- zero-component: $\rho = \frac{1}{c} j^0 = \frac{i\hbar}{2mc^2} \left(\Phi^* \frac{\partial}{\partial t} \Phi - \Phi \frac{\partial}{\partial t} \Phi^* \right)$
- plane wave: $\Phi(t, \vec{x}) = \mathcal{N} e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})}$

  $\Rightarrow \quad \rho = |\mathcal{N}|^2 \, \frac{E}{mc^2}$

  - nonrelativistic limit: $E \to mc^2 \quad \Rightarrow \quad \rho \to |\mathcal{N}|^2 \quad \checkmark$
  - relativistic behavior: zero component of a 4-vector

    $\rho \propto E$ $\leftrightarrow$ Lorentz contraction of the volume $\leftrightarrow$ larger density

Fundamental problems

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- relativistic energy-momentum relation: $E^2 = m^2c^4 + \vec{p}^{\,2}c^2$

  $\Rightarrow$ solution with positive and with negative energy: $E = \pm\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$

  both consistent with the Klein-Gordon equation

$\Rightarrow$ The energy spectrum is not bounded from below:

  - negative energies with arbitrarily large $|E|$ possible
  - E can be lowered by increasing the momentum.

$\Rightarrow$ stability problems!

- probability interpretation

  plane wave: $\rho = |\mathcal{N}|^2 \frac{E}{mc^2}$ negative for $E < 0$

- Resolution of the problems only within quantum field theory

  (negative energies $\leftrightarrow$ antiparticles, ρ = charge density)

3.4 Klein-Gordon equation with electromagnetic field

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- So far: free Klein-Gordon equation (no interactions)
- Including interactions:
  - Schrödinger: add a potential
  - similar procedure in the Klein-Gordon theory:
    $$-\hbar^2 \frac{\partial^2}{\partial t^2} \Phi(t,\vec{x}) = \left(-\hbar^2 c^2 \vec{\nabla}^2 + \left(mc^2 + V(t,\vec{x})\right)^2\right) \Phi(t,\vec{x})$$
  - requirements for a relativistic theorie:
    correct behavior under Lorentz transformations $x \to x' = \Lambda x$
  - fulfilled for scalar fields $V(t,\vec{x}) = V(x)$: $V'(x) = V(\Lambda^{-1} x)$
- Coulomb potential: $\leftrightarrow$ zero-component of the 4-potential $(A^\mu) = \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix}$
  - fieldstrength tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \rightarrow \vec{E}$- and $\vec{B}$ fields

Treatment of electromagnetic fields in classical mechanics

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- Equation of motion of a point charge (Lorentz force):

$$m\ddot{\vec{x}} = \vec{F}_L = q\left(\vec{E} + \frac{\dot{\vec{x}}}{c} \times \vec{B}\right), \qquad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$\hat{=}$ Euler-Lagrange equations $\quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}^k} - \dfrac{\partial L}{\partial x^k} = 0$

for the Lagrange function $\quad L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} m\dot{\vec{x}}^{\,2} - q\phi(t, \vec{x}) + \frac{q}{c}\,\dot{\vec{x}} \cdot \vec{A}(t, \vec{x})$

- Conjugate momentum: $\quad p^k \equiv \dfrac{\partial L}{\partial \dot{x}^k} = m\dot{x}^k + \dfrac{q}{c} A^k$

- Hamilton function: $\quad H \equiv \dot{\vec{x}} \cdot \vec{p} - L = \frac{1}{2} m\dot{\vec{x}}^{\,2} + q\phi = \dfrac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} + q\phi$

$$\Leftrightarrow \quad H - q\phi = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m}$$

without electromagnetic field: $E = H = \frac{\vec{p}^{\,2}}{2m}$

$\Rightarrow$ effect of the electromagnetic field:

$E \to E - q\phi\,, \qquad \vec{p} \to \vec{p} - \frac{q}{c}\vec{A}$ "minimal substitution"

- covariant notation: $p^\mu \to p^\mu - \frac{q}{c}A^\mu$

- analogous substitution in quantum mechanics:

$i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} - q\phi\,, \qquad \frac{\hbar}{i}\vec{\nabla} \to \frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}$

- covariant notation: $D_\mu = \partial_\mu + \frac{iq}{\hbar c}A_\mu$ "covariant derivative"[1]

[1] The name has nothing to do with Lorentz covariance but with the behavior under gauge transformations.

→ Klein-Gordon equation with electromagnetic field:

$$\left[D_\mu D^\mu + \left(\frac{mc}{\hbar}\right)^2\right]\Phi(x) = 0$$

- explicitly:

$$\left[\left(\partial_\mu + \frac{iq}{\hbar c}A_\mu\right)\left(\partial^\mu + \frac{iq}{\hbar c}A^\mu\right) + \left(\frac{mc}{\hbar}\right)^2\right]\Phi(x) = 0$$

- non explicitly Lorentz invariant form:

$$\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)^2\Phi(t, x^k) = \left[\left(\frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2 c^2 + m^2c^4\right]\Phi(t, x^k)$$

- conserved 4-current: (→ exercises)

$$j^\mu(x) = \frac{i\hbar}{2m}\left(\Phi^*(x)\partial^\mu\Phi(x) - \Phi(x)\partial^\mu\Phi^*(x)\right) - \frac{q}{mc}\Phi^*(x)\Phi(x)A^\mu(x)$$

3.5 The Dirac Gleichung

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- Schrödinger equation:

  $E = \frac{\vec{p}^{\,2}}{2m} + V \;\; \longrightarrow$ differential eq. 1st order in t, 2nd order in $\vec{x}$

  $\Rightarrow$ E bounded from below, but equation not Lorentz covariant

- Klein-Gordon-equation:

  $E^2 = \vec{p}^{\,2}c^2 + m^2c^4 \;\; \longrightarrow$ 2nd order in t and $\vec{x}$

  $\Rightarrow$ Lorentz covariant, but $E = \pm\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$ not bounded from below

- Alternatives?

