if reflection positivity is violated due to the pion.

Notation:
In this work we will use several times the residue theorem. In our notation $\oint_{up}$ will mean integration over the closed path paramterized by $u_+(t)$ with r going to infinity and $\oint_{down}$ means integration over the closed path parametrized by $u_-(t)$ with r again going to infinity, where $u_\pm$ are given by

$$u_\pm : [0,1) \to \mathbb{C} \qquad u_\pm(t) = \begin{cases} r(4t-1) & t \in [0,1/2) \\ re^{\pm i\pi(2t-1)} & t \in [1/2,1) \end{cases} .$$

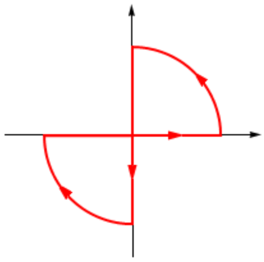

Fig. 2: Closed path for Wick rotation

The Minkowski metric of relativistic quantum field theory causes many technical problems like singularities in the propagators. In pertubation theory, where one does 4-dimensional momentum integrations over the propagators, those singularities can be eliminated by a Wick rotation to imaginary time components. Basically one does an analytical continuation to complex times, substitutes the time component of a 4-vector, $t_0 \to it_4$, and works with a Euclidean metric instead of Minkowski. Usually one also performs some redefinitions of the gamma matrices in order to absorb some negative signs and imaginary factors to obtain the form $S(p) \sim -i\not{p} + m$ for the propagators. We will show as an example a Wick rotation of a propagator. In order to motivate the redefinitions of the gamma matrices we will do a Wick rotation on a fermionic propagator. One should notice that for the Wick rotation of a fermionic propagator one has to do a few more arguments than in the case of a bosonic propagator. In the Wick rotation of a bosonic propagator the crucial point is that one evaluates the integrations over the zeroth momentum component with the residue theorem using the fact that $kS(k) \to 0$ for $k \to \infty$. For a fermionic propagator this limit is not satisfied, since we got an additional factor $\not{k} + m$. However when one does momentum integration over some particle loops, the loops either consist of a closed fermion loop, which means the additional factor $\not{k}$ vanishes due to the trace that occurs for closed fermion loops and the Wick rotation reduces to the case of Wick rotation of a bosonic propagator or the loops consist of two propagators which leads to an additional factor k^{-1} (for a second fermionic propagator) or k^{-2} (for an additional bosonic propagator) and the limit $kS(k) \to 0$ for $k \to \infty$ is satisfied again, which makes use of the residue theorem possible. Since Wick rotation for bosonic propagators can be looked up in any quantum field theory book and in order to motivate the redefinitions of the gamma matrices, we describe the second possibility and do the Wick rotation for a product of a fermionic and a bosonic propagator. The case of two fermionic propagators can be calculated analogously. For that one has to consider the following integral:

$$\int \frac{d^4k}{(2\pi)^4} \frac{\not{k} + m_1}{k^2 - m_1^2 + i\epsilon} \frac{1}{(p-k)^2 - m_2^2 + i\epsilon} \tag{2.1}$$

with singularities at $k_0 = \pm(\sqrt{\vec{k}^2 + m_1^2} - i\tilde{\epsilon})$, $k_0 = p_0 \pm (\sqrt{(\vec{p} - \vec{k})^2 + m_2^2} - i\tilde{\epsilon})$. Using the residue theorem, the time integration over the path shown in Fig. 2 equals zero, since the singularities are excluded from the interior of the path. Pushing the size of the integration path to infinity, the integration over the circular arcs vanishes, since the integrant decreases rapidly for complex numbers with their absolute value going to infinity. So we are left with

$$\int_{-\infty}^{\infty} dk_0 \cdot \frac{\not{k} + m_1}{k^2 - m_1^2 + i\epsilon} \frac{1}{(p-k)^2 - m_2^2 + i\epsilon} + \int_{i\infty}^{-i\infty} dk_0 \cdot \frac{\not{k} + m_1}{k^2 - m_1^2 + \epsilon} \frac{1}{(p-k)^2 - m_2^2 + i\epsilon} = 0 \tag{2.2}$$

or equivalently

$$\int_{-\infty}^{\infty} dk_0 \cdot \frac{\not{k} + m_1}{k^2 - m_1^2 + i\epsilon} \frac{1}{(p-k)^2 - m_2^2 + i\epsilon} = \int_{-i\infty}^{i\infty} dk_0 \cdot \frac{\not{k} + m_1}{k^2 - m_1^2 + i\epsilon} \frac{1}{(p-k)^2 - m_2^2 + i\epsilon}. \tag{2.3}$$

We then can do the substitution $k_4 := ik_0$ and get

$$i \int_{-\infty}^{\infty} dk_4 \cdot \frac{-i\gamma^0 k_4 - \vec{\gamma} \cdot \vec{k} + m_1}{-k_4^2 - \vec{k}^2 - m_1^2 + i\epsilon} \frac{1}{-(p_4 - k_4)^2 - (\vec{p} - \vec{k})^2 - m_2^2 + i\epsilon}. \tag{2.4}$$

The first thing we notice is that we do not need the term $i\epsilon$ in the denominator anymore, since due to the positive mass the denominator is strictly negativ without the ϵ-term. We can just leave this term away and the two negativ signs from both denominators cancel. Then we can redefine the gamma matrices (as shown below) to get the following form

$$i\int_{-\infty}^{\infty} dk_4 \cdot \frac{-i\gamma^4 k_4 - i\vec{\gamma}\cdot\vec{k} + m_1}{k_4^2 + \vec{k}^2 + m_1^2} \frac{1}{(p_4 - k_4)^2 + (\vec{p} - \vec{k})^2 + m_2^2}. \tag{2.5}$$

For the whole propagator in Euclidean space we then get

$$i\int \frac{d^4k}{(2\pi)^4} \frac{-i\not{k} + m_1}{k^2 + m_1^2} \frac{1}{(p-k)^2 + m_2^2} \tag{2.6}$$

with Euclidean notation, i.e. Euclidean gamma matrices, $\not{k} = \sum_{i=1}^{4} \gamma^i k^i$ and $k^2 = k_4^2 + \vec{k}^2$. From comparing eq. (2.4) and (2.5) one immediately sees that the redefinitions of the Dirac matrices have to be the following:

$$\gamma^4 = \gamma^0_{Minkowski} \tag{2.7}$$

$$\gamma^k = -i\gamma^k_{Minkowski} \qquad k = 1,2,3 \tag{2.8}$$

One can extract from these thoughts some substitution rules, which will allow us to switch from field theory in Minkowski space to Euclidean field theory without doing always a full Wick rotation by hand. We will list here the basic rules for substitution (taken from [5]), which will also be used later in this work.

Position Space	Momentum Space
$\int^M d^4x^M \to -i\int^E d^4x^E$	$\int^M d^4k^M \to i\int^E d^4k^E$
$\not{\partial} \to i\gamma^E \cdot \partial^E$	$\not{k} \to -i\gamma^E \cdot k^E$
$\not{A} \to -i\gamma^E \cdot A^E$	$\not{A} \to -i\gamma^E \cdot A^E$
$A_\mu B^\mu \to -A^E \cdot B^E$	$k_\mu q^\mu \to -k^E \cdot q^E$
$x^\mu \partial_\mu \to x^E \cdot \partial^E$	$k_\mu x^\mu \to -k^E \cdot x^E$

One can see that with those substiution rules and with the redefinitions of the gamma matrices one can immediately go from eq. (2.1) to (2.6) without a detailed Wick rotation. There is also a more direct approach to avoid singularities that arise because of the fact that the Minkowski metric is not positiv definite, which is to formulate quantum field theory from the beginning as an Euclidean field theory. The axioms of Euclidean field theory were first constructed by Osterwalder and Schrader [6] such that they are equivalent to the axioms of relativistic quantum field theory in Minkowski space (Wightman axioms).

2.1 Axioms for Euclidean Green's Functions (Schwinger Functions)

A detailed introduction to the axioms for Euclidean Green's functions (Schwinger function) be found in [6]. The basic axioms for real scalar fields (taken from [7]) are listed here:

We can write the Schwinger functions as

$$S^{(n)}(x_1,...,x_n) = w^{(n)}(-ix_1^4, \vec{x_1}, ..., -ix_n^4, \vec{x_n}), \tag{2.9}$$

where

$$w^{(n)}(x_1,...,x_n) = \langle \Omega \mid \phi(x_1)...\phi(x_n) \mid \Omega \rangle \tag{2.10}$$

are the Wightman distributions in Minkowski space, $\mid \Omega\rangle$ denotes the ground state of the intercating theory. The basic axioms are:

1. Distributions:
   The Schwinger functions $S^{(n)}$ are analytical functions on the space $\mathscr{S}(\mathbb{R}^{4n}\backslash\Delta)$, the Schwartz space of smooth functions on $\mathbb{R}^{4n}$ decreasing fast for large x and vanishing with all its derivatives on Δ, where
   $\Delta = \{x_1,...,x_n | x_j = x_k \text{ for some } j \neq k\}$.
   The Schwinger functions may be regarded as distributions with $\mathscr{S}(\mathbb{R}^{4n}\backslash\Delta)$ as the space of test functions.

