3.8 Nonrelativistic limit of the free Dirac equation

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- Dirac equation in non-covariant form: $i\hbar\frac{\partial}{\partial t}\psi = \left(\frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi$
- solutions with positive energy and $\vec{p} = \vec{0}$: $\psi = u_s(\vec{p} = \vec{0})\, e^{-\frac{i}{\hbar}mc^2 t}$
- small nonvanishing momenta: $\vec{p}^{\,2} \ll m^2c^2$

  $\Rightarrow \quad E - mc^2 \approx \frac{\vec{p}^{\,2}}{2m} \ll mc^2 \quad \longrightarrow$ ansatz: $\psi(t,\vec{x}) = e^{-\frac{i}{\hbar}mc^2 t}\begin{pmatrix}\varphi(t,\vec{x})\\ \chi(t,\vec{x})\end{pmatrix}$

  - φ, χ: two-component spinors,
    only slowly varying in time in comparison with $e^{-\frac{i}{\hbar}mc^2 t}$:

    $$|i\hbar\tfrac{\partial}{\partial t}\varphi| \ll |mc^2\,\varphi|\,, \quad |i\hbar\tfrac{\partial}{\partial t}\chi| \ll |mc^2\,\chi|$$

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$$\psi(t,\vec{x}) = e^{-\frac{i}{\hbar}mc^2 t}\begin{pmatrix}\varphi(t,\vec{x})\\ \chi(t,\vec{x})\end{pmatrix}$$

$$\Rightarrow \quad i\hbar\frac{\partial}{\partial t}\psi = \begin{pmatrix} mc^2\varphi + i\hbar\frac{\partial\varphi}{\partial t}\\ mc^2\chi + i\hbar\frac{\partial\chi}{\partial t}\end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t}$$

$$\stackrel{!}{=} \quad \left(\frac{\hbar c}{i}\,\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi = \left[\frac{\hbar c}{i}\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{\nabla}\\ \vec{\sigma}\cdot\vec{\nabla} & 0\end{pmatrix} + \begin{pmatrix} mc^2 & 0\\ 0 & -mc^2\end{pmatrix}\right]\psi$$

$$= \begin{pmatrix} mc^2\varphi + \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\chi\\ -mc^2\chi + \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\varphi\end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t}$$

$$\Rightarrow \quad i\hbar\frac{\partial\varphi}{\partial t} = \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\chi$$

$$i\hbar\frac{\partial\chi}{\partial t} = \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\varphi - 2mc^2\chi$$

$$\psi(t,\vec{x}) = e^{-\frac{i}{\hbar}mc^2 t}\begin{pmatrix}\varphi(t,\vec{x})\\ \chi(t,\vec{x})\end{pmatrix}$$

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$$\Rightarrow \quad i\hbar\frac{\partial\varphi}{\partial t} = \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\chi$$

$$i\hbar\frac{\partial\chi}{\partial t} = \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\varphi - 2mc^2\chi \quad \stackrel{|i\hbar\frac{\partial}{\partial t}\chi| \ll |mc^2\chi|}{\Rightarrow} \quad \chi \approx \frac{\hbar}{2imc}\vec{\sigma}\cdot\vec{\nabla}\varphi$$

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$$\psi(t,\vec{x}) = e^{-\frac{i}{\hbar}mc^2 t}\begin{pmatrix}\varphi(t,\vec{x})\\ \chi(t,\vec{x})\end{pmatrix}$$

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$$\Rightarrow \quad i\hbar\frac{\partial\varphi}{\partial t} = \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\chi \qquad\qquad \Rightarrow \quad i\hbar\frac{\partial\varphi}{\partial t} \approx -\frac{\hbar^2}{2m}\left(\vec{\sigma}\cdot\vec{\nabla}\right)^2\varphi$$

$$i\hbar\frac{\partial\chi}{\partial t} = \frac{\hbar c}{i}\vec{\sigma}\cdot\vec{\nabla}\varphi - 2mc^2\chi \quad \stackrel{|i\hbar\frac{\partial}{\partial t}\chi| \ll |mc^2\chi|}{\Rightarrow} \quad \chi \approx \frac{\hbar}{2imc}\vec{\sigma}\cdot\vec{\nabla}\varphi$$

$$i\hbar\frac{\partial\varphi}{\partial t} \approx -\frac{\hbar^2}{2m}\left(\vec{\sigma}\cdot\vec{\nabla}\right)^2\varphi$$

- reminder: $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

$$\Rightarrow \quad \boxed{i\hbar\frac{\partial}{\partial t}\varphi \approx -\frac{\hbar^2}{2m}\vec{\nabla}^2\varphi}$$

  - Schrödinger-like equation for the two-component spinor φ
    (→ Pauli spinor, spin $\frac{1}{2}$, see later)
  - plane waves: $\varphi \propto e^{-\frac{i}{\hbar}(E_{\text{nr}}t - \vec{p}\cdot\vec{x})}$, $E_{\text{nr}} = \frac{\vec{p}^{\,2}}{2m}$

$$\Rightarrow \quad \chi \approx \frac{\hbar}{2imc}\vec{\sigma}\cdot\vec{\nabla}\varphi = \frac{\vec{\sigma}\cdot\vec{p}}{2mc}\varphi \quad \overset{|\vec{p}|\ll mc}{\Rightarrow} \quad |\chi| \ll |\varphi|$$

