Interpretation by Feynman and Stückelberg

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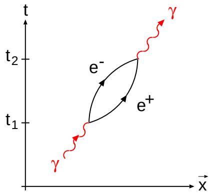
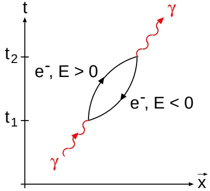
- consider the following process:
  - e^+e^- annihilation at time t_2
  - e^+e^- pair creation at time t_1



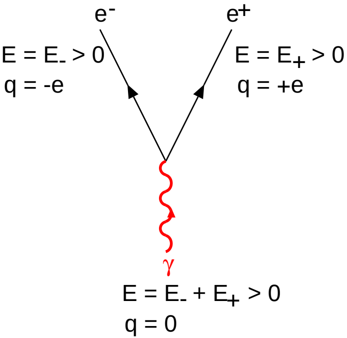
- hole theory:
  - e^+ = missing e^- with negative energy
  - creation of an e^+ = destruction of an e^- with negative energy
  - destruction of an e^+ = creation of an e^- with negative energy



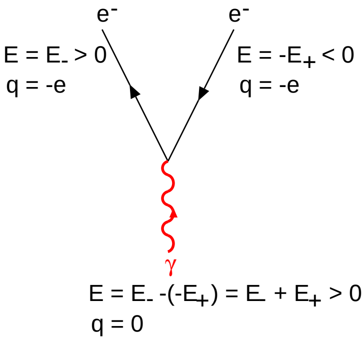
$\rightarrow$ The e^- with negative energy propagates **backward in time!**

Pair creation in detail

Physical process:



Feynman & Stückelberg:



energy and charge conserved. ✓

- Postulate:

  Particles with positive energy travel forward,
  particles with negative energy backward in time.
  The latter correspond to antiparticles with positive energy,
  which travel forward in time.

- In scattering theory this can be achieved by choosing appropriate boundary conditions ($\leftrightarrow$ $i\varepsilon$ prescriptions in the Green's functions).

- works for bosons, too!

3.11 Spinor transformations and the Lorentz covariance of the Dirac equation

- Goal:

  We want to show that the Dirac equation,

  $$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi(x) = 0,$$

  is form invariant under Lorentz transformationen $x'^\mu = \Lambda^\mu{}_\nu x^\nu$,
  i.e., that in the transformed frame it takes the form

  $$\left(i\gamma^\mu\partial'_\mu - \frac{mc}{\hbar}\right)\psi'(x') = 0 \qquad (\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu}).$$

- Remarks:

  - The γ matrices should not be transformed.
    In particular they are not the components of a four-vector
  - We should expect that the spinor does not behave like a scalar field (cf. Klein-Gordon: $\Phi'(x') = \Phi(x)$), but transforms in a non-trivial way:

    $$\psi'(x') = S(\Lambda)\psi(x), \quad S(\Lambda):\ \text{matrix in spinor space}$$

$$0 = \left(i\gamma^\mu\partial_\mu - \tfrac{mc}{\hbar}\right)\psi(x) \qquad\qquad \partial_\mu = \partial'_\nu\Lambda^\nu{}_\mu, \quad \psi(x) = S^{-1}(\Lambda)\psi'(x')$$

$$= \left(i\gamma^\mu\partial'_\nu\Lambda^\nu{}_\mu - \tfrac{mc}{\hbar}\right)S^{-1}(\Lambda)\psi'(x') \qquad \mid S(\Lambda)\times$$

$$= \left(iS(\Lambda)\gamma^\mu\partial'_\nu\Lambda^\nu{}_\mu S^{-1}(\Lambda) - \tfrac{mc}{\hbar}\right)\psi'(x')$$

$$= \left(iS(\Lambda)\Lambda^\nu{}_\mu\gamma^\mu S^{-1}(\Lambda)\,\partial'_\nu - \tfrac{mc}{\hbar}\right)\psi'(x')$$

$$\stackrel{!}{=} \left(i\gamma^\nu\partial'_\nu - \tfrac{mc}{\hbar}\right)\psi'(x') \qquad \Rightarrow \quad S(\Lambda)\Lambda^\nu{}_\mu\gamma^\mu S^{-1}(\Lambda) \stackrel{!}{=} \gamma^\nu$$

$$\Leftrightarrow \quad \boxed{S^{-1}(\Lambda)\,\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu\gamma^\mu}$$

- The right-hand side looks like the transformation of a 4-vector.