- Restriction to positive square root: $E = +\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$

  $$\Rightarrow \quad i\hbar\frac{\partial}{\partial t}\psi = \sqrt{m^2c^4 - \hbar^2c^2\vec{\nabla}^2}\,\psi = \left(mc^2 - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{\hbar^4}{8m^3c^2}\vec{\nabla}^4 - \ldots\right)\psi$$

  $\longrightarrow$ non-local theory (gradients of all orders)

  $\longrightarrow$ violates causality (propagation of signals with $v > c$)

- Ansatz by Dirac: 1st order in time and spatial coordinates

$$i\hbar \frac{\partial}{\partial t}\, \psi(t, x^k) \;=\; H_{\rm D}\, \psi(t, x^k) \;\equiv\; \left(\frac{\hbar c}{i}\, \alpha^k \partial_k + \beta m c^2 \right)\, \psi(t, x^k)$$

  Dirac equation in non-explicitly covariant form (1928)

  - α^k, β: constants to be determined
  - Invariance under rotations:
    At least α^k cannot be simple numbers ➞ matrices

- Apply the equation twice:

$$\left(i\hbar \tfrac{\partial}{\partial t}\right)^2 \psi = \; i\hbar \tfrac{\partial}{\partial t}\, H_{\rm D}\, \psi = \; H_{\rm D}^2\, \psi = \; \left(\tfrac{\hbar c}{i}\, \alpha^k \partial_k + \beta m c^2\right) \left(\tfrac{\hbar c}{i}\, \alpha^l \partial_l + \beta m c^2\right)\, \psi$$

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2\psi = \left(\frac{\hbar c}{i}\,\alpha^k\partial_k + \beta mc^2\right)\left(\frac{\hbar c}{i}\,\alpha^l\partial_l + \beta mc^2\right)\psi$$

$$= \left(-\hbar^2c^2\,\alpha^k\alpha^l\,\partial_k\partial_l + \frac{\hbar c}{i}\,mc^2\,(\alpha^k\beta + \beta\alpha^k)\partial_k + m^2c^4\beta^2\right)\psi$$

$$= \left(-\frac{1}{2}\hbar^2c^2\,\left\{\alpha^k,\alpha^l\right\}\,\partial_k\partial_l + \frac{\hbar c}{i}\,mc^2\,\left\{\alpha^k,\beta\right\}\partial_k + m^2c^4\beta^2\right)\psi$$

with the anti-commutator $\{A, B\} \equiv AB + BA$.

- Explanation for the last step:

$$\alpha^k\alpha^l\,\partial_k\partial_l \equiv \sum_{k,l=1}^{3}\alpha^k\alpha^l\,\partial_k\partial_l = \frac{1}{2}\sum_{k,l=1}^{3}\left(\alpha^k\alpha^l\,\partial_k\partial_l + \alpha^l\alpha^k\,\partial_l\partial_k\right)$$

$$= \frac{1}{2}\sum_{k,l=1}^{3}\left(\alpha^k\alpha^l + \alpha^l\alpha^k\right)\partial_k\partial_l \equiv \frac{1}{2}\left\{\alpha^k,\alpha^l\right\}\partial_k\partial_l$$

- Hence:

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2\psi = \left(-\tfrac{1}{2}\hbar^2c^2\,\left\{\alpha^k,\alpha^l\right\}\,\partial_k\partial_l + \frac{\hbar c}{i}\,mc^2\,\left\{\alpha^k,\beta\right\}\partial_k + m^2c^4\beta^2\right)\psi$$

- Apply to plane wave $\psi(t,\vec{x}) = \psi_0\,e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{x})}$:

$$E^2\psi = \left(\tfrac{1}{2}\left\{\alpha^k,\alpha^l\right\}p_kp_l\,c^2 + \left\{\alpha^k,\beta\right\}p_k mc^3 + m^2c^4\beta^2\right)\psi$$

$$\stackrel{!}{=} \left(\vec{p}^{\,2}c^2 + m^2c^4\right)\psi$$

$\Rightarrow$ α^k and β are matrices with

$$\boxed{\left\{\alpha^k,\alpha^l\right\} = 2\delta^{kl}, \qquad \left\{\alpha^k,\beta\right\} = 0, \qquad \beta^2 = 1}$$

Determination of α^k und β

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- Properties:
  - i) H_D hermitian $\Rightarrow$ α^k, β hermitian
  - ii) ${\alpha^k}^2 = \beta^2 = 1$ $\Rightarrow$ eigenvalues = ± 1
  - iii) $\mathrm{tr}\,\alpha^k = \mathrm{tr}\,\beta = 0$

    e.g.: $\mathrm{tr}\,\alpha^k \overset{\beta^2=1}{=} \mathrm{tr}[\alpha^k\beta\beta] \overset{\text{cycl. perm.}}{=} \mathrm{tr}[\beta\alpha^k\beta] \overset{\{\alpha^k,\beta\}=0}{=} -\mathrm{tr}[\alpha^k\beta\beta] = -\mathrm{tr}\,\alpha^k$

    $\Rightarrow$ $\mathrm{tr}\,\alpha^k = 0$
  - iv) Since the different α^k anticommute with each other and with β, they must all be linear independent.

  $\Rightarrow$ Search for four linear independent hermitian traceless matrices.
- in general: $N^2 - 1$ linear independent hermitian traceless $N \times N$ matrices
- trace = sum over the eigenvalues $\Rightarrow$ N even