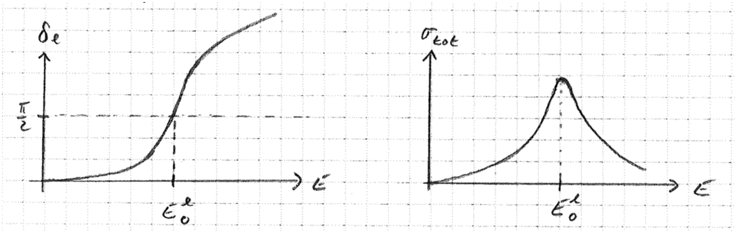
2.9 Resonances

- $\sigma_{tot}(E) = \sum_{\ell} \sigma_\ell(E), \quad \sigma_\ell(E) = \frac{2\pi}{k^2}(2\ell+1)\big(1 - \eta_\ell(E)\cos 2\delta_\ell(E)\big)$
- For $\eta_\ell(E) \approx$ *const*., $\sigma_\ell(E)$ is maximal at $\delta_\ell(E) = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$
  (not exactly true because of the factor $\frac{1}{k^2}$)
- Crossing $\delta_\ell(E) = (2n+1)\frac{\pi}{2}$ with big slope
  → sharp maximum of $\sigma_\ell(E)$ ("resonance"), which often dominates $\sigma_{tot}(E)$



Simplification: purely elastic scattering ($\eta_\ell = 1$)

- $f_\ell(E) = \frac{2\ell+1}{k} \sin\delta_\ell(E)\, e^{i\delta_\ell(E)} = \frac{2\ell+1}{k} \frac{\sin\delta_\ell}{e^{-i\delta_\ell}} = \frac{2\ell+1}{k} \frac{\sin\delta_\ell}{\cos\delta_\ell - i\sin\delta_\ell} = \frac{2\ell+1}{k} \frac{1}{\cot\delta_\ell - i}$
- Resonance: $\delta_\ell(E_0^\ell) = (2n+1)\frac{\pi}{2} \quad \Rightarrow \quad \cot\delta_\ell(E_0^\ell) = 0$

  $\Rightarrow \quad f_\ell(E_0^\ell) = \frac{2\ell+1}{k} i$ (purely imaginary)
- Taylor expansion about the resonance:

$$\cot\delta_\ell(E) = \cot\delta_\ell(E_0^\ell) + \left.\frac{d\,\cot\delta_\ell(E)}{dE}\right|_{E=E_0^\ell} (E - E_0^\ell) + \ldots \approx -\delta_\ell'(E_0^\ell)\,(E - E_0^\ell)$$

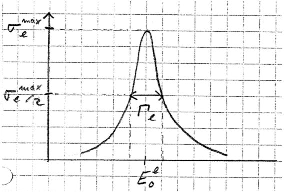
  - $\cot\delta_\ell(E_0^\ell) = 0$
  - $\frac{d\cot x}{dx} = -\frac{1}{\sin^2 x} \quad \Rightarrow \quad \left.\frac{d\,\cot\delta_\ell(E)}{dE}\right|_{E=E_0^\ell} = -\frac{1}{\sin^2 E_0^\ell}\,\delta_\ell'(E_0^\ell) = -\delta_\ell'(E_0^\ell)$

$$\cot\delta_\ell(E) \approx -\delta_\ell'(E_0^\ell)\,(E-E_0^\ell) \quad\Rightarrow\quad f_\ell(E) = \frac{2\ell+1}{k}\,\frac{1}{\cot\delta_\ell - i} \approx -\frac{2\ell+1}{k}\,\frac{1}{\delta_\ell'(E_0^\ell)\,(E-E_0^\ell)+\mathrm{i}}$$

$$\Rightarrow \quad \boxed{f_\ell(E\approx E_0^\ell) = -\frac{2\ell+1}{k}\,\frac{\Gamma_\ell/2}{(E-E_0^\ell)+\mathrm{i}\Gamma_\ell/2}} \quad \text{mit} \quad \Gamma_\ell \equiv \frac{2}{\delta_\ell'(E_0^\ell)}$$

„Breit-Wigner formula"

- $\sigma_\ell = \frac{4\pi}{2\ell+1}|f_\ell|^2 \quad\Rightarrow\quad \boxed{\sigma_\ell(E) = \frac{4\pi}{k^2}\,(2\ell+1)\,\frac{\Gamma_\ell^2/4}{(E-E_0^\ell)^2+\Gamma_\ell^2/4}}$



- $\sigma_\ell(E_0^\ell) = \frac{4\pi}{k^2}\,(2\ell+1) \equiv \sigma_\ell^{\max}$
- $\sigma_\ell(E_0^\ell \pm \frac{\Gamma_\ell}{2}) = \sigma_\ell^{\max}/2$

