N-boson- and N-fermion systems

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- (Anti-) symmetrization operator: $S_\pm |i_1, \ldots, i_N\rangle \equiv \frac{1}{\sqrt{N!}} \sum_{P \in S_N} (\pm 1)^P \, P|i_1, \ldots, i_N\rangle$
- normalized basis states for N identical fermions:

$$S_-|i_1, \ldots, i_N\rangle = \frac{1}{\sqrt{N!}} \sum_{P \in S_N} (-1)^P \, P|i_1, \ldots, i_N\rangle$$

$$= \frac{1}{\sqrt{N!}} \begin{vmatrix} |i_1\rangle_1 & \ldots & |i_1\rangle_N \\ \vdots & & \vdots \\ |i_N\rangle_1 & \ldots & |i_N\rangle_N \end{vmatrix}$$

"Slater determinant"

In particular the Pauli principle holds:

$S_-|i_1, \ldots, i_N\rangle = 0$ if $|i_i\rangle = |i_j\rangle$ for any $i \neq j$

- example $N = 2$: $S_-|i, j\rangle = \frac{1}{\sqrt{2}} \big(|i, j\rangle - |j, i\rangle\big)$

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- normalized basis states for N identical bosons:

$$\frac{1}{\sqrt{n_1!n_2!\ldots}} S_+ |i_1, \ldots, i_N\rangle = \frac{1}{\sqrt{N!n_1!n_2!\ldots}} \sum_{P \in S_N} P |i_1, \ldots, i_N\rangle$$

  - n_i = number of particles in the single-particle state $|i\rangle$
  - reason for the extra normalization factor:

    For n_{j_1} particles in the state $|j_1\rangle$, n_{j_2} particles in the state $|j_2\rangle$, etc., the $N!$ permutations yield only $\frac{N!}{n_{j_1}!n_{j_2}!\ldots}$ different states, each with multiplicitiy $n_{j_1}!n_{j_2}!\ldots$.
  - example:

$$\frac{1}{\sqrt{n_i!n_j!}} S_+ |i, i, j\rangle = \frac{1}{\sqrt{2!1!}} \frac{1}{\sqrt{3!}} \Big(2|i, i, j\rangle + 2|i, j, i\rangle + 2|j, i, i\rangle\Big)$$

$$= \frac{1}{\sqrt{3}} \Big(|i, i, j\rangle + |i, j, i\rangle + |j, i, i\rangle\Big) \quad \checkmark$$

4.2 Occupation-number representation

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- N identical particles: only totally (anti-) symmetric states

  $\Rightarrow$ basis of product states $\{|i_1, \ldots, i_N\rangle\}$
  (much) larger than the dimension of the space of allowed states

$\rightarrow$ take (anti-) symmetrized basis states only

- occupation-number representation: $|n_1, n_2, \ldots\rangle$
  - n_i = number of particles in the single-particle state $|i\rangle$
  - bosons: $n_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$
  - fermions: $n_i \in \{0, 1\}$
  - vacuum state: $|0\rangle \equiv |0, 0, 0, \ldots\rangle$
  - single-particle states: $|1, 0, 0, 0, \ldots\rangle$, $|0, 1, 0, 0, \ldots\rangle$, ...
  - total particle number: $\sum_i n_i = N$

- example: spin-$\frac{1}{2}$ fermions (with no other quantum numbers)
  - $N = 0$: $n_\uparrow = n_\downarrow = 0 \quad \rightarrow \quad |0,0\rangle \equiv |0\rangle$
  - $N = 1$: $\begin{cases} n_\uparrow = 1,\ n_\downarrow = 0 & \rightarrow \quad |1,0\rangle \equiv |\uparrow\rangle \\ n_\uparrow = 0,\ n_\downarrow = 1 & \rightarrow \quad |0,1\rangle \equiv |\downarrow\rangle \end{cases}$
  - $N = 2$: $n_\uparrow = n_\downarrow = 1 \quad \rightarrow \quad |1,1\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle)$
  - There are no states with $N > 2$.
- more realistic systems:

  infinitely many states, but only a finite number of $n_i \neq 0$ ($\Rightarrow N$ finite)

Fock space

- The space spanned by the states $|n_1, n_2, ...\rangle$ is called Fock space.
- preliminary simplification: only discrete spectra
  - e.g., finite volume V with periodic boundary conditions
    → discrete momenta

    At the end one can take the limit $V \to \infty$.
- Orthonormality and completeness:
  - $\langle n_1, n_2, ...|n_1', n_2', ...\rangle = \delta_{n_1,n_1'}\delta_{n_2,n_2'} \cdots$

    $\Rightarrow$ States with different particle numbers are orthogonal.
  - $\sum_{n_1,n_2,...} |n_1, n_2, ...\rangle\langle n_1, n_2, ...| = \mathbb{1}$

