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  - pair distribution function:

$$g(\vec{x} - \vec{x}') = \frac{\langle\phi|\psi^\dagger(\vec{x})\psi^\dagger(\vec{x}')\psi(\vec{x}')\psi(\vec{x})|\phi\rangle}{\langle\phi|\psi^\dagger(\vec{x})\psi(\vec{x})|\phi\rangle\langle\phi|\psi^\dagger(\vec{x}')\psi(\vec{x}')|\phi\rangle}$$

$$= \frac{N-1}{N} + \frac{1}{N^2}\left\{\Big|\sum_{\vec{k}} e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} n_{\vec{k}}\Big|^2 - \sum_{\vec{k}} n_{\vec{k}}^2\right\}$$

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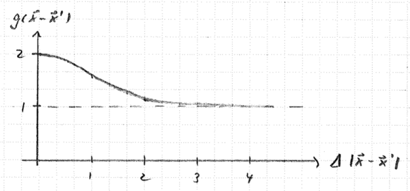
$\Rightarrow\ g(\vec{x}-\vec{x}') = \frac{N-1}{N}$ totally uncorrelated

- example 2: Gaussian distribution: $n_{\vec{k}} \propto e^{-(\vec{k}-\vec{k}_0)^2/\Delta^2}$

$$\Rightarrow \quad g(\vec{x}-\vec{x}\,') = 1 + e^{-\frac{\Delta^2}{2}(\vec{x}-\vec{x}\,')^2}, \quad g(0) = 2$$

→ The probability of finding a second particle close to another one is enhanced.



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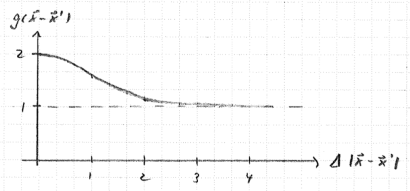

- thermal distribution:

$$n_{\vec{k}} \propto \frac{1}{e^{E_{\vec{k}}/k_BT}-1} \overset{E_{\vec{k}} \gg k_BT}{\approx} e^{-\frac{E_{\vec{k}}}{k_BT}} = e^{-\frac{\hbar^2\vec{k}^2}{2mK_BT}} \quad \Rightarrow \quad \Delta^2 = \frac{2mk_BT}{\hbar^2}$$

4.8 Weakly interacting, dilute boson gases (Bogoliubov theory)

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  → Consider only interaction terms with at most two indices $\vec{k} \neq \vec{0}$.

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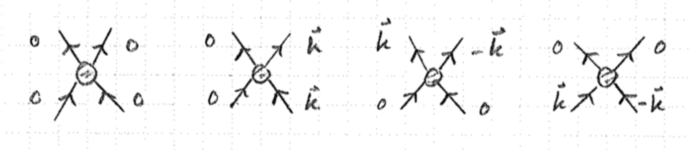

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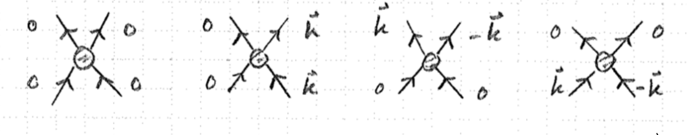

- potential symmetric under reflections: $V(\vec{x}) = V(-\vec{x}) \quad \Leftrightarrow \quad \tilde{V}(\vec{k}) = \tilde{V}(-\vec{k})$

Bogoliubov replacement

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- formal realization by "coherent states"

= states without well-defined particle number $\quad |\phi\rangle = \prod_{\vec{k}} f_{\vec{k}}(a^\dagger_{\vec{k}})|0\rangle$

e.g., $f_0(a_0^\dagger) = e^{\alpha a_0^\dagger} \;\Rightarrow\; a_0|\phi\rangle = \alpha|\phi\rangle$

- Bogoliubov replacement in our Hamiltonian:

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→ exact diagonalization of the approximated Hamiltonian

Bogoliubov-Transformation

- ansatz for new annihilation and creation operators:

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    holds for symmetric coefficients: $u_{\vec{k}} = u_{-\vec{k}}, \quad v_{\vec{k}} = v_{-\vec{k}}$

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  - $\left[\alpha_{\vec{k}}, \alpha_{\vec{k}'}\right] = -\left(u_{\vec{k}} v_{-\vec{k}} - v_{\vec{k}} u_{-\vec{k}}\right) \delta_{\vec{k},-\vec{k}'} \stackrel{!}{=} 0$

    holds for symmetric coefficients: $u_{\vec{k}} = u_{-\vec{k}}, \;\; v_{\vec{k}} = v_{-\vec{k}}$

- further assumption: $u_{\vec{k}}$ und $v_{\vec{k}}$ are real.

