- $N = 2 \;\Rightarrow\;$ only $N^2 - 1 = 3$ such matrices,

  e.g., Pauli matrices: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- lowest possible dimension: $N = 4$

- one choice ("standard representation", "Dirac representation"):

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \text{(with } 2 \times 2 \text{ blocks)}$$

- There is an infinite number of possible representations.

- Observables do not depend on the representation.

- $H_D = \left(\frac{\hbar c}{i}\alpha^k\partial_k + \beta mc^2\right)$, α^k, β: 4×4 matrices

  → wave function $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ "Dirac spinor"

- not a four-vector!

- Although the number of components depends on the space-time dimension, they are in general not identical:

  D space-time dimension

  → D linear independent traceless hermitian $N \times N$ matrices

  → $N^2 - 1 \geq D$

  example: $D = 3$ → $N = 2$ sufficient, e.g., α^k, β: Pauli matrices

Continuity equation

- Dirac equation: $i\hbar\frac{\partial}{\partial t}\psi = \left(\frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi$
  - $\vec{\alpha}\cdot\vec{\nabla} \equiv \alpha^k\partial_k$
- adjoint equation: $-i\hbar\frac{\partial}{\partial t}\psi^\dagger = \left(-\frac{\hbar c}{i}(\vec{\nabla}\psi^\dagger)\cdot\vec{\alpha} + mc^2\psi^\dagger\beta\right)$
  - hermitian adjoint spinor: $\psi^\dagger = \left(\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*\right)$

Continuity equation

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- Dirac equation: $i\hbar\frac{\partial}{\partial t}\psi = \left(\frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi \qquad \Big| \quad \frac{1}{i\hbar}\psi^\dagger \times$
  - $\vec{\alpha}\cdot\vec{\nabla} \equiv \alpha^k\partial_k$
- adjoint equation: $-i\hbar\frac{\partial}{\partial t}\psi^\dagger = \left(-\frac{\hbar c}{i}(\vec{\nabla}\psi^\dagger)\cdot\vec{\alpha} + mc^2\psi^\dagger\beta\right) \qquad \Big| \quad \times -\frac{1}{i\hbar}\psi$
  - hermitian adjoint spinor: $\psi^\dagger = \left(\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*\right)$

$$\Rightarrow \quad \psi^\dagger\frac{\partial}{\partial t}\psi + \left(\frac{\partial}{\partial t}\psi^\dagger\right)\psi = -c\,\psi^\dagger\vec{\alpha}\cdot\vec{\nabla}\psi - c\,(\vec{\nabla}\psi^\dagger)\cdot\vec{\alpha}\,\psi$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t}\left(\psi^\dagger\psi\right) = -\vec{\nabla}\cdot\left(c\,\psi^\dagger\vec{\alpha}\,\psi\right)$$

$\longrightarrow$ continuity equation: $\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$ with $\rho = \psi^\dagger\psi$, $\vec{j} = c\,\psi^\dagger\vec{\alpha}\,\psi$

Probability interpretation

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- probability density:

$$\rho = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0 \qquad \text{always positive} \checkmark$$

- probability current density: $\vec{j} = c\,\psi^\dagger\,\vec{\alpha}\,\psi$

- interpretation:

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{1}{i\hbar}\left[\vec{x}, H\right] \qquad \text{(Heisenberg picture)}$$

evaluate for $H = H_\mathrm{D}$:

$$v^k = \frac{1}{i\hbar}\left[x^k, \frac{\hbar c}{i}\,\alpha^l \partial_l + \beta m c^2\right] = -c\alpha^l \underbrace{\left[x^k, \partial_l\right]}_{=-\delta^k_l} = c\alpha^k$$

$$\Rightarrow \quad \vec{v} = c\vec{\alpha}$$

$$\Rightarrow \quad \vec{j} = \psi^\dagger\,\vec{v}\,\psi \qquad \text{(cf. classical point charge: } \vec{j} = \rho\vec{v}\text{)}$$

3.6 Dirac equation in covariant form

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- Dirac equation: $i\hbar \frac{\partial}{\partial t} \psi = \left(\frac{\hbar c}{i} \vec{\alpha} \cdot \vec{\nabla} + \beta m c^2 \right) \psi$

$$\Leftrightarrow \left(\frac{i}{c} \frac{\partial}{\partial t} + i\alpha^k \partial_k - \frac{mc}{\hbar} \beta \right) \psi = 0$$