2. Euclidean Invarianz of $S^{(n)}$:
$$S^{(n)}(gx_1, ..., gx_n) = S^{(n)}(x_1, ..., x_n),$$
where $gx = Rx + a$ with $R \in SO(4)$ and $a \in \mathbb{R}^4$.

3. Permutation Symmetry:
$$S^{(n)}(x_{\pi(1)}, ..., x_{\pi(n)}) = S^{(n)}(x_1, ..., x_n)$$
for any permutation (element of the symmetric group) π.

Because of the 3rd axiom an Schwinger function is already given if it is known for Euclidean time ordered configurations of the arguments and because of the 2nd axiom it is even sufficient if it is known for arguments with positive time components, i.e. a Schwinger function is already given if it is known on the subset

$$M = \{(x_1, ..., x_n)|x_1^4 > ... > x_n^4 > 0\}.$$

$\mathscr{S}_+(\mathbb{R}^{4n})$ denotes Schwartz space of test functions with support in M.
For $f_n \in \mathscr{S}_+(\mathbb{R}^{4n})$ we can define

$$\Theta f_n(x_1, ..., x_n) := \bar{f}_n(\vartheta x_1, ..., \vartheta x_n),$$

where the bar on the function f denotes the complex conjugate and $\vartheta x = (-x^4, \vec{x})$ gives the Euclidean time reflection. Further we define the direct product

$$\mathscr{T} = \oplus_{n=0}^{\infty} \mathscr{S}_+(\mathbb{R}^{4n})$$

and

$$(f_n \otimes g_m)(x_1, ..., x_n, y_1, ..., y_m) = f_n(x_1, ..., x_n) g_m(y_1, ..., y_m).$$

The 4th axiom can then be formulated as:

4. Reflection positivity:
$$\sum_{n,m} S^{(n+m)}(\Theta f_n \otimes f_m) \geq 0$$
for all $f \in \mathscr{T}$ with $f_n \in \mathscr{S}_+(\mathbb{R}^{4n})$.

The first two axioms are immediately clear. They describe the fact that field operators in quantum field theory are operator-valued distributions and they are invariant under the Poincare group in Minkowski space, which leads to Euclidean invariance in Euclidean space. Permutation symmetry implies the fact that bosonic field operators commute with each other and reflection positivity is needed to obtain a positive definite metric for a probabilistic interpretation. This will be explained in more detail in the next section. For a fermionic Dirac field slight changes for the axioms have to be done. In principle one has to add indices to the field operators in the Schwinger function, which label the component of the Dirac field one considers. So a Schwinger function of n Dirac field operators will carry n indices (each running from 1 to 4). Of course symmetry under permutation has to be changed in anti-symmetry. For a permutation π one gets an additional factor $sgn(\pi)$ in the 3rd axiom, since fermionic field operators anti-commute.

The important axiom for this work is reflection positivity. This axiom is necessary to obtain a postive definite metric. To explain it, we consider the norm of an arbitary one-particle state $\phi(x) \mid \Omega\rangle$ in Minkowski space, which has to be positive definite :

$$\langle\Omega \mid \phi(x)^{\dagger}\phi(x) \mid \Omega\rangle \geq 0 \tag{2.11}$$

In a more mathematical formulation one has to consider an operator $\phi(f_1)$ with a testfunction $f_1 \in \mathscr{S}_+(\mathbb{R}^4)$, since ϕ is an operator-valued distribution. Then one gets

$$w^{(2)}(\bar{f}_1 \otimes f_1) = \langle\Omega \mid \phi(\bar{f}_1)\phi(f_1) \mid \Omega\rangle = \langle\Omega \mid \phi(f_1)^{\dagger}\phi(f_1) \mid \Omega\rangle \geq 0. \tag{2.12}$$

For Euclidean field theory the appropriate condition is

$$S^{(2)}(\Theta f_1 \otimes f_1) \geq 0. \tag{2.13}$$

If one considers multi-particle states the generalization of this condition yields the 4th axiom.

Now we will derive, for the case of a scalar field, from the axiom of reflection positivity a concrete formula, which will allow us to check for a given Schwinger function if the axiom is violated or not. If we choose $f \in \mathscr{T}$ with $f_1 \in \mathscr{S}_+(\mathbb{R}^4)$ and $f_n \equiv 0 \in \mathscr{S}_+(\mathbb{R}^{4n})$ for $n \neq 1$ then we get

$$\int d^4x d^4y \bar{f}_1(-x^4, \vec{x}) S^{(2)}(x-y) f_1(y^4, \vec{y}) \geq 0, \tag{2.14}$$

where the f_i's are testfunctions with support in $\{x \in \mathbb{R}^4 : x^4 > 0\}$. As argued in [8] we can then apply a 3-dimensional Fourier transformation and get

$$\int_0^{\infty} dt \int_0^{\infty} dt' \int d^3p \cdot \bar{\hat{f}}_1(t, \vec{p}) \hat{S}^{(2)}(-(t+t'), \vec{p}) \hat{f}_1(t', \vec{p}) \geq 0. \tag{2.15}$$

(See appendix A.1 for the calculation.) This condition yields

$$\forall \vec{p} \in \mathbb{R}^3, \forall t > 0 : \qquad \hat{S}^{(2)}(-t, \vec{p}) \geq 0. \tag{2.16}$$

To see this we could assume that condition (2.16) is violated for a given t and $\vec{p}$. Then one could choose f_1 such that $\hat{f}_1$ peaks at a $(t', \vec{p})$ and $(t'', \vec{p})$ with $t' + t'' = t$. This would immediately lead to a violation of condition (2.15), i.e. violation of reflection positivity. Since the two-point Schwinger function is symmetric in its arguments condition

$$\forall \vec{p} \in \mathbb{R}^3, \forall t \in \mathbb{R} : \qquad \hat{S}^{(2)}(t, \vec{p}) \geq 0 \tag{2.17}$$

holds. For a fermionic Dirac field, where one can decompose the two-point function into

$$S^{-1}(p) = i\not{p}A(p^2) + B(p^2), \tag{2.18}$$

$$S(p) = -i\not{p}\sigma_V(p^2) + \sigma_S(p^2), \tag{2.19}$$

with the scalar functions $\sigma_V = \frac{A}{p^2A^2+B^2}$ and $\sigma_S = \frac{B}{p^2A^2+B^2}$, one gets analogous conditions for these two scalar functions. For the special case of $\vec{p} = 0$ we then get the following condition:

$$\forall t \in \mathbb{R} : \qquad \Delta_{V,S}(t) := \frac{1}{2\pi} \int dq_4 e^{-iq_4 t} \sigma_{V,S}(q_4^2, \vec{0}^2) = \frac{1}{\pi} \int_0^{\infty} dq_4 cos(q_4 t) \sigma_{V,S}(q_4^2, \vec{0}^2) \geq 0 \tag{2.20}$$

Since one can write the Schwinger function above also as

$$\Delta_{V,S}(t) = \int d^3x \int d^4q e^{-i(tq_4+\vec{x}\vec{q})} \sigma_{V,S}(q^2) = \int d^3x \Delta(t,\vec{x}), \tag{2.21}$$

the Schwinger function for the special case of $\vec{p} = 0$ can be regarded as an spatially averaged Schwinger function.

To get a better understandig of condition (2.20) we now discuss its connection to the Källen-Lehmann spectral representation of an n-point function in an interacting theory. In a theory with asymptotic states where the fermions correspond to stable one-particle states with the physical mass m_{phys}, one has in Minkowski space the following Källen-Lehmann representation for the two-point function:

$$S_F(p) = iZ_2 \frac{\not{p} + m_{phys}}{p^2 - m_{phys}^2} + i \int_{\tilde{m}}^{\infty} ds \frac{\not{p}\rho_1(s) + \rho_2(s)}{p^2 - s + i\epsilon} \tag{2.22}$$

with two real, nonnegative functions ρ_i and the treshold $\tilde{m} \sim 2m_{phys}$ for multiparticle states. As we can see from this the two-point function of a stable particle has in the p^2-plane a real singularity at $p^2 = -m_{phys}^2$ in Euclidean field theory. An abitary two-point function has not necessarily such a Källen-Lehmann representation, as shown in eq. (2.22) and it does not need to have a real singulartiy in the p^2-plane. A discussion what the full propgator then might look like in Eucdlidean field theory can be found in [9]. In general the propagator takes the form written in equations (2.18), (2.19) and for unstable particles one expects pairs of complex conjugate poles (with non-vanishing imaginary part) in the p^2-plane. In a theory of quarks complex conjugate poles in the quark propagators are interpreted as a sign for confinement, since the quarks do not correspond to stable one-particle states. In principle one could search in the p^2-plane for the poles of the two-point function, by solving the equation

$$\begin{aligned} p^2 + M^2(p^2) &= 0, \\ M(p^2) &= B(p^2)/A(p^2), \end{aligned} \tag{2.23}$$

to investigate confinement, but a numerically more efficient way is to study the functions $\Delta_{V,S}$. One can show that for a real pole in the p^2-plane $\Delta_{V,S}(t)$ decay exponentially

$$\Delta_{V,S}(t) \sim e^{-|m_{phys}t|}, \tag{2.24}$$

whereas for a complex conjugate pair of poles, $M = a \pm ib$, one gets an oscillating function

$$\Delta_{V,S}(t) \sim e^{-at} cos(bt + \delta). \tag{2.25}$$

From this one can see that confinement due to complex conjugate poles in the p^2-plane leads to violation of the condition (2.20).