Explicit construction of S(Λ)

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- infinitesimal proper orthochronous Lorentz transformation:

  $\Lambda^{\nu}{}_{\mu} = g^{\nu}{}_{\mu} + \Delta\omega^{\nu}{}_{\mu}\,, \qquad \Delta\omega^{\nu}{}_{\mu}:$ infinitesimal

- We have (see section 3.2): $\Lambda_{\mu}{}^{\alpha}\Lambda^{\mu}{}_{\beta} = g^{\alpha}{}_{\beta}$

  $\Rightarrow \quad g^{\alpha}{}_{\beta} = \left(g_{\mu}{}^{\alpha} + \Delta\omega_{\mu}{}^{\alpha}\right)\left(g^{\mu}{}_{\beta} + \Delta\omega^{\mu}{}_{\beta}\right) = g^{\alpha}{}_{\beta} + \Delta\omega^{\alpha}{}_{\beta} + \Delta\omega_{\beta}{}^{\alpha} + \mathcal{O}(\Delta\omega^2)$

  $\Rightarrow \quad \Delta\omega^{\alpha}{}_{\beta} + \Delta\omega_{\beta}{}^{\alpha} = 0 \quad \Leftrightarrow \quad \Delta\omega^{\alpha\beta} + \Delta\omega^{\beta\alpha} = 0$

  ➞ $(\Delta\omega^{\alpha\beta})$ is an antisymmetric 4×4 matrix

  ➞ 6 independent components: $\Delta\omega^{01}, \Delta\omega^{02}, \Delta\omega^{03}, \Delta\omega^{23}, \Delta\omega^{13}, \Delta\omega^{12}$

  $\hat{=}$ 6 generators of the Lorentz group:

  - 3 boosts in *x*-, *y*- and *z* direction
  - 3 rotations around the *x*-, *y*- and *z* axis

- **Ansatz:** $S(\Lambda) = \mathbb{1} + \tau\,, \quad \tau =$ infinitesimal 4×4 matrix

  $\Rightarrow \quad S^{-1}(\Lambda) = \mathbb{1} - \tau \qquad (\Rightarrow\; S^{-1}S = \mathbb{1} + \mathcal{O}(\tau^2) \quad \checkmark)$

- $S^{-1}\gamma^\nu S \stackrel{!}{=} \Lambda^\nu{}_\mu \gamma^\mu$

  $\Rightarrow \quad (\mathbb{1} - \tau)\gamma^\nu(\mathbb{1} + \tau) \;=\; \left(g^\nu{}_\mu + \Delta\omega^\nu{}_\mu\right)\gamma^\mu$

  $\Leftrightarrow \quad \gamma^\nu + \gamma^\nu\tau - \tau\gamma^\nu \;=\; \gamma^\nu + \Delta\omega^\nu{}_\mu\gamma^\mu$

  $\Leftrightarrow \quad \left[\gamma^\nu, \tau\right] = \Delta\omega^\nu{}_\mu\gamma^\mu$

- **Solution:** $\tau = -\frac{i}{4}\Delta\omega^{\alpha\beta}\,\sigma_{\alpha\beta}$ with $\sigma_{\alpha\beta} \equiv \frac{i}{2}\left[\gamma_\alpha, \gamma_\beta\right]$

  Proof: explicit evaluation of $\left[\gamma^\nu, \tau\right]$ ($\rightarrow$ exercises)

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- infinitesimal transformations:

$$\boxed{S(\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \Delta\omega^{\mu}{}_{\nu}) = \mathbb{1} - \frac{i}{4}\Delta\omega^{\alpha\beta}\,\sigma_{\alpha\beta}}$$

- finite transformations

  = series of infinitely many infinitesimal transformations.

- 1st example: Boost along the x axis

$$\Lambda(\chi) \equiv (\Lambda^\mu{}_\nu) = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

divide χ into N intervals $\frac{\chi}{N}$, $N \to \infty$

$$\to \Lambda(\tfrac{\chi}{N}) = \begin{pmatrix} 1 & -\frac{\chi}{N} & 0 & 0 \\ -\frac{\chi}{N} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \tfrac{\chi}{N}\mathcal{I} \equiv (g^\mu{}_\nu) + (\Delta\omega^\mu{}_\nu)$$

with $\mathcal{I} \equiv (\mathcal{I}^\mu{}_\nu) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ($\equiv iK_x$, K_x: generator of the boost)

$$\Rightarrow \Lambda(\chi) = \lim_{N\to\infty} \Lambda^N(\tfrac{\chi}{N}) = \lim_{N\to\infty} \left(\mathbb{1} + \tfrac{\chi}{N}\mathcal{I}\right)^N = \exp(\chi\mathcal{I}) = \sum_{n=0}^{\infty} \frac{\chi^n}{n!}\mathcal{I}^n$$

