4. Many-particle theory

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4.1 Multi-particle states, bosons and fermions

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- arbitrary single-particle state: $|\psi\rangle = \sum_i c_i |i\rangle$
  - $\{|i\rangle\}$: complete basis of the single-particle Hilbert space (usually orthonormalized)
  - continuous spectra: $\sum \to \int$
- basis of the N-particle Hilbert space: $|i_1, i_2, \ldots, i_N\rangle \equiv |i_1\rangle_1 |i_2\rangle_2 \ldots |i_N\rangle_N$
  - direct product of single-particle basis states
  - $|i\rangle_j$: The j^{th} particle is in the state $|i\rangle$.

→ general N-particle state: $|\psi\rangle = \sum_{i_1,\ldots i_N} c_{i_1,\ldots i_N} |i_1,\ldots i_N\rangle$

- example: position-space representation

$$|\psi\rangle = \int d^3x_1 \ldots \int d^3x_N \, \psi(\vec{x}_1, \ldots, \vec{x}_N) \, |\vec{x}_1, \ldots, \vec{x}_N\rangle$$

$$\Rightarrow \; \langle \vec{x}_1, \ldots, \vec{x}_N | \psi \rangle = \int d^3x'_1 \ldots \int d^3x'_N \, \psi(\vec{x}'_1, \ldots, \vec{x}'_N) \, \langle \vec{x}_1, \ldots, \vec{x}_N | \vec{x}'_1, \ldots, \vec{x}'_N \rangle$$

$$\begin{aligned} \langle \vec{x}_1, \ldots, \vec{x}_N | \vec{x}'_1, \ldots, \vec{x}'_N \rangle &= \langle \vec{x}_1 |_1 \ldots \langle \vec{x}_N |_N \; |\vec{x}'_1\rangle_1 \ldots |\vec{x}'_N\rangle_N \\ &= \langle \vec{x}_1 | \vec{x}'_1 \rangle \ldots \langle \vec{x}_N | \vec{x}'_N \rangle \\ &= \delta^3(\vec{x}_1 - \vec{x}'_1) \ldots \delta^3(\vec{x}_N - \vec{x}'_N) \end{aligned}$$

$$\Rightarrow \; \langle \vec{x}_1, \ldots, \vec{x}_N | \psi \rangle = \psi(\vec{x}_1, \ldots, \vec{x}_N) \qquad \checkmark$$

- short-hand notation often used in the following:

$$|\psi\rangle = \sum_{\vec{x}_1, \ldots, \vec{x}_N} \psi(\vec{x}_1, \ldots, \vec{x}_N) \, |\vec{x}_1, \ldots, \vec{x}_N\rangle$$

- transposition operator P_{ij}: interchanges the i^{th} and the j^{th} particle

$$
\begin{aligned}
P_{ij}\,|i_1,\dots,i_i,\dots,i_j,\dots,i_N\rangle &= P_{ij}\big(|i_1\rangle_1\dots|i_i\rangle_i\dots|i_j\rangle_j\dots|i_N\rangle_N\big)\\
&= \quad |i_1\rangle_1\dots|i_i\rangle_j\dots|i_j\rangle_i\dots|i_N\rangle_N\\
&= \quad |i_1\rangle_1\dots|i_j\rangle_i\dots|i_i\rangle_j\dots|i_N\rangle_N\\
&= |i_1,\dots,i_j,\dots,i_i,\dots,i_N\rangle
\end{aligned}
$$

- position-space representation:

$$
\begin{aligned}
P_{ij}\,|\psi\rangle = P_{ij}\sum_{\vec{x}_1,\dots,\vec{x}_N}\psi(\dots,\vec{x}_i,\dots,\vec{x}_j,\dots)\,|\dots,\vec{x}_i,\dots,\vec{x}_j,\dots\rangle\\
= \sum_{\vec{x}_1,\dots,\vec{x}_N}\psi(\dots,\vec{x}_i,\dots,\vec{x}_j,\dots)\,|\dots,\vec{x}_j,\dots,\vec{x}_i,\dots\rangle\\
\overset{\vec{x}_i\leftrightarrow\vec{x}_j}{=}\sum_{\vec{x}_1,\dots,\vec{x}_N}\psi(\dots,\vec{x}_j,\dots,\vec{x}_i,\dots)\,|\dots,\vec{x}_i,\dots,\vec{x}_j,\dots\rangle
\end{aligned}
$$

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$$P_{ij} |\psi\rangle = P_{ij} \sum_{\vec{x}_1,\dots,\vec{x}_N} \psi(\dots, \vec{x}_i, \dots, \vec{x}_j, \dots) \, |\dots, \vec{x}_i, \dots, \vec{x}_j, \dots\rangle$$

$$= \sum_{\vec{x}_1,\dots,\vec{x}_N} \psi(\dots, \vec{x}_j, \dots, \vec{x}_i, \dots) \, |\dots, \vec{x}_i, \dots, \vec{x}_j, \dots\rangle$$

→ If $|\psi\rangle$ has the position-space wave function $\psi(\dots, \vec{x}_i, \dots, \vec{x}_j, \dots)$,
then $P_{ij} |\psi\rangle$ has the position-space wave function $\psi(\dots, \vec{x}_j, \dots, \vec{x}_i, \dots)$.