3 The Nambu-Jona-Lasinio model

3.1 Introduction

In 1961 the NJL-model was constructed by Nambu and Jona-Lasinio, as a theory of the interaction betweens nucleons, to describe the dynamical mass generation of the nucleons in analogy with the energy gap in the theory of superconductivity ([10], [11]). Today we know that nucleons are made of quarks and the accepted theory describing the interaction between them via exchange of gluons is quantum chromodynamics (QCD). One feature of QCD is asymptotic freedom, which means that the running coupling constant of QCD is small at short distances or equivalently at large momenta. For cases of weak coupling calculations can be done with pertubation theory. However at large distances or at small momenta pertubation theory breaks down, since the quark-gluon coupling is too strong. Under these circumstances QCD ist not well understood and due to the breakdown of pertubation theory new techniques had to be developed to do calculations if the coupling is strong. One approach to this problem is lattice gauge theory. However lattice gauge theory has its own problems, e.g. high computer powers are needed, which make lattice calculations, in the case of strong coupling, very complicated again. Another method to do calculations is to consider models that are simpler than QCD but share several important features with it. One might investigate the behavier of QCD in special cases by investigation of the more simple model. Taking this path the NJL-model has been re-interpreted as a model of interaction between quarks. An introduction into the NJL-model of quantum chromodynamics can be found in [12]. In this work we consider quarks of two flavors (up and down). A simple form of the Lagrangian density for this model can be written as

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + G((\bar{\psi}\psi)^2 + \sum_{a=1}^{3}(\bar{\psi}\{i\gamma_5 \otimes \tau_a \otimes \mathbb{1}\}\psi)^2), \tag{3.1}$$

where ψ describes the quark field in $Diracspace \otimes Isospinspace \otimes Colorspace$, m_0 the bare mass of the up and down quarks (the masses of up and down quark are set to be equal since their small mass difference is negligable) and τ_a ($a = 1, 2, 3$) the 3 Pauli matrices in isospin space. Although this model has several shortcomings, e.g. it is non-renormizable (which makes regularization schemes necessary), it is not a gauge theory (hence it does not including gluons) and it does not realize confinement, it is still an important model, since it shares some essantial features with QCD. Symmetries of QCD, for example chiral symmerty and the breaking of it, are also part of the NJL-model.

3.2 Symmetries of the NJL model

According to Noether's theorem each continous symmetry (transformation of the fields under which the Lagrange equations are invariant) of the Lagrangian density $\mathscr{L}$ is associated with a conserved current (a current j_μ that satisfies $\partial^\mu j_\mu = 0$) and a conserved charge $Q = \int_{\mathbb{R}^3} j_0 d^3x$ that is constant in time. In this section we discuss symmetries of the NJL model and their conserved currents.

3.2.1 Conservation of baryon number

The Lagrange equations are invariant under the $U_V(1)$ transformation

$$\psi \rightarrow exp(-i\alpha)\psi, \qquad \alpha \in \mathbb{R}, \tag{3.2}$$

which leads to the conserved current

$$j_\mu = \bar{\psi}\gamma_\mu\psi \tag{3.3}$$

and conservation of baryon number as the associated charge. This symmetry is an exact one of nature.

3.2.2 Isospin symmetry

Since we neglected the mass difference of up and down quarks and set their masses equal, the Lagrange equations are invariant under rotation in isospin space. The $SU_V(2)$ symmetry transformation

$$\psi \rightarrow exp\left(-i\frac{\vec{\tau}\cdot\vec{\theta}}{2}\right)\psi, \qquad \vec{\theta}\in\mathbb{R}^3 \tag{3.4}$$

leads to the conserved current

$$J^k_\mu = \bar{\psi}\gamma_\mu\tau^k\psi, \qquad k=1,2,3. \tag{3.5}$$

In this notation the τ_i are the generators of rotation and Θ_i are the rotation parameters. Since the real masses of up and down quarks are not exactly equal, isospin symmetry is only approximately realized in nature.

3.2.3 $SU_A(2)$ symmetry

For $m_0 = 0$ the $SU_A(2)$ transformation

$$\psi \rightarrow exp\left(-i\frac{\vec{\tau}\vec{\theta}}{2}\gamma_5\right)\psi, \qquad \vec{\theta}\in\mathbb{R}^3 \tag{3.6}$$

is a symmetry and the conserved current is

$$J^k_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\tau^k\psi. \tag{3.7}$$

Since $m_0 \neq 0$ the symmetry is explicitly broken by the part $m_0\bar{\psi}\psi$ in the Lagrangian density $\mathscr{L}$. The symmetry is also broken spontaneously in the vacuum, as we will show later. According to the Goldstone theorem the spontaneous breaking of a symmetry leads to the appearance of massless bosons, which are given by the massless pions in the chiral limit.

3.2.4 Chiral symmetry

Chiral symmetries are symmetries under which the left- and right-handed parts of the field transform independently. For a Dirac field one defines the left- and right- handed operator by the projection operators $\frac{1\pm\gamma_5}{2}$ as:

$$\psi_L = \frac{1-\gamma_5}{2}\psi \tag{3.8}$$

$$\psi_R = \frac{1+\gamma_5}{2}\psi \tag{3.9}$$

In the case $m_u = m_d = m_0 = 0$ both transformations $SU_V(2)$ and $SU_A(2)$ are symmetries and $SU_V(2)\otimes SU_A(2)$ is a chiral symmetry, since it can be written as $SU_V(2)\otimes SU_A(2) \cong SU_L(2)\otimes SU_R(2)$ with $SU_L(2)$ and $SU_R(2)$ acting seperately on ψ_L and ψ_R.

3.3 Regularization

The divergent integrals that appear in the NJL-model have to be regulized in order to get quantitative results. There are several different regularization schemes and in general, since the NJL-model is non-renormizable, the results one gets depend on the scheme one chooses. To mention some covariant regularization shemes, there are the Pauli-Villars method, 4-momentum cut-off in Euclidean space and regularization in proper time. In this work we will use the non-covariant method of 3-momentum cut-off.

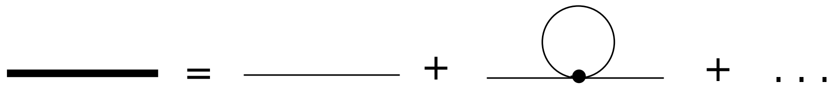

Fig. 3.4a: The full propagator (left hand side) as an infinite sum of Feynman diagrams (right hand side).(The lines on the right side denote the bare quark propagator)

Fig. 3.4 b: Hartree approximation (The continous lines denote the full propagator and the dashed lines the bare propagator)

Because in quantum field theory particles always emit and re-absorb virtual particles, there is an additional mass, which the virtual particles contribute to the bare mass. As a result the effective mass one can measure differs from the bare mass m_0 in the Lagrangian density. In this section we calculate the effective mass m_H in Hartree approximation, only taking closed quark loops as self-interaction terms into account.