1. Bosons

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- $|n_1, n_2, ...\rangle = \frac{1}{\sqrt{n_1! n_2! ...}} S_+ |i_1, ..., i_N\rangle$ (see section 4.1)
  - $|i_1, ..., i_N\rangle$: N-particle state ($N = \sum n_i$) with n_i particles in the single-particle state $|i\rangle$
- creation operator: $a_i^\dagger |..., n_i, ...\rangle = \sqrt{n_i + 1}\, |..., n_i + 1, ...\rangle$
  - raises the occupation number of the single-particle state $|i\rangle$ by 1
- $\langle ...\, n_i' \,...|a_i|...\, n_i \,...\rangle = \langle ...\, n_i \,...|a_i^\dagger|...\, n_i' \,...\rangle^* = \sqrt{n_i' + 1}\, \langle ...\, n_i \,...|...\, n_i' + 1 \,...\rangle^*$

$$= \sqrt{n_i' + 1}\; ...\, \delta_{n_i, n_i'+1} \,... = \sqrt{n_i}\; ...\, \delta_{n_i - 1, n_i'} \,... = \sqrt{n_i}\, \langle ...\, n_i' \,...|...\, n_i - 1 \,...\rangle$$

$\Rightarrow \quad a_i |..., n_i, ...\rangle = \sqrt{n_i}\, |..., n_i - 1, ...\rangle$

  - a_i destroys a particle in the state $|i\rangle$
  - Also: $a_i |..., n_i = 0, ...\rangle = 0$

- creation operator: $a_i^\dagger |\dots, n_i, \dots\rangle = \sqrt{n_i + 1}\, |\dots, n_i + 1, \dots\rangle$

  annihilation operator: $a_i |\dots, n_i, \dots\rangle = \sqrt{n_i}\, |\dots, n_i - 1, \dots\rangle$

  analogous to ladder operators of the harmonic oscillator!

- $a_i^\dagger a_i |\dots, n_i, \dots\rangle = \sqrt{n_i}\, a_i^\dagger |\dots, n_i - 1, \dots\rangle = n_i\, |\dots, n_i, \dots\rangle$

  → particle-number operator: $\hat{n}_i \equiv a_i^\dagger a_i \quad \Rightarrow \quad \hat{n}_i |\dots, n_i, \dots\rangle = n_i |\dots, n_i, \dots\rangle$

  → total particle number: $\hat{N} \equiv \sum_i \hat{n}_i = \sum_i a_i^\dagger a_i$

  $$\Rightarrow \quad \hat{N} |n_1, n_2, \dots\rangle = (n_1 + n_2 + \dots) |n_1, n_2, \dots\rangle = N |n_1, n_2, \dots\rangle$$

- $a_i a_i^\dagger |\dots, n_i, \dots\rangle = \sqrt{n_i + 1}\, a_i |\dots, n_i + 1, \dots\rangle = (n_i + 1)\, |\dots, n_i, \dots\rangle$

  $$\Rightarrow \quad \left[a_i, a_i^\dagger\right] |\dots, n_i, \dots\rangle = |\dots, n_i, \dots\rangle \quad \Rightarrow \quad \boxed{\left[a_i, a_i^\dagger\right] = 1}$$

- $a_i^\dagger |\ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}\, |\ldots, n_i + 1, \ldots\rangle, \qquad a_i |\ldots, n_i, \ldots\rangle = \sqrt{n_i}\, |\ldots, n_i - 1, \ldots\rangle$

$$\Rightarrow \; a_i a_j^\dagger |\ldots, n_i, \ldots n_j, \ldots\rangle \overset{i \neq j}{=} \sqrt{n_i(n_j + 1)}\, |\ldots, n_i - 1, \ldots n_j + 1, \ldots\rangle$$

$$= a_j^\dagger a_i |\ldots, n_i, \ldots n_j, \ldots\rangle$$

- similar for $a_i a_j$ or $a_i^\dagger a_j^\dagger$ (both for $i = j$ and $i \neq j$)

→ commutation relations for bosons: $\left[a_i, a_j\right] = \left[a_i^\dagger, a_j^\dagger\right] = 0, \quad \left[a_i, a_j^\dagger\right] = \delta_{ij}$

- normalized basis states of the Fock space:

$$|n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots}} \left(a_1^\dagger\right)^{n_1} \left(a_2^\dagger\right)^{n_2} \ldots |0\rangle$$