3.6 Dirac equation in covariant form

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- Dirac equation: $i\hbar \frac{\partial}{\partial t} \psi = \left(\frac{\hbar c}{i} \vec{\alpha} \cdot \vec{\nabla} + \beta m c^2 \right) \psi$

$$\Leftrightarrow \quad \left(i\partial_0 + i\alpha^k \partial_k - \tfrac{mc}{\hbar}\beta \right) \psi = 0 \qquad | \, \beta \times$$

$$\Leftrightarrow \quad \left(i\beta\partial_0 + i\beta\alpha^k \partial_k - \tfrac{mc}{\hbar} \right) \psi = 0$$

- Def.: $\boxed{\gamma^0 \equiv \beta , \qquad \gamma^k \equiv \beta\alpha^k}$ "γ matrices"

$$\Rightarrow \quad \left(i\gamma^0\partial_0 + i\gamma^k \partial_k - \tfrac{mc}{\hbar} \right) \psi = 0$$

$$\Leftrightarrow \quad \boxed{\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right) \psi = 0}$$ Dirac equation in covariant form

(We still have to investigate its exact transformation properties.)

- "Feynman slash": $\not{a} \equiv \gamma^\mu a_\mu$ for arbitrary four-vectors (a^μ)

$$\Rightarrow \quad \boxed{\left(i\not{\partial} - \frac{mc}{\hbar} \right) \psi = 0}$$

γ matrices

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- Explicit form in Dirac representation:

$$\gamma^0 = \beta = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$$

$$\gamma^k = \beta\alpha^k = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

- General properties (represenation independent):
  - Hermiticity: $\alpha^{k\dagger} = \alpha^k, \quad \beta^\dagger = \beta$

    $\Rightarrow \quad \gamma^{0\dagger} = \beta^\dagger = \beta = \gamma^0$ hermitian

    $\gamma^{k\dagger} = \left(\beta\alpha^k\right)^\dagger = \alpha^{k\dagger}\beta^\dagger = \alpha^k\beta = -\beta\alpha^k = -\gamma^k$ anti-hermitian

  - Anti-commutator relations:

$$\{\alpha^k, \alpha^l\} = 2\delta^{kl}, \; \{\alpha^k, \beta\} = 0, \; \beta^2 = 1 \quad \Rightarrow \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Continuity equation

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Dirac equation: $\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right) \psi = 0$

$$\Rightarrow \quad 0 = -i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} - \psi^\dagger \frac{mc}{\hbar} \equiv \psi^\dagger \left(-i\gamma^{\mu\dagger} \overleftarrow{\partial}_\mu - \frac{mc}{\hbar}\right) \quad \Big| \times \gamma^0$$

$$\Rightarrow \quad \psi^\dagger \left(-i\gamma^{\mu\dagger}\gamma^0 \overleftarrow{\partial}_\mu - \frac{mc}{\hbar}\gamma^0\right) = 0$$

$$\gamma^{\mu\dagger}\gamma^0 = \left\{ \begin{array}{lll} \gamma^{0\dagger}\gamma^0 = & \gamma^0\gamma^0 & \\ \gamma^{k\dagger}\gamma^0 = & -\gamma^k\gamma^0 & = \gamma^0\gamma^k \end{array} \right\} = \gamma^0\gamma^\mu$$

$$\Rightarrow \quad \psi^\dagger\gamma^0 \left(-i\gamma^\mu \overleftarrow{\partial}_\mu - \frac{mc}{\hbar}\right) = 0$$

Def.: $\overline{\psi} \equiv \psi^\dagger\gamma^0$ "adjoint spinor"

$$\Rightarrow \quad \boxed{\overline{\psi}\left(-i\gamma^\mu \overleftarrow{\partial}_\mu - \frac{mc}{\hbar}\right) = 0}$$ adjoint Dirac equation

- Dirac equation: $\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi = 0$
- Adjoint equation: $\bar{\psi}\left(-i\gamma^\mu\overleftarrow{\partial}_\mu - \frac{mc}{\hbar}\right) = 0$

$$\Rightarrow \quad 0 = \bar{\psi}\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi - \bar{\psi}\left(-i\gamma^\mu\overleftarrow{\partial}_\mu - \frac{mc}{\hbar}\right)\psi = i\bar{\psi}\gamma^\mu\left(\partial_\mu + \overleftarrow{\partial}_\mu\right)\psi$$

$$= i\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right)$$

→ Continuity equation in covariant form: $\boxed{\partial_\mu j^\mu(x) = 0}$

with the conserved 4-current $\boxed{j^\mu(x) = c\,\bar{\psi}(x)\gamma^\mu\psi(x)}$

$$\Rightarrow \quad \rho = \frac{1}{c}j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi \quad \checkmark$$

$$j^k = c\bar{\psi}\gamma^k\psi = c\psi^\dagger\gamma^0\gamma^0\alpha^k\psi = c\psi^\dagger\alpha^k\psi \quad \checkmark$$