The full quark propagtor (or two-point function)

$$iS(p) = \int \frac{d^4x}{(2\pi)^4} e^{ipx} \langle \Omega \mid T[\psi(x)\bar{\psi}(0)] \mid \Omega \rangle \tag{3.10}$$

is equal to the sum of all possibile interaction terms of the bare propagator

$$iS_0(p) = i\frac{\not{p} + m_0}{p^2 - m_0^2 + i\epsilon} = i(\not{p} - m_0 + i\epsilon)^{-1} \tag{3.11}$$

(Fig. 3.4a). If one sums up all one-particle irreducible self-interaction terms into $-i\Sigma$ then the full propagator iS can be written as

$$iS(p) = iS_0(p) + iS_0(p)(-i\Sigma(p))iS(p). \tag{3.12}$$

We then get

$$S = (S_0^{-1} - \Sigma)^{-1} = (\not{p} - m_0 - \Sigma + i\epsilon)^{-1}. \tag{3.13}$$

If we apply the Hartree or mean-field approximation (Fig. 3.4b), where one only takes self-interaction terms due to closed quark loops into account, we get from this

$$iS_H(p) = i\frac{\not{p} + m_H}{p^2 - m_H^2 + i\epsilon} = i(\not{p} - m_H + i\epsilon)^{-1} \tag{3.14}$$

with

$$m_H = m_0 + \Sigma_H, \tag{3.15}$$

where Σ_H is given by

$$-i\Sigma_H = -2iG\left\{\int \frac{d^4k}{(2\pi)^4} Tr[iS_H(k)] + \sum_{a=1}^{3}(i\gamma_5 \otimes \tau_a \otimes \mathbb{1})\int \frac{d^4k}{(2\pi)^4} Tr[(i\gamma_5 \otimes \tau_a \otimes \mathbb{1})iS_H(k)]\right\}. \tag{3.16}$$

Since the traces of the Pauli matrices in isospin space vanish, we are left only with the first term of the sum. The trace of an odd number of Dirac matrices is also zero, so the part of the propagtor S_H proportional to $\not p$ drops out too. For the remaining part, we get a factor 4 from Dirac space, a factor $N_f = 2$ from isospin space (we only consider up and down quarks) and a factor $N_c = 3$ from color space. So the remaining part can be written as

$$\Sigma_H = 8GN_f N_c m_H I_1(m_H) \tag{3.17}$$

with

$$I_1(M) = i\int \frac{d^4k}{(2\pi)^4} \frac{1}{k_0^2 - \vec{k}^2 - M^2 + i\epsilon}. \tag{3.18}$$

The integral $I_1(m_H)$ can be evaluated in spherical coordinates and we get

$$\begin{aligned} I_1(m_H) &= i\int \frac{d^4k}{(2\pi)^4} \frac{1}{k_0^2 - E_k^2 + i\epsilon} = i\int \frac{d^3k}{(2\pi)^4} \oint_{up} dk_0 \frac{1}{2E_k}\left(\frac{1}{k_0 - (E_k - i\epsilon)} - \frac{1}{k_0 + (E_k - i\epsilon)}\right) \\ &= i\int \frac{d^3k}{(2\pi)^4} \frac{-2\pi i}{2E_k} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} = 4\pi \int_0^\infty \frac{dk}{(2\pi)^3} \frac{k^2}{2E_k} = \int_0^\infty \frac{dk}{(2\pi)^2} \frac{k^2}{E_k}, \end{aligned} \tag{3.19}$$

where $k = |\vec{k}|$ and $E_k^2 = \vec{k}^2 + m_H^2$.
Since this integral diverges, we apply the 3-momentum cut-off Λ mentioned in section 3.5 and get

$$I_1(m_H) = \int_0^\Lambda \frac{dk}{(2\pi)^2} \frac{k^2}{E_k} = \frac{1}{(2\pi)^2} \frac{1}{2}\left(\Lambda\sqrt{\Lambda^2 + m_H^2} - m_H^2 arcsinh[\frac{\Lambda}{m_H}]\right). \tag{3.20}$$

We get for the gap equation 3.15

$$m_H = m_0 + 8N_f N_c G m_H I_1(m_H), \tag{3.21}$$

which we can now solve iteratively, starting with an arbitrary m_H on the right hand side.

3.5 Choice of Paramters and Numerical Result for the Hartree Mass

In the NJL-model we got 3 yet undetermined parameters: the bare quark mass m_0, the coupling constant G, and the cut-off parameter Λ. As done in [13] one usually fixes the parameters such that they fit the pion mass $m_\pi = 135$ MeV, the pion decay constant $f_\pi = (92.4 \pm 0.2)$ MeV and the quark condensate $\langle \bar{u}u \rangle$. The value of the quark condensate is not known that accurately. For this reason in [13] there are given different sets of parameters. For this work we will use:

$$\Lambda = 587.9 \text{ MeV} \tag{3.22}$$

$$m_0 = 5.6 \text{ MeV} \tag{3.23}$$

$$G = 2.44/\Lambda^2 \tag{3.24}$$

With this parameters we calculate the Hartree mass iteratively from eq. (3.15) and get

$$m_H = 399.44 \text{ MeV}. \tag{3.25}$$

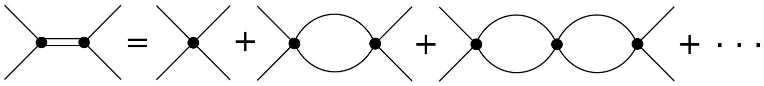

Fig. 3.6a: Infinite Bethe-Salpeter equation for quark-antiquark scattering

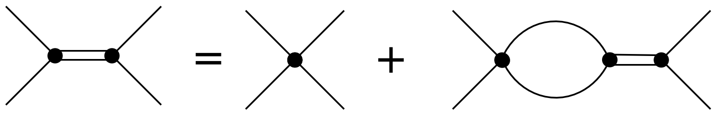

Fig. 3.6b: Summarized Bethe-Salpeter equation for quark-antiquark scattering

The basic NJL-model described by the Lagrangian density (3.1) only includes quark fields as its elementary degrees of freedom. In order to describe mesons one could regard mesons as point like and add meson field to the Lagrangian, as we will do in the next section. In the basic definition of the NJL-model one usually describes mesons as resonances in the quark-antiquark scattering, i.e. mesons are described by a sum of any possible number of quark-antiquark loops (Fig. 3.6a). We can write this infinite sum as shown in Fig. 3.6b, using the fact that the sum of all possible numbers of quark-antiquark loops on the right hand side in Fig 3.6a describes the meson propagator. From this we get for the quark-antiquark scattering matrix T the following equation:

$$iT(q^2) = iK + iK(-i\Pi(q^2))iT(q^2) \tag{3.26}$$

with the scatterin kernel

$$K = 2G\sum_M(\Gamma_M \otimes \Gamma_M) \tag{3.27}$$

and the polarisation functions

$$J_{MN} = \Gamma_M \Pi(q^2)\Gamma_N = i\int \frac{d^4k}{(2\pi)^4} Tr[\Gamma_M S(k_0+q_0, \vec{k}+\vec{q}/2)\Gamma_N S(k_0, \vec{k}-\vec{q}/2)]. \tag{3.28}$$

Since the scalar and pseudoscalar channels do not mix ($J_{sp} = 0$), one can use the ansatz

$$T_M(q^2) = -\Gamma_M D_M(q^2)\Gamma_M \tag{3.29}$$

and gets

$$D_M(q^2) = \frac{-2G}{1-2GJ_M(q^2)} \tag{3.30}$$

with

$$J_s(q^2) = 4N_cN_fI_1(m_H) - (q^2-4m^2)2N_cN_fI_2(q;m_H), \tag{3.31}$$
$$J_p(q^2) = 4N_cN_fI_1(m_H) - q^2 2N_cN_fI_2(q;m_H), \tag{3.32}$$

where

$$I_2(q;m_H) = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m_H^2+i\epsilon)((k+q)^2-m_H^2+i\epsilon)}. \tag{3.33}$$

One can interprete D_M as an effective pion exchange between quarks to determine the properties of the described meson. Near its pole D_M behaves like a pion propagator, i.e.

$$D_M(q^2) = \frac{-2G}{1 - 2GJ_M(q^2)} \approx \frac{g^2_{Mqq}}{q^2 - m^2_{Mes} + i\epsilon}, \tag{3.34}$$

where $g_{\pi qq}$ is the quark-pion coupling constant. Since in general the mass of a particle is given by the singularity of its propagator, one can calculate the mass m_{Mes} of the meson by calculating the singularity of D_M, which leads to the equation

$$D^{-1}_M(q^2)|_{q^2=m^2_{Mes}} = 0 \Longleftrightarrow 1 - 2GJ_M(q^2)|_{q=m_{Mes}} = 0. \tag{3.35}$$

To determine the coupling constant $g_{\pi qq}$ one can and do a 1st order Taylor expansion (in q^2) of the denominator of D_M around the meson mass m_{Mes} and plug the expansion into eq. (3.34). For D_M with a Taylor expanded denominator we get, by using eq. (3.35) the following equation:

$$D_M(q^2) = \frac{-2G}{1 - 2GJ_M(q^2)|_{q=m_{Mes}} - 2G\frac{dJ_M}{d(q^2)}|_{q^2=m^2_{Mes}}(q^2 - m^2_{Mes})} = \frac{1}{\frac{dJ_M}{d(q^2)}|_{q^2=m^2_{Mes}}(q^2 - m^2_{Mes})} \tag{3.36}$$

With eq. (3.34) one then gets

$$g^2_{Mqq} = \frac{1}{\frac{dJ_M}{d(q^2)}|_{q^2=m^2_{Mes}}}. \tag{3.37}$$

With this one can show that in the NJL model the Goldberger-Treiman relation

$$f_\pi g_{\pi qq} = m_H \tag{3.38}$$

is satisfied. From this relation and the pion decay constant $f_\pi = (92.4 \pm 0.2)$ MeV we determine the coupling constant $g_{\pi qq}$, which we will need in the next section.