- inverse transformation:

$$a_{\vec{k}} = u_{\vec{k}}\,\alpha_{\vec{k}} + v_{\vec{k}}\,\alpha^\dagger_{-\vec{k}}\,, \qquad a^\dagger_{\vec{k}} = u_{\vec{k}}\,\alpha^\dagger_{\vec{k}} + v_{\vec{k}}\,\alpha_{-\vec{k}}$$

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- insert into the Hamiltonian:

$$\begin{aligned} H \approx\ & \frac{N^2}{2V}\tilde{V}(\vec{0}) \\ & + \sum_{\vec{k}\neq\vec{0}} \Big\{ \Big(\frac{\hbar^2\vec{k}^2}{2m} + \frac{N}{V}\tilde{V}(\vec{k}) \Big) \Big(u^2_{\vec{k}}\,\alpha^\dagger_{\vec{k}}\alpha_{\vec{k}} + v^2_{\vec{k}}\,\alpha_{\vec{k}}\alpha^\dagger_{\vec{k}} + u_{\vec{k}}v_{\vec{k}}\,(\alpha^\dagger_{\vec{k}}\alpha^\dagger_{-\vec{k}} + \alpha_{\vec{k}}\alpha_{-\vec{k}}) \Big) \\ & \qquad + \frac{N}{2V}\tilde{V}(\vec{k}) \Big((u^2_{\vec{k}} + v^2_{\vec{k}})(\alpha^\dagger_{\vec{k}}\alpha^\dagger_{-\vec{k}} + \alpha_{\vec{k}}\alpha_{-\vec{k}}) + 2u_{\vec{k}}v_{\vec{k}}\,(\alpha^\dagger_{\vec{k}}\alpha_{\vec{k}} + \alpha_{\vec{k}}\alpha^\dagger_{\vec{k}}) \Big) \Big\} \end{aligned}$$

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- For the non-diagonal terms to vanish, it must hold:

$$\underbrace{\Big(\frac{\hbar^2\vec{k}^2}{2m} + \frac{N}{V}\tilde{V}(\vec{k})\Big)}_{A} u_{\vec{k}}v_{\vec{k}} + \underbrace{\frac{N}{2V}\tilde{V}(\vec{k})}_{B}(u^2_{\vec{k}} + v^2_{\vec{k}}) \stackrel{!}{=} 0$$

$$A\,uv + B(u^2 + v^2) = 0 \quad \Rightarrow \quad A^2\,u^2 v^2 = B^2(u^2 + v^2)^2$$

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(non-interacting limit: $\tilde{V} = 0 \;\Rightarrow\; B = 0 \;\Rightarrow\; v = 0, \quad u^2 = 1 \quad \checkmark$)

- Insertion into the Hamiltonian finally yields:

$$H \approx \underbrace{\frac{N^2}{2V}\tilde{V}(\vec{0}) + \sum_{\vec{k}\neq\vec{0}} \frac{1}{2}\left(\hbar\omega_{\vec{k}} - \frac{\hbar^2\vec{k}^{\,2}}{2m} - \frac{N}{V}\tilde{V}(\vec{k})\right)}_{\text{ground-state energy (= constant number)}} + \underbrace{\sum_{\vec{k}\neq\vec{0}} \hbar\omega_{\vec{k}}\,\alpha^\dagger_{\vec{k}}\,\alpha_{\vec{k}}}_{\text{quasiparticle excitations}}$$

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$$\hbar\omega_{\vec{k}} = \sqrt{A - 4B} = \hbar|\vec{k}|\sqrt{\frac{N}{mV}\tilde{V}(\vec{k}) + \frac{\hbar^2\vec{k}^{\,2}}{4m^2}} \qquad (\text{for } |\vec{k}| \to 0 \text{ linear in } |\vec{k}|)$$

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→ The ground state is defined as the state which contains no quasiparticles:

$\alpha_{\vec{k}}\,|\text{g.s.}\rangle = 0$ für alle $\vec{k} \neq \vec{0}$ ground state $\hat{=}$ "quasiparticle vacuum"

- But the ground state contains particles (not quasiparticles) outside of the condensate:

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5. Outlook on quantum field theory

5.1 Quantization of field theories

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- Hamiltonian function: $H = \sum_k p_k \dot{q}_k - L$

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  - Schrödinger picture: operators time independent
  - Heisenberg picture: operators time dependent

    $\rightarrow$ commutators at equal times

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Examples

1. electromagnetic field:

   $\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$, field-strength tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

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$$\Leftrightarrow\ \Big(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V\Big)\psi = i\hbar \frac{\partial}{\partial t}\psi$$ (Schrödinger equation)

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- canonical conjugate momentum densities: $\pi(x) = \frac{\partial\mathscr{L}}{\partial\dot{\phi}}$

- Hamiltonian function: $H = \int d^3x\, \mathscr{H}$, $\mathscr{H} = \pi(x)\dot{\phi}(x) - \mathscr{L}$

  Hamiltonian density

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- notation: From now on we write ϕ, π, ... for the operators $\hat{\phi}$, $\hat{\pi}$ ...

Schrödinger theory

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- Lagrangian density:

$$\mathscr{L} = -\frac{\hbar^2}{2m}\vec{\nabla}\psi \cdot \vec{\nabla}\psi^\dagger + i\hbar\,\psi^\dagger\dot{\psi} - V\psi^\dagger\psi \qquad (\psi\text{:}\quad \text{operator} \;\Rightarrow\; \psi^* \to \psi^\dagger)$$

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  new point of view: classical field $\xrightarrow{\text{quantization}}$ field operator

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$$\pi(x) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\Big(a_{\vec{p}}\, e^{-ip\cdot x} - a^\dagger_{\vec{p}}\, e^{+ip\cdot x}\Big)\Big|_{p_0=E_p}$$

- canonical quantization conditions for ϕ and π:

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  reminder: $\quad E_p \equiv +\sqrt{\vec{p}^{\,2} + m^2} > 0 \quad \Rightarrow$ no states with negative energy!

5.3 Dirac theory

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