→ sensible non-relativistic limit ✓

3.9 Dirac equation with elektromagnetic field

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- minimal substitution: $\partial_\mu \;\to\; D_\mu = \partial_\mu + \frac{iq}{\hbar c} A_\mu$

  insert into the free Dirac equation:

$$\Rightarrow \quad \boxed{\left(i\not{D} - \frac{mc}{\hbar}\right)\psi \;\equiv\; \left(i\not{\partial} - \frac{q}{\hbar c}\not{A} - \frac{mc}{\hbar}\right)\psi = 0}$$

- non-covariant form:

$$i\hbar\frac{\partial}{\partial t} \;\to\; i\hbar\frac{\partial}{\partial t} - q\phi\,, \qquad \frac{\hbar}{i}\vec{\nabla} \;\to\; \frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}$$

$$\Rightarrow \quad \left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\psi = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}\right) + \beta mc^2\right]\psi$$

$$\Leftrightarrow \quad i\hbar\frac{\partial}{\partial t}\psi = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}\right) + \beta mc^2 + q\phi\right]\psi$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\vec{\alpha}\cdot\frac{\hbar c}{i}\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}\right) + \beta mc^2 + q\phi\right]\psi$$

- nonrelativistic approximation + weak electric fields: $|\vec{p}|c,\ q\phi \ll mc^2$

→ ansatz as before: $\psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t}$

$$\Rightarrow \quad i\hbar\frac{\partial}{\partial t}\varphi = \left[-\frac{\hbar^2}{2m}\vec{\sigma}\cdot\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}\right)\vec{\sigma}\cdot\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}\right) + q\phi\right]\varphi$$

$$\stackrel{\text{exercises}}{\Rightarrow} \quad \boxed{i\hbar\frac{\partial}{\partial t}\varphi = \left[-\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}\right)^2 - \frac{\hbar q}{2mc}\vec{\sigma}\cdot\vec{B} + q\phi\right]\varphi} \qquad (\vec{B} = \vec{\nabla}\times\vec{A})$$

Pauli equation for nonrelativistic spin-$\frac{1}{2}$ particles in an electromagnetic field

- Simplifications:
  - homogeneous magnetic field: $\vec{B} = const.$
    (e.g., choose $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$)
  - weak magnetic field: neglect $\vec{B}^2$ terms

$$\overset{\text{exercises}}{\Rightarrow} \quad \boxed{i\hbar\frac{\partial}{\partial t}\varphi = \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{q}{2mc}\left(\vec{L}+2\vec{S}\right)\cdot\vec{B} + q\phi\right]\varphi}$$

with $\vec{L} = \vec{x} \times \vec{p} = \vec{x} \times \frac{\hbar}{i}\vec{\nabla}$ orbital angular momentum

$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ spin

Spin

- Reminder of nonrelativistic quantum mechanics:

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \quad \Rightarrow \quad S_z = \frac{\hbar}{2}\sigma_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{S}^{\,2} = \frac{\hbar^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = \frac{3}{4}\hbar^2 \mathbb{1} \;=\; s(s+1)\hbar^2 \mathbb{1} \quad \text{with } s = \frac{1}{2}$$

- simultaneous eigenstates of $\vec{S}^{\,2}$ and S_z:

$$|s = \tfrac{1}{2}, m_s = +\tfrac{1}{2}\rangle \equiv \varphi_\uparrow \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad S_z\varphi_\uparrow = \frac{\hbar}{2}\varphi_\uparrow$$

$$|s = \tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle \equiv \varphi_\downarrow \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad S_z\varphi_\downarrow = -\frac{\hbar}{2}\varphi_\downarrow$$

- We had: $i\hbar\frac{\partial}{\partial t}\varphi = \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{q}{2mc}\left(\vec{L}+2\vec{S}\right)\cdot\vec{B} + q\phi\right]\varphi$

→ magnetic interaction energy:

$$E_{\text{magn}} = -\frac{q}{2mc}\left(\vec{L}+2\vec{S}\right)\cdot\vec{B} \equiv -\vec{\mu}\cdot\vec{B}$$

- magnetic moment: $\vec{\mu} = \vec{\mu}_{\text{orb}} + \vec{\mu}_{\text{spin}}$
  - $\vec{\mu}_{\text{orb}} = \frac{q}{2mc}\vec{L}$
  - $\vec{\mu}_{\text{spin}} = \frac{q}{2mc}\,2\vec{S} \equiv \frac{q}{2mc}\,g\vec{S}$, $\quad g = 2$ "gyromagnetic ratio"

- experimental value: $g_{e^-} = 2\cdot(1.001\,159\,652\,180\,73(28))$

  ↑ QED corrections

  $g_p = 2\cdot 2.79$ not a pointlike "Dirac particle"

3.10 Interpretation of the solutions with negative energy

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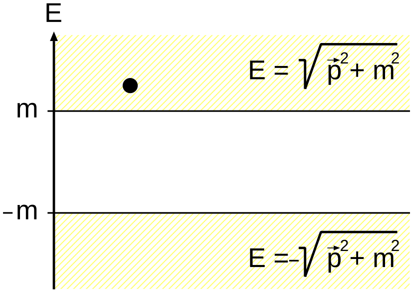
- Instability problem:



An electron could lower its energy further and further by emitting photons.