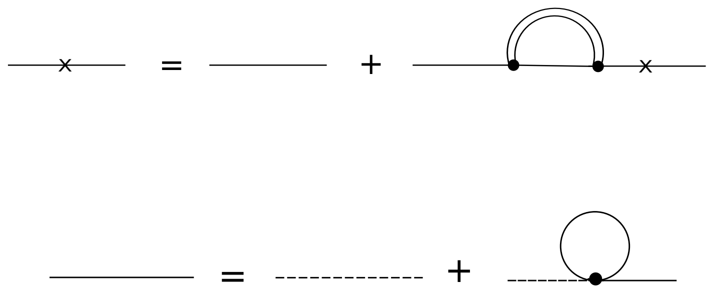

Fig. 3.7: Approximation of Dyson-Equation for the quark propagator with pion loops

In this section we also take pion-loops, as self-interaction terms in the quark propagator, into account. As discussed in the previous section, in the NJL model one usually describes pions by quark-antiquark scattering. However this makes calculations of the Schwinger function complicated if one wants to apply a 3-momentum cut-off. However we mentioned that this situation can be approximated by writing down an pion propagator with its constant for the coupling to the quark field determined by the Goldberger-Treiman relation. For this purpose we add to the Lagrangian of the NJL-model a pion field and minimal local interaction term with the coupling constant $g_{\pi qq}$, which we calculated in the last section.

$$\mathscr{L}_{\pi qq} = -g_{\pi qq}\bar{\psi}(x)(i\gamma_5 \otimes \vec{\tau}\cdot\vec{\pi}\otimes \mathbb{1})\psi(x) \tag{3.39}$$

The quark propagtor with pion-loops is approximated by the Dyson-Equations Fig. 3.7, which can be written as

$$\begin{aligned} iS_{NLO} &= iS_H + iS_H(-i\Sigma_{NLO})iS_{NLO}, &(3.40)\\ iS_H &= iS_0 + iS_0(-i\Sigma_H)iS_H. &(3.41)\end{aligned}$$

From these two equations we then get

$$\begin{aligned} S_{NLO} &= (S_H^{-1} - \Sigma_{NLO})^{-1} = (S_0^{-1} - \Sigma_H - \Sigma_{NLO})^{-1} &(3.42)\\ &= (\not{p} - m_0 - \Sigma_H - \Sigma_{NLO})^{-1} = (\not{p} - m_H - \Sigma_{NLO})^{-1}. &(3.43)\end{aligned}$$

From our modified Lagrangian with the additional part (3.39) we get for each quark-pion-vertex a factor

$$\Gamma_{\pi qq} = -ig_{\pi qq} i\gamma_5 \otimes \vec{\tau}\otimes \mathbb{1}. \tag{3.44}$$

With this we can calculate the selfenergy term $-i\Sigma_{NLO}$:

$$\begin{aligned} -i\Sigma_{NLO}(p^2) &= \int \frac{d^4k}{(2\pi)^4}\Gamma_{\pi qq} i\frac{1}{(k-p)^2 - m_\pi^2 + i\epsilon} i\frac{m_H + \not{k}}{k^2 - m_H^2 + i\epsilon}\Gamma_{\pi qq} &(3.45)\\ &= 3g_{\pi qq}^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k-p)^2 - m_\pi^2 + i\epsilon}\frac{m_H - \not{k}}{k^2 - m_H^2 + i\epsilon} &(3.46)\end{aligned}$$

The factor 3 comes from the term $\vec{\tau}^2$ and we used the relation $\gamma_5\gamma_\mu\gamma_5 = -\gamma_\mu$ to get the negative sign for k in the denominator of the quark propagator.

$$\Sigma_{NLO} = -3ig_{\pi qq}^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k-p)^2 - m_\pi^2 + i\epsilon}\frac{m_H - \not{k}}{k^2 - m_H^2 + i\epsilon} \tag{3.47}$$

4 Schwinger functions of the Quark Propagators in the NJL-model

In this secton we calculate the Schwinger functions for the quark propagator in the NJL-model and investigate, if reflection positivity is violated or not. First we do this in mean-field approximation and in the next step pion loops are taken into account, using the approximation shown in Fig.3.7 and the ansatz (3.39) to add a pion field to the NJL model. In order to do this, we first switch (by the substitution rules given in sec. 2) calculations to Euclidean quantum field theory, where the quark propagators can be decomposed in

$$S^{-1}(p) = i\not{p}A(p^2) + B(p^2), \tag{4.1}$$
$$S(p) = -i\not{p}\sigma_V(p^2) + \sigma_S(p^2), \tag{4.2}$$

with $\sigma_V = \frac{A}{p^2A^2+B^2}$ and $\sigma_S = \frac{B}{p^2A^2+B^2}$. From this we first calculate the functions A and B by

$$A = \frac{1}{4ip^2}Tr_D[\not{p}S^{-1}] = 1 + \frac{1}{4ip^2}Tr_D[\not{p}\Sigma_{NLO}], \tag{4.3}$$
$$B = \frac{1}{4}Tr_D[S^{-1}] = m_H + \frac{1}{4}Tr_D[\Sigma_{NLO}], \tag{4.4}$$

where we used that the trace over the product of an odd number of Dirac matrices vanishes and $Tr_D[\not{p}^2] = 4p^2$, since the square of each Dirac matric equals one. As discussed before (eq. 2.20), reflection positivity yields to the Schwinger functions that the condition

$$\Delta_{V,S}(t) = \frac{1}{\pi}\int_0^\infty dp_4 cos(p_4 t)\sigma_{V,S}(p_4,\vec{0}) \geq 0 \tag{4.5}$$

has to be satisfied. The quark propagotor approximated as shown in Fig.3.7 becomes in Euclidean field theory

$$S^{-1}_{NLO}(p) = i\not{p} + m_0 + \Sigma_H + \Sigma_{NLO}(p^2) = i\not{p} + m_H + \Sigma_{NLO}(p^2) \tag{4.6}$$

4.1 Schwinger function in mean field approximation

In mean field approximation we immediately get from eq. (4.6) for the quark propagator in Euclidean space

$$S^{-1} = i\not{p} + m_H \tag{4.7}$$
$$\tag{4.8}$$

and with eq. (4.3) and (4.4)

$$A = 1, \tag{4.9}$$
$$B = m_H. \tag{4.10}$$

From this we obtain

$$\sigma_V(p^2) = \frac{1}{p^2+m_H^2}, \tag{4.11}$$
$$\sigma_S(p^2) = \frac{m_H}{p^2+m_H^2}. \tag{4.12}$$

We can now calculate the Schwinger functions. For positive masses and positive times we get

$$\begin{aligned}
\Delta_V(t) &= \frac{1}{2\pi}\int dp_4 cos(p_4 t)\frac{1}{p_4^2+m_H^2} \\
&= \frac{1}{2}\frac{1}{2\pi}\int dp_4(e^{ip_4t}+e^{-ip_4t})\frac{1}{p_4^2+m_H^2} \\
&= \frac{1}{2}\frac{1}{2\pi}\oint_{up} dp_4 e^{ip_4t}\frac{1}{2im_H}\{\frac{1}{p_4-im_H}-\frac{1}{p_4+im_H}\} \\
&\quad + \frac{1}{2}\frac{1}{2\pi}\oint_{down} dp_4 e^{-ip_4t}\frac{1}{2im_H}\{\frac{1}{p_4-im_H}-\frac{1}{p_4+im_H}\} \\
&= \frac{1}{4im_H}\frac{1}{2\pi}(2\pi i e^{ip_4t}|_{p_4=im_H}+2\pi i e^{-ip_4t}|_{p_4=-im_H}) \\
&= \frac{1}{4im_H}\frac{1}{2\pi}2\pi i 2 e^{-m_H t} \\
&= \frac{1}{2m_H}e^{-m_H t}.
\end{aligned} \tag{4.13}$$

For each negative masses or negative times one would get an additional negative sign in the exponential function. Considering this and $\sigma_S = m_H\sigma_V$ one gets the analytical results

$$\Delta_V(t) = \frac{1}{2m_H}e^{-|m_H t|}, \tag{4.14}$$

$$\Delta_S(t) = \frac{1}{2}e^{-|m_H t|}. \tag{4.15}$$

As expected one gets an exponentially deacreasing Schwinger function, since the propagator has in the p^2-plane a real timelike pole at $p^2 = -m_H^2 < 0$ and describes a real quark of effective mass m_H.