- $P_{ij}^2 |\psi\rangle = |\psi\rangle \quad \Rightarrow \quad P_{ij}^2 = 1\!\!1 \quad \Rightarrow$ eigenvalues of $P_{ij} = \pm 1$
- Any permutation $P \in S_N$ of N particles can be obtained as a product of transpositions.

→ notation:

$$(-1)^P = \left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\} \text{ if the number of transpositions is } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}.$$

Identical particles

- Consider:
  - N identical (= indistinguishable) particles
  - observable $\leftrightarrow$ operator $\hat{\mathcal{O}}$
  - eigenstate $|\psi_\lambda\rangle$ of $\hat{\mathcal{O}}$ with eigenvalue λ: $\quad \hat{\mathcal{O}}|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$
  - permutation $P \in S_N, \quad P|\psi_\lambda\rangle \equiv |P\psi_\lambda\rangle$
- particles indistinguishable $\Rightarrow \hat{\mathcal{O}}|P\psi_\lambda\rangle = \lambda|P\psi_\lambda\rangle$

$$\Leftrightarrow \quad \hat{\mathcal{O}}P|\psi_\lambda\rangle = \lambda P|\psi_\lambda\rangle = P\lambda|\psi_\lambda\rangle = P\hat{\mathcal{O}}|\psi_\lambda\rangle \quad \Rightarrow \quad [P, \hat{\mathcal{O}}] = 0$$

$\Rightarrow \quad \hat{\mathcal{O}}$ must be symmetric under exchange of identical particles.

In particular this holds for $\hat{\mathcal{O}} = H$.

- example:

  N electrons in the (static) Coulomb potential V of an atomic nucleus
  + mutual Coulomb repulsion

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i=1}^{N} V(\vec{x}_i) + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|\vec{x}_i - \vec{x}_j|},$$

  $\vec{p}_i, \vec{x}_i$: momentum and position operators acting on the i^{th} particle only

  ➞ H is symmetric under exchange of electrons.

- further properties of permutations of identical particles:

  P is unitary: $P^\dagger = P^{-1} \;\Rightarrow\; P^\dagger P = PP^\dagger = 1$

  (reminder: $\langle P\phi|\psi\rangle = \langle\phi|P^\dagger|\psi\rangle = \langle\phi|P^\dagger\psi\rangle$)

  $\Rightarrow \langle P\phi|P\psi\rangle = \langle\phi|\psi\rangle$

Bosons and fermions

Quantum statistics: two classes of particles,
which differ by the symmetry of their many-body states

- bosons:
  - integer spin (0, 1, 2, ...)
  - states symmetric under the interchange of two particles:

    $$P_{ij}|\psi\rangle = |\psi\rangle \quad \forall\, i, j \quad \Rightarrow \quad P|\psi\rangle = |\psi\rangle \quad \forall\, P \in S_N$$

- fermions:
  - half-integer spin ($\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, ...)
  - states antisymmetric under the interchange of two particles:

    $$P_{ij}|\psi\rangle = -|\psi\rangle \quad \forall\, i, j \quad \Rightarrow \quad P|\psi\rangle = (-1)^P|\psi\rangle \quad \forall\, P \in S_N$$

  $\Rightarrow$ Pauli principle:
  States with two or more fermions in the same single-particle state vanish.

- The relation between spin and statistics can be proven within QFT.

Two-particle spin-systems

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1. two distinguishable spin-$\frac{1}{2}$ particles (e.g., $e^- p$):
   - four basis states of the two-particle system:

$$|\uparrow,\uparrow\rangle \equiv |\uparrow\rangle_e\,|\uparrow\rangle_p$$
$$|\uparrow,\downarrow\rangle \equiv |\uparrow\rangle_e\,|\downarrow\rangle_p$$
$$|\downarrow,\uparrow\rangle \equiv |\downarrow\rangle_e\,|\uparrow\rangle_p$$
$$|\downarrow,\downarrow\rangle \equiv |\downarrow\rangle_e\,|\downarrow\rangle_p$$

   - alternative basis: states of good total spin $|SM_s\rangle$

$$\left.\begin{aligned} |1,1\rangle &= |\uparrow,\uparrow\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}\big(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle\big) \\ |1,-1\rangle &= |\downarrow,\downarrow\rangle \end{aligned}\right\} \quad \text{spin 1 (symmetric)}$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\big) \qquad \text{spin 0 (antisymmetric)}$$

2. two indistinguishable spin-$\frac{1}{2}$ particles (e.g., $e^- e^-$):

   fermions ➞ two-particle systems antisymmetric

   ➞ only total spin 0 allowed: $|\psi\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\right)$

   (as long as the particles do not differ by other quantum numbers)

3. two distinguishable spin-1 particles (e.g., $\rho^+\rho^-$):

   ➞ $3^2 = 9$ basis states: $(m_1, m_2) = (1,1),\ (1,0),\ (1,-1),\ \ldots,\ (-1,-1)$

   - alternative basis: $|S=2, M_S\rangle$, $|S=1, M_S\rangle$, $|S=0, M_S=0\rangle$

     5 symmetric + 3 antisymm. + 1 symmetric

4. two indistinguishable spin-1 particles (e.g., $\omega\omega$):

   bosons ➞ 6 symmetric basis states,

   e.g., $|S=2, M_S\rangle$ and $|S=0, M_S=0\rangle$