(all figures in this section in “natural units”: $c = 1$)

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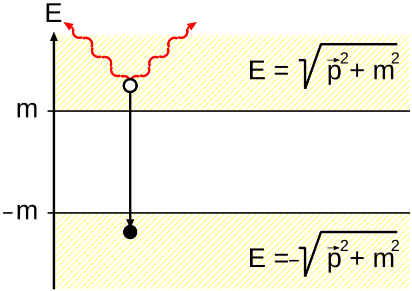

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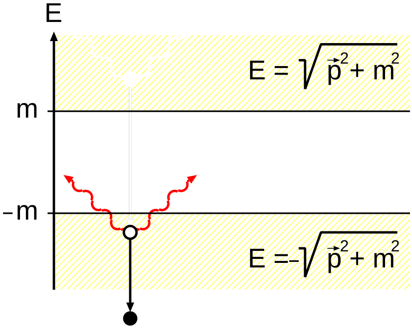

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$\Rightarrow$ All atoms would be unstable (live time τ = 0)!

Dirac's solution: "hole theory"

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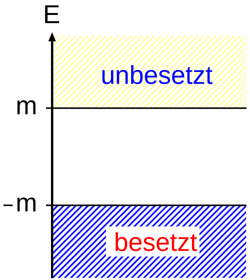
- physical vacuum ("Dirac sea"):
  - states with $E > 0$ empty
  - states with $E < 0$ occupied



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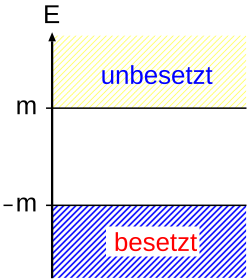

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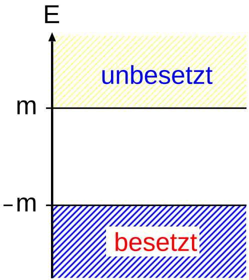

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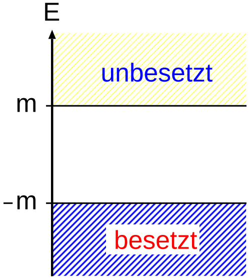


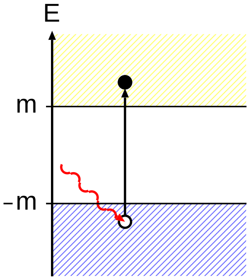
- occupied Dirac sea:
  - energy and charge of the vacuum = $-\infty$ (for electrons)
  - "renormalization":

    Energy and charge are measured relative to the occupied Dirac sea.

- Consequence:

  Adding an amount of energy $\Delta E > 2mc^2$ to the system (e.g., by radiation), an electron in the Dirac sea can be lifted into a positive-energy state:

  Creation of a "particle" with energy $E > 0$ and a "hole" in the Dirac sea



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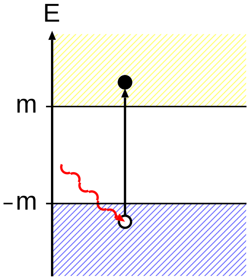

- Interpretation of the holes:

  missing particle with energy $E_h < 0$, momentum $\vec{p}_h$, spin s_h, charge q_h, ...

  = antiparticle w/ energy $-E_h > 0$, momentum $-\vec{p}_h$, spin $-s_h$, charge $-q_h$, ...

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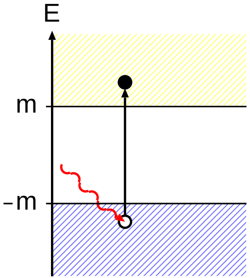

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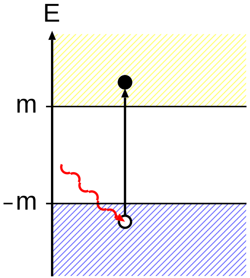

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  $\rightarrow$ pair creation: $\gamma^* \rightarrow e^+ e^-$

- inverse reaction: $e^+ e^- \rightarrow \gamma^*$ (pair annihilation)

  recombination of particle and hole

Discussion

- most spectacular success:

  Based on these considerations Dirac predicted the positron before its detection by Anderson 1932.

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  - However, the QFT vacuum has certain similarities to the Dirac sea. QFT extends this concept in a way that is also applicable for bosons.
  - The infinite many-body problem remains.

Fermi sea

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- The concept of the hole theory is still used in many-body theory, e.g., in condensed-matter physics or nuclear physics.

  There one considers particle-hole excitations in the Fermi sea, whereas the Dirac sea is often neglected.