3.12 Spin and angular momentum

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- so far: passive transformations

  The same physical process is described in two inertial frames K and K', which can be transformed into each other by a Lorentz transformation.

  K $\to$ K' $\quad \longrightarrow \quad x \to x' = \Lambda x\,, \quad \psi(x) \to \psi'(x') = S\psi(x)$

- active transformations = transformation of the physical system (equivalent to passive transformations in the opposite direction)

- examples:
  - clockwise rotation of the physical system
    $\hat{=}$ counter-clockwise rotation of the frame
  - boost of the physical system with velocity $\vec{v}$
    $\hat{=}$ boost of the framewith velocity $-\vec{v}$

Consider the active rotation of a physical system in a frame K, and

- ψ: original spinor described in K
- $\tilde{\psi}$: rotated spinor described in K
- K': frame that was rotated in the same way as the physical system
- $\tilde{\psi}'$: rotated spinor described in K', must be identical to ψ

$$\Rightarrow \quad \underset{\displaystyle S\tilde{\psi}(x)}{\underset{\shortparallel}{\tilde{\psi}'(x')}} = \underset{\displaystyle \psi(\Lambda x)}{\underset{\shortparallel}{\psi(x')}} \qquad \text{(same argument on both sides!)}$$

$$\Rightarrow \quad \tilde{\psi}(x) = S^{-1}\psi(\Lambda x)$$

with the matrices S and Λ of the passive transformations

- $\tilde{\psi}(x) = S^{-1}\psi(\Lambda x)$

- infinitesimal transformations:

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \Delta\omega^{\mu}{}_{\nu} \quad \Rightarrow \quad \Lambda^{\mu}{}_{\nu}x^{\nu} = x^{\mu} + \Delta\omega^{\mu}{}_{\nu}x^{\nu}$$

$$S = 1 - \tfrac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} \quad \Rightarrow \quad S^{-1} = 1 + \tfrac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}$$

$$\Rightarrow \psi(\Lambda x) = \psi\big((x^{\mu} + \Delta\omega^{\mu}{}_{\nu}x^{\nu})\big)$$

$$= \psi(x) + \big(\partial_{\mu}\psi(x)\big)\cdot\Delta\omega^{\mu}{}_{\nu}x^{\nu} = \big(1 + \Delta\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\big)\,\psi(x)$$

$$\Rightarrow \tilde{\psi}(x) = \big[1 + \Delta\omega^{\mu\nu}(x_{\nu}\partial_{\mu} + \tfrac{i}{4}\sigma_{\mu\nu})\big]\,\psi(x)$$

$x_{\nu}\partial_{\mu}$ ↗ due to Λ → always present, e.g., for scalar fields

$\tfrac{i}{4}\sigma_{\mu\nu}$ ↖ due to S^{-1} → specific for Dirac spinors

- inifinitesimal rotation around the z-axis: $\Delta\omega^{12} = -\Delta\omega^{21} = -\delta\varphi$

$$\Rightarrow \quad \tilde{\psi}(x) = \left[1 - \delta\varphi(x_2\partial_1 - x_1\partial_2 + \tfrac{i}{2}\sigma_{12})\right]\psi(x)$$

$$= \left[1 - \tfrac{i}{\hbar}\delta\varphi\left(x^1\,\tfrac{\hbar}{i}\,\tfrac{\partial}{\partial x^2} - x^2\,\tfrac{\hbar}{i}\,\tfrac{\partial}{\partial x^1} + \tfrac{\hbar}{2}\,\Sigma^3\right)\right]\psi(x)$$

$$= \left[1 - \tfrac{i}{\hbar}\delta\varphi\, J^3\right]\psi(x)$$

with

$\vec{J} = \vec{L} + \vec{S}$	total angular momentum
$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times\frac{\hbar}{i}\vec{\nabla}$	orbital angular momentum
$\vec{S} = \frac{\hbar}{2}\vec{\Sigma} = \frac{\hbar}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$	spin

- finite rotation about an arbitrary axis: $$\tilde{\psi}(x) = \exp\left(-\frac{i}{\hbar}\vec{\varphi}\cdot\vec{J}\right)\psi(x)$$

Total angular momentum: further properties

- $[H_\text{D}, \vec{J}] = 0 \quad \Rightarrow$ conserved quantity, simultaneous eigenstates

  (remains true in the presence of radially symmetric potentials)

  in contrast: $[H_\text{D}, \vec{L}] = -[H_\text{D}, \vec{S}] \neq 0$

- $[J^i, J^j] = i\hbar\epsilon^{ijk} J^k, \quad [L^i, L^j] = i\hbar\epsilon^{ijk} L^k, \quad [S^i, S^j] = i\hbar\epsilon^{ijk} S^k$

- $H_\text{D}, \vec{L}^{\,2}, \vec{S}^{\,2}, \vec{J}^{\,2}$ and J^k mutually commute

  → simultaneous eigenstates: $|E, \ell, s = \frac{1}{2}, j, m_j\rangle$ (but not m_ℓ, m_s)

Helicity

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- helicity operator: $\hat{h} = \vec{\Sigma} \cdot \frac{\vec{p}}{|\vec{p}|}$
  - $\vec{p} \equiv \frac{\hbar}{i}\vec{\nabla}$ = momentum operator,
    $|\vec{p}|$ can be defined via expansion in momentum eigenstates
- $[H_D, \hat{h}] = 0 \quad \rightarrow$ simultaneous eigenstates
  but: $[H_D, \vec{\Sigma} \cdot \vec{n}] \neq 0$, unless $\vec{n}$ is parallel to $\vec{p}$.
- $\hat{h}^2 = \mathbb{1} \quad \Rightarrow$ eigenvalues of $\hat{h}$: ± 1
  - righthanded spinors: $\hat{h}\psi = +\psi$ (momentum and spin parallel)
  - lefthanded spinors: $\hat{h}\psi = -\psi$ (momentum and spin anti-parallel)