4.2 Schwinger function of the quark propagator with pion loop

For calculating the Schwinger function we first take equation (3.47) to Euclidean field theory. We then get

$$\Sigma_{NLO}(p^2) = 3g_{\pi qq}^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k-p)^2+m_\pi^2}\frac{i\not{k}+m_H}{k^2+m_H^2}. \tag{4.16}$$

Since we want to regularize this integral by a 3-momentum cut-off, we first use the residue theorem to carry out the integral over p_4 and get

$$\begin{aligned}
\Sigma_{NLO} = -3g_{\pi qq}^2\int\frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}(&\{i\not{k}|_{k_4=iE_H}+m_H\}\frac{1}{iE_H-p_4-iE_\pi}+\{i\not{k}|_{k_4=p_4+iE_\pi}+m_H\}\frac{1}{iE_\pi+p_4-iE_H} \\
&-\{i\not{k}|_{k_4=p_4+iE_\pi}+m_H\}\frac{1}{iE_\pi+p_4+iE_H}-\{i\not{k}|_{k_4=iE_H}+m_H\}\frac{1}{iE_H-p_4+iE_\pi}),
\end{aligned} \tag{4.17}$$

with $E_H^2 = \vec{k}^2+m_H^2$ and $E_\pi^2(\vec{p}) = (\vec{k}-\vec{p})^2+m_\pi^2$. (The calculation can be found in the appendix A.2)

With this we now calculate the functions A and B. Since we later only need the case $\vec{p}=0$ for calculating the Schwinger functions, we already calculate $A(p_4,\vec{0})$ and $B(p_4,\vec{0})$. We then get

$$\frac{1}{4}Tr_D[\not{p}\Sigma_{NLO}]|_{\vec{p}=0} = \frac{1}{4}Tr_D[\gamma_4 p_4 \Sigma_{NLO}(p_4,\vec{0})] = 3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi}\frac{ip_4^2}{(E_H+E_\pi)^2+p_4^2}, \tag{4.18}$$

$$\frac{1}{4}Tr_D[\Sigma_{NLO}]|_{\vec{p}=0} = \frac{1}{4}Tr_D[\Sigma_{NLO}(p_4,\vec{0})] = 3m_H g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi E_H}\frac{(E_H+E_\pi)}{(E_H+E_\pi)^2+p_4^2}, \tag{4.19}$$

(4.20)

with $E_H^2=\vec{k}^2+m_H^2$ and $E_\pi^2(\vec{p})=\vec{k}^2+m_\pi^2$. (The full calcualtions can be found in the appendix A.3 and A.4.) With this we get

$$A(p_4,\vec{0}) = 1+3g_{\pi qq}^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi}\frac{1}{p_4^2+(E_H+E_\pi)^2}, \tag{4.21}$$

$$B(p_4,\vec{0}) = m_H+3m_H g_{\pi qq}^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi E_H}\frac{(E_H+E_\pi)}{p_4^2+(E_H+E_\pi)^2} \tag{4.22}$$

or in spherical coordinates

$$A(p_4,\vec{0}) = 1+3g_{\pi qq}^2\int_0^\infty\frac{dk}{(2\pi)^2}k^2\frac{1}{E_\pi}\frac{1}{p_4^2+(E_H+E_\pi)^2}, \tag{4.23}$$

$$B(p_4,\vec{0}) = m_H+3m_H g_{\pi qq}^2\int_0^\infty\frac{dk}{(2\pi)^2}k^2\frac{1}{E_\pi E_H}\frac{(E_H+E_\pi)}{p_4^2+(E_H+E_\pi)^2} \tag{4.24}$$

with $k=|\vec{k}|$ and $E_{H,\pi}=\sqrt{k^2+m_{H,\pi}^2}$.
Applying a 3-momentum cut-off Λ on the divergent functions A and B (by replacing $\infty\to\Lambda$), we can calculate the Schwinger functions $\Delta_{V,S}$ to check condition 4.5. We apply a cut-off at 600 MeV. This value has been taken from [14].

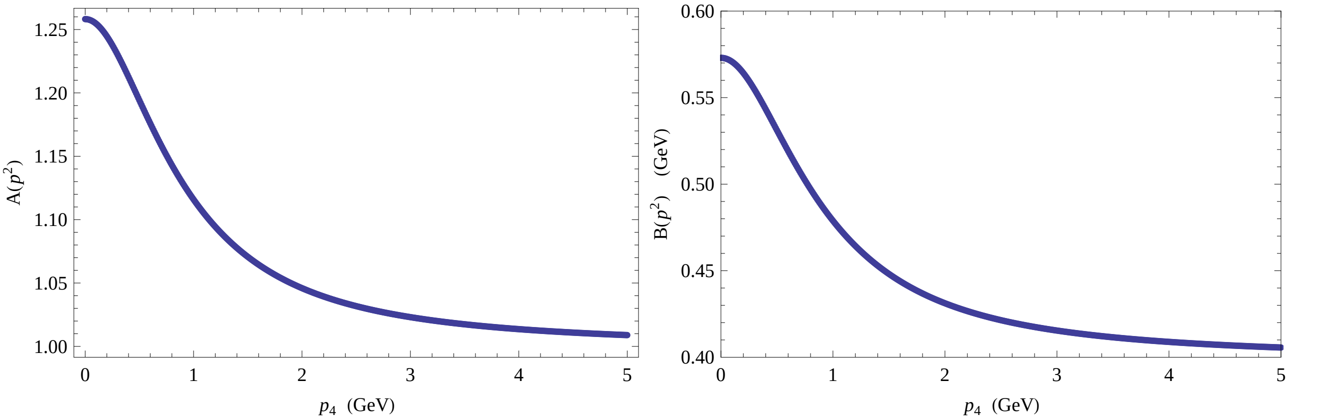

Fig.5.1: This Figure shows part $A(p^2)$ of the quark propagator with pion loop (left hand side) and part $B(p^2)$ (right hand side) vs. the the time-component of the external momentum of the quark.

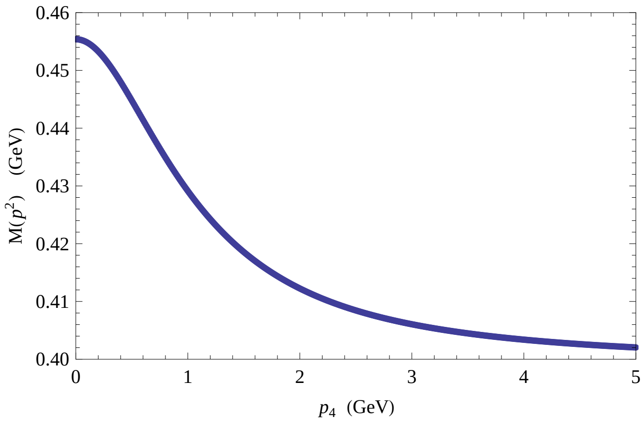

Fig. 5.2: The mass function $M(p^2) = B(p^2)/A(p^2)$ of the quark propagator with pion loop vs. the time-component of the external momentum of the quark.

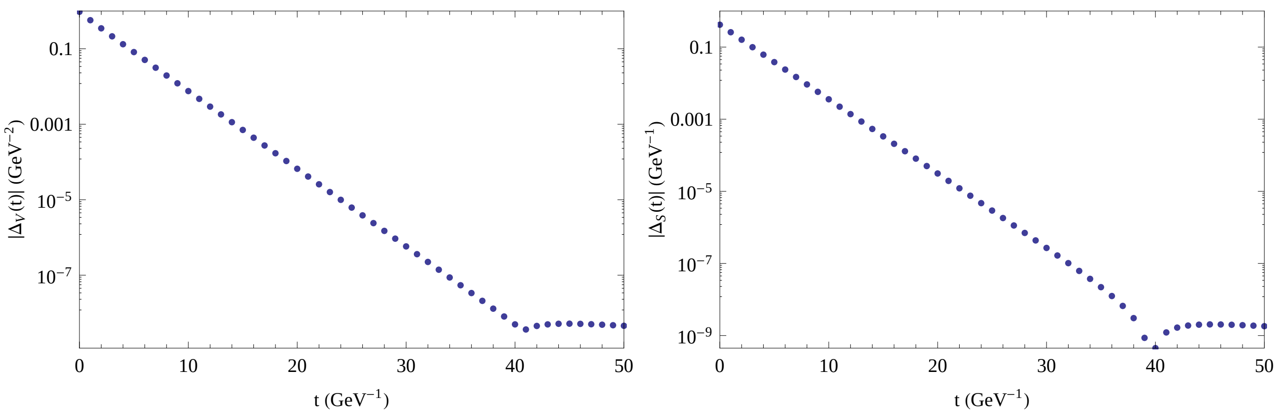

Fig 5.3: Logarithmic plot of the Schwinger functions Δ_V (left hand side) and Δ_S (right hand side).