Argand diagrams

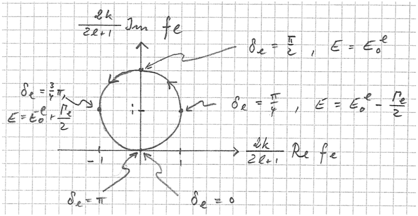
- scattering amplitude for elastic scattering (with inelastic channels present):

$$f_\ell^{\mathrm{el}} = \frac{2\ell+1}{2ik}\left(\eta_\ell\, e^{2i\delta_\ell} - 1\right) \quad \Rightarrow \quad \frac{2k}{2\ell+1} f_\ell^{\mathrm{el}} = \mathrm{i} + \eta_\ell\, e^{i(2\delta_\ell - \frac{\pi}{2})}$$

- purely elastic scattering ($\eta_\ell = 1$): $\frac{2k}{2\ell+1} f_\ell(E) = \mathrm{i} + e^{i(2\delta_\ell(E) - \frac{\pi}{2})}$

  → unit circle in the complex plane with center i

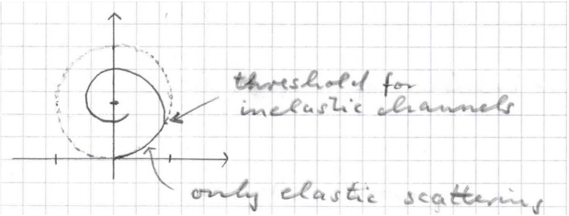
"Argand diagram"



- on resonance: $\delta_\ell = (2n+1)\frac{\pi}{2} \;\Rightarrow\; 2\delta_\ell - \frac{\pi}{2} = 2\pi n + \frac{\pi}{2} \;\Rightarrow\; \frac{2k}{2\ell+1} f_\ell(E_0^\ell) = 2\mathrm{i}$
- Breit-Wigner: $\frac{2k}{2\ell+1} f_\ell(E_0^\ell \pm \frac{\Gamma_\ell}{2}) = -2\,\frac{1}{\pm 1 + \mathrm{i}} = \mp 2\,\frac{1}{1 \pm \mathrm{i}} = \mp 2\,\frac{1 \mp \mathrm{i}}{1+1} = \mathrm{i} \mp 1$

$$\frac{2k}{2\ell+1} f_\ell^{\text{el}}(E) = \mathrm{i} + \eta_\ell(E)\, e^{i(2\delta_\ell(E) - \frac{\pi}{2})}$$

- presence of inelastic channels: $\eta_\ell < 1$
  - $\frac{2k}{2\ell+1} f_\ell^{\text{el}}(E)$ lies inside the unit circle
  - distance from the center: $\eta_\ell(E)$, depends on energy
  - otherwise similar behavior



3. Relativistic quantum mechanics

TECHNISCHE UNIVERSITÄT DARMSTADT

3.1 Motivation

- Nonrelativistic quantum mechanics works well in many areas of physics, e.g., hydrogen atom.

  → So why do we need relativistic quantum mechanics?

- Already in classical physics, Newton's mechanics is only a (often very good) approximation to special and general relativity.

  → In order to develop a theory as correct as possible, we should try to formulate quantum mechanics consistently with the principles of relativity. So far this has only been achieved for special relativity.

- The relativistic formulation has led to new insights even for the nonrelativistic limit:
  - existence of antiparticles
  - nature of spin

- In the "microcosmos", where quantum mechanics is inevitable, one also finds the highest velocities, e.g., cosmic rays, particle accelerators.
- estimate: Bohr model

  Electrons move on circular orbits with $L = n\hbar$.

  $n = 1$: $\quad L = r\,p = rmv = \hbar \quad \Leftrightarrow \quad \frac{v}{c} = \frac{\hbar c}{r\,mc^2} \approx \frac{2\,\text{keV Å}}{0.5\text{Å}\cdot 500\,\text{keV}} = \frac{1}{125}$

  $\hbar c \approx 200\,\text{MeV fm} = 2\,\text{keV Å}$

  electron: $mc^2 = 511\,\text{keV}$

  Bohr radius (hydrogen): $r_B \approx 0.5\text{Å}$

- In the “microcosmos”, where quantum mechanics is inevitable, one also finds the highest velocities, e.g., cosmic rays, particle accelerators.
- estimate: Bohr model

  Electrons move on circular orbits with $L = n\hbar$.