3.13 Discrete symmetry transformations

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  We have shown: $\quad S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu \gamma^\mu$

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  Probability conservation:

  $$\psi'^{\dagger}(x')\psi'(x') \stackrel{!}{=} \psi^{\dagger}(x)\psi(x) \quad \Rightarrow \quad P^{\dagger}P = \mathbb{1} \quad \Rightarrow \quad \eta_P = e^{i\varphi} \quad \Rightarrow \quad P = e^{i\varphi}\gamma^0$$

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The phase φ is not observable. $\rightarrow$ choose $\varphi = 0$ $\quad \Rightarrow \quad \boxed{P = \gamma^0}$

- $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $\rightarrow$ opposite sign for the transformation of spinors with positive and negative energy:

$$P\psi^{(+)}_{p,s}(x) = +\psi^{(+)}_{p',s}(x'), \quad P\psi^{(-)}_{p,s}(x) = -\psi^{(-)}_{p',s}(x'), \quad p' = \begin{pmatrix} E/c \\ -\vec{p} \end{pmatrix}$$

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- In interacting theories with several different particle species the relative phases are in general fixed.

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  $\det\Lambda = +1$ for proper orthochronous Lorentz transformations

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- important definition: $\gamma_5 = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

  property: $\{\gamma^\mu, \gamma_5\} = 0$ for all $\mu = 0, \dots, 3$

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- Using the γ matrices one can construct so-called "covariant bilinears" from $\overline{\psi}$ and ψ, which transform in different well-defined ways under Lorentz transformations.

Covariant bilinears

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$S(x) \equiv \overline{\psi}(x)\psi(x)$	$S'(x') = S(x)$	scalar
$P(x) \equiv \overline{\psi}(x)i\gamma_5\psi(x)$	$P'(x') = \det\Lambda\, P(x)$	pseudoscalar
$V^\mu(x) \equiv \overline{\psi}(x)\gamma^\mu\psi(x)$	$V'^{\,\mu}(x') = \Lambda^\mu{}_\nu\, V^\nu(x)$	vector
$A^\mu(x) \equiv \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)$	$A'^{\,\mu}(x') = \det\Lambda\, \Lambda^\mu{}_\nu\, A^\nu(x)$	axialvector
$T^{\mu\nu}(x) \equiv \overline{\psi}(x)\sigma^{\mu\nu}\psi(x)$	$T'^{\,\mu\nu}(x') = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\, T^{\rho\sigma}(x)$	2nd rank tensor

(proof: exercises)

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- The bilinears are hermitian.
- $\{\mathbb{1}, i\gamma_5,\ \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$: 16 linearly independent 4×4 matrices

  $1 + 1 + 4\ +\ 4\ +\ 6\ = 16$

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  → complete basis for arbitrary 4×4 matrices in Dirac space

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2. Charge conjugation

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- Dirac equation with electromagnetic field
  - electron ($q = -e$): $$\left(i\not{\partial} + \frac{e}{\hbar c}\not{A} - \frac{mc}{\hbar}\right)\psi(x) = 0 \qquad (1)$$
  - positron ($q = +e$): $$\left(i\not{\partial} - \frac{e}{\hbar c}\not{A} - \frac{mc}{\hbar}\right)\psi_C = 0 \qquad (2)$$

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(in Dirac representation, can be different in other representations!)

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3. Time reversal

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- Dirac equation with electromagnetic field (non-covariant form):

$$i\hbar \frac{\partial}{\partial t}\,\psi(t,\vec{r}) = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}(t,\vec{r})\right) + \beta mc^2 + q\phi(t,\vec{r})\right]\psi(t,\vec{r})$$

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- Consider a "backward running movie":
  - coordinates: $(t,\vec{r}) \to (-t,\vec{r})$
  - electromagnetic charges and currents: $\rho \to \rho, \;\; \vec{j} \to -\vec{j}$

$$\Rightarrow \;\vec{E} \to \vec{E}, \;\; \vec{B} \to -\vec{B} \quad \Rightarrow \;\; \phi(t,\vec{r}) \to \phi(-t,\vec{r}), \;\; \vec{A}(t,\vec{r}) \to -\vec{A}(-t,\vec{r})$$

- Dirac equation with electromagnetic field:

$$i\hbar\frac{\partial}{\partial t}\,\psi(t,\vec{r}) = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}(t,\vec{r})\right) + \beta mc^2 + q\phi(t,\vec{r})\right]\psi(t,\vec{r})$$

- Dirac equation with time-reversed electromagnetic field:

$$i\hbar\frac{\partial}{\partial t}\,\psi_T(t,\vec{r}) = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} + \frac{iq}{\hbar c}\vec{A}(-t,\vec{r})\right) + \beta mc^2 + q\phi(-t,\vec{r})\right]\psi_T(t,\vec{r})$$

- Dirac equation with electromagnetic field:

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{r}) = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}(t,\vec{r})\right) + \beta mc^2 + q\phi(t,\vec{r})\right]\psi(t,\vec{r})$$

- Dirac equation with time-reversed electromagnetic field:

$$i\hbar\frac{\partial}{\partial t}\psi_T(t,\vec{r}) = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} + \frac{iq}{\hbar c}\vec{A}(-t,\vec{r})\right) + \beta mc^2 + q\phi(-t,\vec{r})\right]\psi_T(t,\vec{r})$$

- Solution:

$$\boxed{\psi_T(t,\vec{r}) = \sigma^{13}\psi^*(-t,\vec{r}) = i\gamma^1\gamma^3\psi^*(-t,\vec{r})}$$