In this section we present the numerical results for the Schwinger functions. As a first step we calculate the functions $A(p^2)$, $B(p^2)$, $M(p^2)$ and eventually the two Schwinger functions $\Delta_{V,S}(t)$. The results for the functions $A(p^2)$, $B(p^2)$ can be seen in Fig. 5.1. One can see that for $p_4 \to \infty$ both functions approach their Hartree values, which are $A_H = 1$ and $B_H = m_H$, as one would expect. For $p_4 = 0$ function A is increased by a factor 1.25 which means it received an additional value of 25% and B is increased by about 157 MeV which is slightly more than the pion mass itself, which is about 140 MeV.

The numerical result for the mass function $M(p^2)$ is displayed in Fig. 5.2. One can see in this figure that the mass of the quark with included quark and pion loops has to be at least about 456 MeV, probably even slighty higher. This can be concluded from the fact that $M(p_4)$ approaches about 456 MeV for $p_4^2 \to 0$. The mass is, as discussed, defined by the equation

$$-p^2 + m_{phys}(-p^2) = 0. \tag{5.1}$$

So for calculating the mass we would have to take p_4^2 to negative values and from the plot for $M(p^2)$ one can suggest a value that is slightly higher than 456 MeV for the mass.

The major result is the calculation of the Schwinger functions. We display the absolute value of the Schwinger functions $\Delta_{V,S}$ in a logharithmic plot, which is commonly done when one is working on confinement with Schwinger functions. A possible oscillatory, cosine like behavior of the Schwinger function would, in this kind of plot, result a series of spikes in a decreasing function, whereas an exponential deacresing function will be displayed as a straight line with negative slope. The logarithmic plots of the Schwinger functions can be seen in Fig. 5.3. One can see in Fig. 5.3 that the Schwinger functions behave until about 37 GeV^{-1} like straight lines in the logarithmic plot, which one would expect for a Schwinger function of a stable particle with a real pole in the propagator in the p^2-plane. If one goes to even higher energies, one can see a spike in each Schwinger function at about 40 GeV, but the Schwinger functions have clearly no oscillatory behavior. The spikes in the two Schwinger functions are due to numerical effects in the Fourier transformation. So for physical interpretation we only consider the part of the Schwinger functions until about 37 MeV. In this area the Schwinger functions decrease exponentially (linearly in the logharithmic plot), which means no violation of reflection positivity due to the pion loop. To calculate the mass of the quark with pion loops, we use eq. (2.25) and fit the data for the Schwinger functions by such an exponentially deacresing function. For the mass we get from the Schwinger function Δ_V

$$m_{phys,V} = (498.85 \pm 1.35)\ \text{MeV} \tag{5.2}$$

and from Δ_S

$$m_{phys,S} = (478.10 \pm 0.13)\ \text{MeV}. \tag{5.3}$$

These two values agree with our expectation that the mass should be slighty higher than 456 MeV, which we concluded from Fig. 5.2

6 Conclusion and Outlook

The major task of this work was to investigate if the scenario that Gribov supposed as an explanation of confinement is realized. This scenario suggests that free quarks might be excluded from the physical subspace of the Hilbert space due to the fact that their Schwinger functions violate reflection positivity, if one takes pions, the Goldstone bosons of chiral symmetry into account. For this purpose we gave at the beginning of this work an introduction to Euclidean field theory, in particular to its axioms and explained the relationship between reflection positivity, positivity of the spatially averaged Schwinger function and a possible confinement mechanism. We also discussed the absence of a Källen-Lehmann representation for the two-point function of unstable particles and the possible occurance of complex conjugate poles instead in the context of positivity violation of the Schwinger function.

In the next chapter of this work we introduced the NJL model of quantum chromodynamics with some basic features and symmetries. In particular we discussed the chiral symmetry and its explicit breaking. From the gap equation, which we solved in Hartree approximation in order to calculate the effective mass, we could see that the chiral symmetry is also broken spontaneously, which is essential for Gribov's confinement scenario. In the next step we discussed the possibilities to describe mesons, in particular pion loops, in the NJL model, which elementary degrees of freedom are quarks, in order to take pion loops into account. Those pions, as the Goldstone bosons of chiral symmetry breaking, might lead to the confining mechanism described by Gribov. We described the usual way to describe mesons in the NJL model, which is based on quark-antiquark scattering. Since calculations of the Schwinger function become complicated if ones wants to apply a 3-momentum cut-off with pions described by quark-antiquark scattering we then discussed an approximation where pions are regarded as point-like elementary particles and introduced an pion field into the NJL Lagrangian density. With this describtion of pions, we calculated the quark propagator with self energies due to closed quark and pion loops.

Finally we calculated in the next chapter the 2-point Schwinger functions, first in Hartree approximation and after that we took pion loops into account. The numerical results were shown in the next chapter. We could show that Gribov's scenario does not occur. Taking pion loops in the quark propagator into account did not lead to an oscillating Schwinger function, which means the confinement mechanism supposed by Gribov is not realized in the NJL model. However we want to stress that this non-oscillating Schwinger function does not mean that free quarks are part of the physical subspace of the Hilbert space, since other confinement mechanism might exclude them from the physical subspace. Since the NJL model is expected to have no confinement one would not expect other confinement mechanisms to exist anyway, but for QCD the latter comment is important, since it also happens in QCD that quark propagators do not violate reflection positivity due to pion loops. But in contrast to the NJL model one expects that in QCD other confinement mechanism exist that exclude free quarks from the physical subspace.

A Appendix

A.1

We apply a 3 dimensional Fourier transformation on eq. (2.14) in order to obtain eq. (2.15).

$$
\begin{aligned}
&\int d^4x d^4y \bar{f}_1(-x_4,\vec{x}) S^{(2)}(x-y) f_1(y_4,\vec{y}) \sim \\
&\int d^4x \cdot d^4y \cdot d^3p \cdot d^3q \cdot d^3r \cdot e^{i\vec{p}\vec{x}} e^{-i\vec{q}(\vec{x}-\vec{y})} e^{-i\vec{r}\vec{y}} \hat{\bar{f}}_1(-x_4,\vec{p}) \hat{S}^{(2)}(x_4-y_4,\vec{q}) \hat{f}_1(y_4,\vec{r}) = \\
&\int d^4x \cdot d^4y \cdot d^3p \cdot d^3q \cdot d^3r \cdot e^{-i\vec{x}(\vec{q}-\vec{p})} e^{-i\vec{y}(\vec{r}-\vec{q})} \hat{\bar{f}}_1(-x_4,\vec{p}) \hat{S}^{(2)}(x_4-y_4,\vec{q}) \hat{f}_1(y_4,\vec{r}) \sim \\
&\int dx_4 \cdot dy_4 \cdot d^3p \cdot d^3q \cdot d^3r \cdot \delta(\vec{q}-\vec{p}) \delta(\vec{r}-\vec{q}) \hat{\bar{f}}_1(-x_4,\vec{p}) \hat{S}^{(2)}(x_4-y_4,\vec{q}) \hat{f}_1(y_4,\vec{r}) = \\
&\int dx_4 \cdot dy_4 \cdot d^3p \cdot \hat{\bar{f}}_1(-x_4,\vec{p}) \hat{S}^{(2)}(x_4-y_4,\vec{p}) \hat{f}_1(y_4,\vec{p}) = \\
&\int_{-\infty}^{0} dx_4 \cdot \int_{0}^{\infty} dy_4 \cdot \int d^3p \cdot \hat{\bar{f}}_1(-x_4,\vec{p}) \hat{S}^{(2)}(x_4-y_4,\vec{p}) \hat{f}_1(y_4,\vec{p}) = \\
&\int_{0}^{\infty} dt \cdot \int_{0}^{\infty} dt' \cdot \int d^3p \cdot \hat{\bar{f}}_1(t,\vec{p}) \hat{S}^{(2)}(-(t+t'),\vec{p}) \hat{f}_1(t',\vec{p})
\end{aligned}
$$

The x_4 and y_4 integrations above reduce to integrations on the intervals $(-\infty,0)$ and $(0,\infty)$ because $f_1 \in \mathscr{S}_+(\mathbb{R}^4)$, i.e. $\hat{\bar{f}}_1(x_4,\vec{p}) = \hat{f}_1(x_4,\vec{p}) = 0$ for $x_4 \leq 0$.

