Lorentz transformations

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- The laws of nature are invariant under Poincaré transformations

  = inhomogeneous Lorentz transformations: $$\boxed{x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}}$$

  = (homogeneous) Lorentz transformations + translations in space and time

- Lorentz transformations: $$\boxed{x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}}$$

  $$= \underbrace{\text{Lorentz boosts + rotations}}_{\text{proper orhochronous Lorentz transformations}} \quad + \text{parity transformations + time reversal}$$

- Example: Boost along the x axis

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Lorentz transformation of the covariant components:

$$\boxed{x'_\mu = \Lambda_\mu{}^\nu\, x_\nu} \qquad (= \text{definition of } \Lambda_\mu{}^\nu)$$

$$= g_{\mu\alpha}\, x'^\alpha = g_{\mu\alpha}\, \Lambda^\alpha{}_\beta\, x^\beta = g_{\mu\alpha}\, \Lambda^\alpha{}_\beta\, g^{\beta\nu}\, x_\nu \quad \Rightarrow \quad \boxed{\Lambda_\mu{}^\nu = g_{\mu\alpha}\, \Lambda^\alpha{}_\beta\, g^{\beta\nu}}$$

i.e., the indices of Λ can be raised and lowered by the metric tensor as well.

- Invariance of the proper time:

$$\Lambda_\mu{}^\nu\, x_\nu\, \Lambda^\mu{}_\lambda\, x^\lambda = x'_\mu\, x'^\mu \stackrel{!}{=} x_\nu\, x^\nu = x_\nu\, g^\nu{}_\lambda\, x^\lambda \quad \Rightarrow \quad \boxed{\Lambda_\mu{}^\nu\, \Lambda^\mu{}_\lambda = g^\nu{}_\lambda = \delta^\nu_\lambda}$$

- backtransformation:

$$x^\nu = g^\nu{}_\lambda\, x^\lambda = \Lambda_\mu{}^\nu\, \Lambda^\mu{}_\lambda\, x^\lambda = \Lambda_\mu{}^\nu\, x'^\mu \quad \Rightarrow \quad \boxed{x^\nu = x'^\mu\, \Lambda_\mu{}^\nu}$$

analogously: $\boxed{x_\nu = x'_\mu\, \Lambda^\mu{}_\nu}$

Four-vectors

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- Contra- and covariant four-vectors

  = objects a with four components which under Lorentz transformations behave like (x^μ) or (x_μ), respectively:

$$\begin{aligned} a'^{\mu} &= \Lambda^{\mu}{}_{\nu}\, a^{\nu}\,, & a^{\nu} &= a'^{\mu}\, \Lambda_{\mu}{}^{\nu}\,, \\ a'_{\mu} &= \Lambda_{\mu}{}^{\nu}\, a_{\nu}\,, & a_{\nu} &= a'_{\mu}\, \Lambda^{\mu}{}_{\nu}\,, \end{aligned}$$

- Scalar product: $a \cdot b \equiv a^\mu b_\mu = a_\mu b^\mu$

  $\Rightarrow \quad a' \cdot b' = a \cdot b \qquad$ (because $x'^{\,\mu} x'_\mu = x^\mu x_\mu$)

- Example: four-momentum $p = (p^\mu) = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$

  with the relativistic energy $E = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$

  $\Rightarrow \quad p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2 \quad \checkmark$

- Four-gradients:

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}} = \Lambda_{\mu}{}^{\nu}\frac{\partial}{\partial x^{\nu}} \quad \text{like } x_{\mu} \quad \rightarrow \quad \text{covariant} \quad \rightarrow \quad \frac{\partial}{\partial x^{\mu}} \equiv \partial_{\mu}$$

$$\frac{\partial}{\partial x'_{\mu}} = \frac{\partial x_{\nu}}{\partial x'_{\mu}}\frac{\partial}{\partial x_{\nu}} = \Lambda^{\mu}{}_{\nu}\frac{\partial}{\partial x_{\nu}} \quad \text{like } x^{\mu} \quad \rightarrow \quad \text{contravariant} \quad \rightarrow \quad \frac{\partial}{\partial x_{\mu}} \equiv \partial^{\mu}$$