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Spinor transformation:

$$S(\Lambda(\chi)) = \lim_{N\to\infty} S^N\left(\Lambda(\tfrac{\chi}{N})\right) = \lim_{N\to\infty}\left(1 - \tfrac{i}{4}\underbrace{\Delta\omega^{\alpha\beta}}_{\frac{\chi}{N}\mathcal{I}^{\alpha\beta}}\sigma_{\alpha\beta}\right)^N = \exp\left(-i\tfrac{\chi}{4}\mathcal{I}^{\alpha\beta}\,\sigma_{\alpha\beta}\right)$$

$$\mathcal{I}^{\alpha\beta}\,\sigma_{\alpha\beta} = \mathcal{I}^{01}\,\sigma_{01} + \mathcal{I}^{10}\,\sigma_{10} = 2\mathcal{I}^{01}\,\sigma_{01} = 2\mathcal{I}^{0}{}_{1}\,\sigma^{01} = -2\sigma^{01}$$

$$= -i\,[\gamma^0, \gamma^1] = -2i\,\gamma^0\gamma^1 = -2i\alpha^1$$

$$\Rightarrow\; S(\Lambda(\chi)) = \exp\left(-\tfrac{\chi}{2}\alpha^1\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\tfrac{\chi}{2}\right)^n\left(\alpha^1\right)^n, \quad \alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \Rightarrow \left(\alpha^1\right)^2 = \mathbb{1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(-\tfrac{\chi}{2}\right)^{2n}\mathbb{1} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\tfrac{\chi}{2}\right)^{2n+1}\alpha^1$$

$$= \cosh\left(\tfrac{\chi}{2}\right)\mathbb{1} - \sinh\left(\tfrac{\chi}{2}\right)\alpha^1 = \begin{pmatrix} \cosh\left(\frac{\chi}{2}\right) & -\sinh\left(\frac{\chi}{2}\right)\sigma^1 \\ -\sinh\left(\frac{\chi}{2}\right)\sigma^1 & \cosh\left(\frac{\chi}{2}\right) \end{pmatrix}$$

- 2nd example: rotation around the z axis

$$\Lambda(\varphi) \equiv (\Lambda^\mu{}_\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(for a passive transformation: rotation of the frame by φ)

analogous procedure $\longrightarrow$ $S(\Lambda(\varphi)) = \exp\left(\mathrm{i}\frac{\varphi}{2}\sigma^{12}\right) = \cos\left(\frac{\varphi}{2}\right) + \mathrm{i}\sin\left(\frac{\varphi}{2}\right)\,\sigma^{12}$

$$\sigma^{12} = \frac{\mathrm{i}}{2}[\gamma^1,\gamma^2] = \mathrm{i}\gamma^1\gamma^2 = \mathrm{i}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \equiv \Sigma^3$$

- interesting feature: $\varphi = 2\pi \;\Rightarrow\; S = -\mathbb{1} \;\Rightarrow\; \psi' = -\psi$

  $\varphi = 4\pi \;\Rightarrow\; S = +\mathbb{1} \;\Rightarrow\; \psi' = \psi$

  - related to spin $\frac{1}{2}$
  - observables: bilinear in ψ (e.g., $j^\mu = \overline{\psi}\gamma^\mu\psi$) $\Rightarrow$ sign drops out

- Rotation about an arbitrary axis:

$$S = \exp\left(\mathrm{i}\frac{\vec{\varphi}}{2}\cdot\vec{\Sigma}\right) = \cos\left(\frac{\varphi}{2}\right) + \mathrm{i}\sin\left(\frac{\varphi}{2}\right)\,\vec{\Sigma}\cdot\frac{\vec{\varphi}}{\varphi}$$

  - $\varphi = |\vec{\varphi}|$ = rotation angle, $\frac{\vec{\varphi}}{\varphi}$ = direction of the rotation axis
  - $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

- further properties of S:
  - rotations: $S^\dagger = S^{-1}$ (unitary)
  - boosts: $S^\dagger = S$ (hermitian)
  - both: $\gamma^0 S^\dagger \gamma^0 = S^{-1}$

$\longrightarrow\ \overline{\psi} = \psi^\dagger\gamma^0 \ \Rightarrow\ (S\psi)^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0 = \overline{\psi}\gamma^0 S^\dagger\gamma^0 = \overline{\psi}S^{-1}$