A.2

Here we calculate eq. (4.17), the self-engergy term Σ_{NLO}, in detail.

$$
\begin{aligned}
\Sigma_{NLO} &= 3g_{\pi qq}^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k-p)^2+m_\pi^2} \frac{i\not{k}+m_H}{k^2+m_H^2} \\
&= -3g_{\pi qq}^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\not{k}+m_H}{2E_\pi 2E_H} \left(\frac{1}{k_4-p_4-iE_\pi} - \frac{1}{k_4-p_4+iE_\pi}\right)\left(\frac{1}{k_4-iE_H} - \frac{1}{k_4+iE_H}\right) \\
&= -3g_{\pi qq}^2 \oint_{up} \frac{d^4k}{(2\pi)^4} \frac{i\not{k}+m_H}{4E_\pi E_H} \left(\frac{1}{k_4-p_4-iE_\pi} - \frac{1}{k_4-p_4+iE_\pi}\right)\left(\frac{1}{k_4-iE_H} - \frac{1}{k_4+iE_H}\right) \\
&= -3g_{\pi qq}^2 \oint_{up} \frac{d^4k}{(2\pi)^4} \frac{i\not{k}+m_H}{4E_\pi E_H} \left(\frac{1}{k_4-p_4-iE_\pi}\frac{1}{k_4-iE_H} - \frac{1}{k_4-p_4-iE_\pi}\frac{1}{k_4+iE_H} - \frac{1}{k_4-p_4+iE_\pi}\frac{1}{k_4-iE_H}\right) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^4} \frac{2\pi i}{4E_\pi E_H} \Big(\{i\not{k}|_{k_4=iE_H}+m_H\} \frac{1}{k_4-p_4-iE_\pi}|_{k_4=iE_H} + \{i\not{k}|_{k_4=p_4+iE_\pi}+m_H\} \frac{1}{k_4-iE_H}|_{k_4=p_4+iE_\pi} \\
&\quad - \{i\not{k}|_{k_4=p_4+iE_\pi}+m_H\} \frac{1}{k_4+iE_H}|_{k_4=p_4+iE_\pi} - \{i\not{k}|_{k_4=iE_H}+m_H\} \frac{1}{k_4-p_4+iE_\pi}|_{k_4=iE_H}\Big) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3} \frac{i}{4E_\pi E_H} \Big(\{i\not{k}|_{k_4=iE_H}+m_H\} \frac{1}{iE_H-p_4-iE_\pi} + \{i\not{k}|_{k_4=p_4+iE_\pi}+m_H\} \frac{1}{iE_\pi+p_4-iE_H} \\
&\quad - \{i\not{k}|_{k_4=p_4+iE_\pi}+m_H\} \frac{1}{iE_\pi+p_4+iE_H} - \{i\not{k}|_{k_4=iE_H}+m_H\} \frac{1}{iE_H-p_4+iE_\pi}\Big)
\end{aligned}
$$

A.3

Here we present the calulation of eq. (4.18).

$$
\begin{aligned}
\frac{1}{4}Tr_D[\gamma_4 p_4 \Sigma_{NLO}(p_4,\vec{0})] &= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}(\{-p_4E_H\}\frac{1}{iE_H-p_4-iE_\pi}+\{ip_4(p_4+iE_\pi)\}\frac{1}{iE_\pi+p_4-iE_H} \\
&\quad -\{ip_4(p_4+iE_\pi)\}\frac{1}{iE_\pi+p_4+iE_H}-\{-p_4E_H\}\frac{1}{iE_H-p_4+iE_\pi}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}(\{-p_4E_H\}\frac{1}{iE_H-p_4-iE_\pi}-\{ip_4(p_4+iE_\pi)\}\frac{1}{iE_H-p_4-iE_\pi} \\
&\quad -\{ip_4(p_4+iE_\pi)\}\frac{1}{iE_\pi+p_4+iE_H}-\{-p_4E_H\}\frac{1}{iE_H-p_4+iE_\pi}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}(ip_4\frac{iE_H-p_4-iE_\pi}{iE_H-p_4-iE_\pi}-\{ip_4(p_4+iE_\pi)\}\frac{1}{iE_\pi+p_4+iE_H} \\
&\quad +\{p_4E_H\}\frac{1}{iE_H-p_4+iE_\pi}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}ip_4(1-(p_4+iE_\pi)\frac{1}{i(E_H+E_\pi)+p_4}-iE_H\frac{1}{i(E_H+E_\pi)-p_4}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}ip_4(1+(p_4+iE_\pi)\frac{i(E_H+E_\pi)-p_4}{(E_H+E_\pi)^2+p_4^2}+iE_H\frac{i(E_H+E_\pi)+p_4}{(E_H+E_\pi)^2+p_4^2}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}ip_4(1+\frac{ip_4E_H+ip_4E_\pi-p_4^2-E_\pi E_H-E_\pi^2-ip_4E_\pi-E_H^2-E_HE_\pi+ip_4E_H}{(E_H+E_\pi)^2+p_4^2}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}ip_4(1+\frac{2ip_4E_H-p_4^2-E_\pi^2-E_H^2-2E_HE_\pi}{(E_H+E_\pi)^2+p_4^2}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}ip_4(1+\frac{2ip_4E_H-(E_H+E_\pi)^2-p_4^2}{(E_H+E_\pi)^2+p_4^2}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{4E_\pi E_H}ip_4\frac{2ip_4E_H}{(E_H+E_\pi)^2+p_4^2} \\
&= 3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi}\frac{ip_4^2}{(E_H+E_\pi)^2+p_4^2}
\end{aligned}
$$

A.4

In this section we present the calcultation of eq. (4.19).

$$
\begin{aligned}
\frac{1}{4}Tr_D[\Sigma_{NLO}(p_4,\vec{0})] &= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{im_H}{4E_\pi E_H}(\frac{1}{iE_H-p_4-iE_\pi}+\frac{1}{iE_\pi+p_4-iE_H} \\
&\quad -\frac{1}{iE_\pi+p_4+iE_H}-\frac{1}{iE_H-p_4+iE_\pi}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{im_H}{4E_\pi E_H}(-\frac{1}{iE_\pi+p_4+iE_H}-\frac{1}{iE_H-p_4+iE_\pi}) \\
&= 3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{im_H}{4E_\pi E_H}(\frac{1}{i(E_H+E_\pi)+p_4}+\frac{1}{i(E_H+E_\pi)-p_4}) \\
&= -3g_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{im_H}{4E_\pi E_H}\frac{2i(E_H+E_\pi)}{(E_H+E_\pi)^2+p_4^2} \\
&= 3m_Hg_{\pi qq}^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi E_H}\frac{(E_H+E_\pi)}{(E_H+E_\pi)^2+p_4^2}
\end{aligned}
$$

Bibliography

[1] C. S. Fischer, D. Nickel and R. Williams, Eur. Phys. J. C **60** (2009) 47 [arXiv:0807.3486 [hep-ph]].

[2] V. Gribov, Eur. Phys. J. C **10** (1999) 71 [arXiv:hep-ph/9807224].

[3] V. N. Gribov, Eur. Phys. J. C **10** (1999) 91 [arXiv:hep-ph/9902279].

[4] V. N. Gribov, arXiv:hep-ph/9905285.

[5] C. D. Roberts and S. M. Schmidt, Prog. Part. Nucl. Phys. **45** (2000) S1 [arXiv:nucl-th/0005064].

[6] K. Osterwalder, R. Schrader, Commun. Math. Phys. **31** (1973) 83-112.

[7] R. Haag, "Local Quantum Physics: Fields, Particles, Algebras", Springer (2008).

[8] R. Alkofer, W. Detmold, C. S. Fischer and P. Maris, Phys. Rev. D **70** (2004) 014014 [arXiv:hep-ph/0309077].

[9] P. Maris, Phys. Rev. D **52** (1995) 6087 [arXiv:hep-ph/9508323].

[10] Y. Nambu, G. Jona-Lasinio, Phys. Rev. D **122** (1961) 345-358.

[11] Y. Nambu, G. Jona-Lasinio, Phys. Rev. D **124** (1961) 246-254.

[12] S. P. Klevansky, Rev. Mod. Phys. **64** (1992) 649-708.

[13] M. Buballa, Phys. Rept. **407** (2005) 205 [arXiv:hep-ph/0402234].

[14] M. Oertel, M. Buballa and J. Wambach, Nucl. Phys. A **676** (2000) 247 [arXiv:hep-ph/0001239].

[15] Y. L. Dokshitzer and D. E. Kharzeev, Ann. Rev. Nucl. Part. Sci. **54** (2004) 487 [arXiv:hep-ph/0404216].

[16] C. Ewerz, arXiv:hep-ph/0601271.

[17] K. Heckmann, "Die Scherviskosität im Nambu-Jona-Lasinio-Modell", Diplom-Arbeit, TU Darmstadt (2007).

[18] J. I. Kapusta, C. Gale, "Finite-Temperature Field Theory: Principles and Applications", Cambridge Monographs on Mathematical Physics (2006).

[19] D. Müller "Investigation of the Sigma Meson in the NJL Model at the Critical Endpoint of the QCD Phase Diagram", BSc thesis, TU Darmstadt (2007).

[20] D. Müller, "The Quark Propagator in a selfconsistent 1/Nc Expansion in the NJL Model", MSc thesis, TU Darmstadt (2009).

[21] D. Scheffler, "NJL model study of the QCD phase diagram using the Taylor series expansion technique", BSc thesis, TU Darmstadt (2007).