- Relation to the usual three-gradient:

$$\nabla^{k} = \frac{\partial}{\partial x^{k}} = -\frac{\partial}{\partial x_{k}}$$

$$\Rightarrow \quad (\partial_{\mu}) = \begin{pmatrix} \frac{\partial}{\partial x^{0}} \\ (\frac{\partial}{\partial x^{k}}) \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla} \end{pmatrix}, \quad (\partial^{\mu}) = \begin{pmatrix} \frac{\partial}{\partial x_{0}} \\ (\frac{\partial}{\partial x_{k}}) \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\vec{\nabla} \end{pmatrix}$$

- d'Alambert operator: $\Box \equiv \partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ Lorentz scalar!

- Tensors of rank 2: $A'^{\,\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\,A^{\alpha\beta}$ (like $x^{\mu}x^{\nu}$)

  Examples: fieldstrength tensor $F^{\mu\nu}$, $g^{\mu\nu}$, $\Lambda^{\mu}{}_{\nu}$

- Tensors of rank n: $A'^{\,\mu_1\dots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\dots\Lambda^{\mu_n}{}_{\nu_n}\,A^{\nu_1\dots\nu_n}$

Classification of Lorentz transformations

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- Using $\Lambda_\mu{}^\nu \, \Lambda^\mu{}_\lambda \; = \; \delta^\nu_\lambda$ (see above) one can show:
  - $\det(\Lambda_\mu{}^\nu) = \pm 1$
  - $\Lambda^0{}_0 \; \geq \; 1 \quad$ or $\quad \Lambda^0{}_0 \; \leq \; -1$

→ classification of Lorentz transformations by the sign of $\det(\Lambda_\mu{}^\nu)$ and $\Lambda^0{}_0$

- Boosts and rotations: $\quad \det(\Lambda_\mu{}^\nu) = +1, \quad \Lambda^0{}_0 \; \geq \; 1$

  (can be generated continuously from the identity)

- parity transform.: $\quad x' = \begin{pmatrix} ct' \\ \vec{x}\,' \end{pmatrix} = \begin{pmatrix} ct \\ -\vec{x}\,' \end{pmatrix} \quad \Rightarrow \quad (\Lambda^\mu{}_\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

  $\Rightarrow \quad \det(\Lambda_\mu{}^\nu) = -1, \quad \Lambda^0{}_0 \; \geq \; 1$

- time reversal: $\quad x' = \begin{pmatrix} ct' \\ \vec{x}\,' \end{pmatrix} = \begin{pmatrix} -ct \\ \vec{x}\,' \end{pmatrix} \quad \Rightarrow \quad (\Lambda^\mu{}_\nu) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  $\Rightarrow \quad \det(\Lambda_\mu{}^\nu) = -1, \quad \Lambda^0{}_0 \; \leq \; -1$

3.3 The Klein-Gordon equation

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- Analogous procedure to the “derivation” of the Schrödinger equation:
  - relativistic energy-momentum relation: $E^2 = \vec{p}^{\,2}c^2 + m^2c^4$
  - replace by operators: $E \to i\hbar\frac{\partial}{\partial t}, \quad \vec{p} \to \frac{\hbar}{i}\vec{\nabla}$

$$\Rightarrow \quad \boxed{-\hbar^2\frac{\partial^2}{\partial t^2}\Phi(\vec{r},t) = \left(-\hbar^2c^2\vec{\nabla}^2 + m^2c^4\right)\Phi(\vec{r},t)}$$

“Klein-Gordon equation” (Schrödinger, Fock, Klein, Gordon 1926)

  - partial diff. equation of 2nd order in time and spatial derivatives
    (Schrödinger equation: 1st order in time, 2nd order in space)

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  - partial diff. equation of 2nd order in time and spatial derivatives (Schrödinger equation: 1st order in time, 2nd order in space)

- Lorentz invariant form:

$$\Leftrightarrow \quad \boxed{\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + \left(\frac{mc}{\hbar}\right)^2\right] \Phi(\vec{r}, t) = 0}$$

