# 3.7 Solutions of the free Dirac equation

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- linear independent solutions:

$$\psi^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t}, \quad \psi^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t},$$

$$\psi^{(3)}(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+\frac{i}{\hbar}mc^2 t}, \quad \psi^{(4)}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+\frac{i}{\hbar}mc^2 t}$$

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$\Rightarrow$ The problem of negative energies remains!

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(But Dirac found a way out, see later ... )

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### i) positive energies

Ansatz: $\psi(x) = \psi_p^{(+)}(x) \equiv \begin{pmatrix} \varphi(p) \\ \chi(p) \end{pmatrix} e^{-\frac{i}{\hbar}p\cdot x}$, $\quad E \equiv p^0 > 0$,

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

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$$i\partial\!\!\!/\, e^{-\frac{i}{\hbar}p\cdot x} = i\gamma^\mu \partial_\mu e^{-\frac{i}{\hbar}p_\nu x^\nu} = i\gamma^\mu \left(-\frac{i}{\hbar}p_\nu\, \delta^\nu_\mu\right) e^{-\frac{i}{\hbar}p\cdot x} = \frac{1}{\hbar}\, p\!\!\!/\, e^{-\frac{i}{\hbar}p\cdot x}$$

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$$\Rightarrow \quad 0 = \left(i\partial\!\!\!/ - \frac{mc}{\hbar}\right)\psi_p^{(+)}(x) = \frac{1}{\hbar}\left(p\!\!\!/ - mc\right)\psi_p^{(+)}(x)$$

$$\Rightarrow \quad \left(\not{p} - mc\right) \begin{pmatrix} \varphi(p) \\ \chi(p) \end{pmatrix} = 0$$

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limit $\vec{p} \to 0$: $\quad (E - mc^2)\, \varphi(E, \vec{0}) = (E + mc^2) \chi(E, \vec{0}) = 0$

$$E > 0 \quad \Rightarrow \quad E = mc^2, \quad \chi = 0 \quad \checkmark$$

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$$\Rightarrow \quad \left( E - mc^2 - \tfrac{(\vec{\sigma} \cdot \vec{p})^2 c^2}{E + mc^2} \right) \varphi(p) = 0$$

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→ two linear independent solutions:

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- corresponding lower components of the Dirac spinor: $\chi(p) \;=\; \frac{\vec{\sigma}\cdot\vec{p}c}{E+mc^2}\,\varphi(p)$
  (nonrelativistically suppressed → "small components")

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  - we still write: $p \cdot x = Et - \vec{p} \cdot \vec{x}$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} \psi_p^{(-)}(x) = -E \, \psi_p^{(-)}(x) \quad \to \quad \text{energy} = -E$$

$$\frac{\hbar}{i} \vec{\nabla} \psi_p^{(-)}(x) = -\vec{p} \, \psi_p^{(-)}(x) \quad \to \quad \text{momentum} = -\vec{p}$$

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$\Rightarrow$ lin. indep. solutions: $\chi_\uparrow = \mathcal{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\chi_\downarrow = \mathcal{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\varphi(p) = \frac{\vec{\sigma} \cdot \vec{p} c}{E + mc^2} \chi(p)$

## Summary of the solutions:

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- positive energy:

$$\psi^{(+)}_{p,s}(x) = u_s(\vec{p})\, e^{-\frac{i}{\hbar} p\cdot x}$$

$$u_{1,2}(\vec{p}) = \begin{pmatrix} \varphi_{\uparrow,\downarrow}(\vec{p}) \\ \frac{\vec{\sigma}\cdot\vec{p}c}{E+mc^2}\, \varphi_{\uparrow,\downarrow}(\vec{p}) \end{pmatrix}, \quad \varphi_\uparrow(\vec{p}) = \mathcal{N}(\vec{p}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_\downarrow(\vec{p}) = \mathcal{N}(\vec{p}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- negative energy:

$$\psi^{(-)}_{p,s}(x) = v_s(\vec{p})\, e^{\frac{i}{\hbar} p\cdot x}$$

$$v_{1,2}(\vec{p}) = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}c}{E+mc^2}\, \chi_{\uparrow,\downarrow}(\vec{p}) \\ \chi_{\uparrow,\downarrow}(\vec{p}) \end{pmatrix}, \quad \chi_\uparrow(\vec{p}) = \mathcal{N}(\vec{p}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_\downarrow(\vec{p}) = \mathcal{N}(\vec{p}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- in both cases: $p \cdot x = Et - \vec{p}\cdot\vec{x}$ mit $E = +\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$

- explicitly:

$$\vec{\sigma}\cdot\vec{p} \;=\; \sigma^1 p_x + \sigma^2 p_y + \sigma^3 p_z \;=\; \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$\Rightarrow \quad u_1(\vec{p}) \;=\; \mathcal{N}(\vec{p}) \begin{pmatrix} 1 \\ 0 \\ \frac{p_z c}{E+mc^2} \\ \frac{(p_x+ip_y)c}{E+mc^2} \end{pmatrix}, \quad u_2(\vec{p}) \;=\; \mathcal{N}(\vec{p}) \begin{pmatrix} 0 \\ 1 \\ \frac{(p_x-ip_y)c}{E+mc^2} \\ \frac{-p_z c}{E+mc^2} \end{pmatrix}$$

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# Further relations

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With the normalization

- $u_s(\vec{p})^\dagger u_s(\vec{p}) \;=\; v_s^\dagger(\vec{p}) v_s(\vec{p}) \;=\; \frac{E}{mc^2}$

one finds the orthonormalization relations

- $\bar{u}_r(\vec{p}) u_s(\vec{p}) \;=\; -\bar{v}_r(\vec{p}) v_s(\vec{p}) \;=\; \delta_{rs} \quad \Rightarrow \quad \boxed{\bar{u}_s(\vec{p}) u_s(\vec{p}) \;=\; 1, \quad \bar{v}_s(\vec{p}) v_s(\vec{p}) \;=\; -1}$
- $\bar{u}_r(\vec{p}) v_s(\vec{p}) \;=\; \bar{v}_r(\vec{p}) u_s(\vec{p}) \;=\; 0$
- $u_r(\vec{p})^\dagger u_s(\vec{p}) \;=\; v_r(\vec{p})^\dagger v_s(\vec{p}) \;=\; \frac{E}{mc^2}\,\delta_{rs}$
- $u_r(\vec{p})^\dagger v_s(-\vec{p}) \;=\; v_r(\vec{p})^\dagger u_s(-\vec{p}) \;=\; 0$

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analogously: $\Lambda_-(\vec{p}) = -\sum_{r=1}^{2} v_r(\vec{p})\bar{v}_r(\vec{p})$

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- Often one still needs to normalize the probability to find a particle within a certain volume (or the entire space) in a given frame:

  $$\int_V d^3x\, \psi^\dagger\psi \stackrel{!}{=} 1$$

  In this case one still keeps the normalization of the $u_s$ and $v_s$ and introduces additional normalization factors.

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$$\psi(x) = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{mc^2}{E} \sum_{s=1}^{2} \left( b(\vec{p}, s)\, u_s(\vec{p})\, e^{-\frac{i}{\hbar} p\cdot x} + d^*(\vec{p}, s)\, v_s(\vec{p})\, e^{\frac{i}{\hbar} p\cdot x} \right)$$

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