2. Fermions

- Antisymmetrized N-fermion states (Slater determinant):

$$S_-|i_1,\dots,i_N\rangle = \frac{1}{\sqrt{N!}}\sum_{P\in S_N}(-1)^P\,P|i_1,\dots,i_N\rangle = \frac{1}{\sqrt{N!}}\begin{vmatrix} |i_1\rangle_1 & \dots & |i_1\rangle_N \\ \vdots & & \vdots \\ |i_N\rangle_1 & \dots & |i_N\rangle_N \end{vmatrix}$$

- The order of the particles matters:

$$S_-P|i_1,\dots,i_N\rangle = (-1)^P S_-|i_1,\dots,i_N\rangle,$$

e.g., $S_-|i,j\rangle = \frac{1}{\sqrt{2}}(|i,j\rangle - |j,i\rangle) = -S_-|j,i\rangle$

- Define creation operators by:

$S_-|i_1,\dots,i_N\rangle = a^\dagger_{i_N}\dots a^\dagger_{i_1}|0\rangle$ in this order!

  - first create a particle in the state $|i_1\rangle$ and last a particle in the state $|i_N\rangle$.
  - convention: some authors write it the other way around!

- $S_- \left| i_1, \ldots, i_N \right\rangle = a^\dagger_{i_N} \ldots a^\dagger_{i_1} \left| 0 \right\rangle \quad \Rightarrow \quad a^\dagger_i a^\dagger_j = -a^\dagger_j a^\dagger_i$

  e.g., $a^\dagger_i a^\dagger_j \left| 0 \right\rangle = S_- \left| j, i \right\rangle = -S_- \left| i, j \right\rangle = -a^\dagger_j a^\dagger_i \left| 0 \right\rangle$

  $\Rightarrow \quad \left(a^\dagger_i \right)^2 = 0$ (Pauli principle)

- basis states of the Fock space:

  $$\left| n_1, n_2, \ldots \right\rangle = \ldots \left(a^\dagger_2 \right)^{n_2} \left(a^\dagger_1 \right)^{n_1} \left| 0 \right\rangle$$ (in this order)

- example:

  $$\left| 1, 1, 0, 0, \ldots \right\rangle = a^\dagger_2 a^\dagger_1 \left| 0 \right\rangle = S_- \left| 1, 2 \right\rangle = \frac{1}{\sqrt{2}} \left(\left| 1, 2 \right\rangle - \left| 2, 1 \right\rangle \right)$$

- $|n_1, n_2, ...\rangle = ... \left(a_2^\dagger\right)^{n_2} \left(a_1^\dagger\right)^{n_1} |0\rangle$

  $$\Rightarrow \; a_i^\dagger |..., n_i, ...\rangle = (-1)^{\sum_{j>i} n_j} (1 - n_i) |..., n_i + 1, ...\rangle$$

  - $\sum_{j>i} n_j$ = number of transpositions to move $a_i^\dagger$ to the “correct” place.
  - $(1 - n_i) = \begin{cases} 1 & \text{if } n_i = 0, \quad \text{i.e., if } |i\rangle \text{ was not occupied} \\ 0, & \text{if } n_i = 1, \quad \text{i.e., if } |i\rangle \text{ was occupied} \end{cases}$

- analogous to the boson case:

  $$\langle ... n_i' ... | a_i | ... n_i ... \rangle = \langle ... n_i ... | a_i^\dagger | ... n_i' ... \rangle^* = ...$$

  $$\Rightarrow \; a_i |..., n_i, ...\rangle = (-1)^{\sum_{j>i} n_j} n_i |..., n_i - 1, ...\rangle \qquad \Rightarrow \; a_i |..., n_i = 0, ...\rangle = 0$$

- $a_i^\dagger |\dots, n_i, \dots\rangle = (-1)^{\sum_{j>i} n_j} (1 - n_i) |\dots, n_i + 1, \dots\rangle$

  $a_i |\dots, n_i, \dots\rangle = (-1)^{\sum_{j>i} n_j} n_i |\dots, n_i - 1, \dots\rangle$

$\Rightarrow$ $a_i^\dagger a_i |\dots, n_i, \dots\rangle = \big(1 - (n_i - 1)\big) n_i |\dots, n_i, \dots\rangle \overset{n_i \in \{0,1\}}{=} n_i |\dots, n_i, \dots\rangle$

  → particle-number operator: $\hat{n}_i \equiv a_i^\dagger a_i$ (as for bosons)

- $a_i a_i^\dagger |\dots, n_i, \dots\rangle = (n_i + 1)(1 - n_i) |\dots, n_i, \dots\rangle \overset{n_i \in \{0,1\}}{=} (1 - n_i) |\dots, n_i, \dots\rangle$

$\Rightarrow$ $\{a_i, a_i^\dagger\} = a_i a_i^\dagger + a_i^\dagger a_i = 1$

→ anti-commutator relations for fermions:

$$\boxed{\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \quad \{a_i, a_j^\dagger\} = \delta_{ij}}$$