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- Lorentz invariant form:

$$\Leftrightarrow \quad \boxed{\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\Phi(x) = 0} \qquad \frac{\hbar}{mc} = \frac{\hbar c}{mc^2} \equiv \lambda_C \quad \text{“Compton wavelength”}$$

- plane-wave ansatz for the solution: $\Phi(t,\vec{x}) = \mathcal{N} e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})}$

$$\Rightarrow \quad \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + \left(\frac{mc}{\hbar}\right)^2\right]\Phi(t,\vec{x}) = \left[-\frac{E^2}{\hbar^2 c^2} + \frac{\vec{p}^{\,2}}{\hbar^2} + \frac{m^2c^2}{\hbar^2}\right]\Phi(t,\vec{x}) \overset{!}{=} 0$$

$\Rightarrow \quad E^2 = m^2c^4 + \vec{p}^{\,2}c^2$ relativistic energy-momentum relation $\checkmark$

- using four-vectors:

$$x = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}, \quad p = \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix} \quad \Rightarrow \quad p\cdot x = Et - \vec{p}\cdot\vec{x}$$

$\Rightarrow$ The solution is Lorentz invariant: $\Phi(t,\vec{x}) = \mathcal{N} e^{-\frac{i}{\hbar}p\cdot x} \equiv \Phi(x)$

insert into the Klein-Gordon equation:

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\Phi(x) = \left[\partial_\mu\partial^\mu + \left(\frac{mc}{\hbar}\right)^2\right]\Phi(x) = \left[-\frac{p^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2}\right]\Phi(x) \overset{!}{=} 0$$

$$\Rightarrow \quad p^2 = \frac{E^2}{c^2} - \vec{p}^{\,2} \overset{!}{=} m^2c^2 \quad \Leftrightarrow \quad E^2 = m^2c^4 + \vec{p}^{\,2}c^2 \quad \checkmark$$

Current conservation

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- Klein-Gordon equation: $\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right] \Phi(x) = 0$

  complex-conjugate equation: $\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right] \Phi^*(x) = 0$

$$\Rightarrow \quad 0 = \Phi^*(x)\Box\Phi(x) - \Phi(x)\Box\Phi^*(x)$$
$$= \Phi^*(x)\partial_\mu\partial^\mu\Phi(x) - \Phi(x)\partial_\mu\partial^\mu\Phi^*(x)$$
$$= \partial_\mu \left[\Phi^*(x)\partial^\mu\Phi(x) - \Phi(x)\partial^\mu\Phi^*(x)\right] \quad \Rightarrow \quad \boxed{\partial_\mu j^\mu(x) = 0}$$

with the conserved 4-current

$$j^\mu(x) = \alpha \left(\Phi^*(x)\partial^\mu\Phi(x) - \Phi(x)\partial^\mu\Phi^*(x)\right)$$
$$\equiv \alpha\, \Phi^*(x) \left(\partial^\mu - \overleftarrow{\partial}^\mu\right) \Phi(x),$$

α: arbitrary constant

- time- and space-like components: $(j^\mu) = \begin{pmatrix} j^0 \\ \vec{j} \end{pmatrix} \equiv \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$

$$\Rightarrow \quad 0 = \partial_\mu j^\mu = \partial_0 j^0 + \partial_k j^k = \frac{1}{c}\frac{\partial}{\partial t} c\rho + \vec{\nabla} \cdot \vec{j}$$

$$\Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0}$$ continuity equation

interpretation in the nonrelativistic case: ρ = probability density

$\vec{j}$ = probability current density

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- determination of the constant: $\vec{j} \overset{!}{=} \vec{j}_{\text{Schrödinger}}$

$$\vec{j} \equiv \left(j^k\right) = \alpha \left(\Phi^*(\partial^k)\Phi - \Phi(\partial^k)\Phi^*\right)$$

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$$\vec{j} \equiv (j^k) = \alpha \left(\Phi^*(-\vec{\nabla})\Phi - \Phi(-\vec{\nabla})\Phi^*\right) \overset{!}{=} \frac{\hbar}{2mi}\left(\Phi^*\vec{\nabla}\Phi - \Phi\vec{\nabla}\Phi^*\right)$$

$$\Rightarrow \quad \alpha = \frac{i\hbar}{2m}$$