  $n = 1$: $\quad L = r\,p = rmv = \hbar \quad \Leftrightarrow \quad \frac{v}{c} = \frac{\hbar c}{r\,mc^2} = Z\alpha = \frac{Z}{137}$

  Bohr radius (atomic number Z): $\quad r = \frac{\hbar^2}{Z\,e^2\,m} = \frac{\hbar c}{Z\alpha\,mc^2}$

  finestructure constant: $\quad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$

  - hydrogen: $Z = 1 \quad \Rightarrow \quad \frac{v}{c} = \frac{1}{137}$ nonrelativistic approx. OK
  - hydrogen-like uranium: $Z = 92 \quad \Rightarrow \quad \frac{v}{c} = 0.67$ nonrel. app. questionable

- In the “microcosmos”, where quantum mechanics is inevitable, one also finds the highest velocities, e.g., cosmic rays, particle accelerators.

- estimate: Bohr model

  Electrons move on circular orbits with $L = n\hbar$.

  $n = 1$: $\quad L = r\,p = rmv = \hbar \quad \Leftrightarrow \quad \frac{v}{c} = \frac{\hbar c}{r\,mc^2} = Z\alpha = \frac{Z}{137}$

  Bohr radius (atomic number Z): $\quad r = \frac{\hbar^2}{Z\,e^2\,m} = \frac{\hbar c}{Z\alpha\,mc^2}$

  finestructure constant: $\quad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$

  - hydrogen: $Z = 1 \quad \Rightarrow \quad \frac{v}{c} = \frac{1}{137}$ nonrelativistic approx. OK
  - hydrogen-like uranium: $Z = 92 \quad \Rightarrow \quad \frac{v}{c} = 0.67$ nonrel. app. questionable

  - quarks in proton (quark model): $mc^2 \sim 300$ MeV, $r \sim 1$ fm $\rightarrow \frac{v}{c} \sim 0.67$

- In the "microcosmos", where quantum mechanics is inevitable, one also finds the highest velocities, e.g., cosmic rays, particle accelerators.
- estimate: Bohr model

  Electrons move on circular orbits with $L = n\hbar$.

  $n = 1$: $L = r\,p = rmv = \hbar \quad \Leftrightarrow \quad \frac{v}{c} = \frac{\hbar c}{r\,mc^2} = Z\alpha = \frac{Z}{137}$

  Bohr radius (atomic number Z): $r = \frac{\hbar^2}{Z\,e^2\,m} = \frac{\hbar c}{Z\alpha\,mc^2}$

  finestructure constant: $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$

  - hydrogen: $Z = 1 \Rightarrow \frac{v}{c} = \frac{1}{137}$ nonrelativistic approx. OK
  - hydrogen-like uranium: $Z = 92 \Rightarrow \frac{v}{c} = 0.67$ nonrel. app. questionable
  - quarks in proton (QCD): $mc^2 \sim 5$ MeV, $r \sim 1$ fm $\rightarrow \frac{v}{c} \sim 40$ ↯

3.2 Basics of special relativity (brief reminder)

- Fundamental axiom of special relativity:

  In all inertial frames the laws of nature are equal.
  In particular this holds for the speed of light.

- Lorentz Boosts:
  - Two inertial frames: $\mathcal{I}$: (t, x, y, z), $\mathcal{I}$': (t', x', y', z')
  - $\mathcal{I}$' moves relative to $\mathcal{I}$ with constant velocity $\vec{v}$.
  - We assume that at $t = 0$ the origins of both frames coincide,

    $(t, x, y, z) = (0, 0, 0, 0) \quad \hat{=} \quad (t', x', y', z') = (0, 0, 0, 0)$,

    and that at this space-time point a light pulse is emitted.
  - Then for the wave front holds:

    $\mathcal{I}$: $\quad x^2 + y^2 + z^2 = (ct)^2 \qquad$ (c: speed of light)

    $\mathcal{I}$': $\quad x'^2 + y'^2 + z'^2 = (ct')^2$

- Lorentz boost in x direction:
  - $\mathcal{I}$' moves relative to $\mathcal{I}$ with constant speed v in x direction.
  - The directions of the axes in $\mathcal{I}$ and $\mathcal{I}$' coincide.

Then one finds:

$$ct' = \gamma\,(ct - \beta x)\,, \qquad \beta \equiv \tfrac{v}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$x' = \gamma\,(x - \beta ct)\,,$$

$$y' = y\,,$$

$$z' = z$$

$$\Rightarrow \quad (c\tau)^2 \equiv (ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2 = \text{invariant}$$

τ: "proper time"

- Rapidity: $\chi = \frac{1}{2} \ln \frac{1+\beta}{1-\beta} \quad \Rightarrow \quad \gamma = \cosh\chi, \quad \beta\gamma = \sinh\chi$

$$\Rightarrow \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

→ formal similarities with (imaginary) rotations

Minkowski space

TECHNISCHE UNIVERSITÄT DARMSTADT

- contravariant four-vector: $$\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \equiv (x^\mu) \equiv x$$
- covariant four-vector: $$(x_\mu) \equiv \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -\vec{x} \end{pmatrix} = \begin{pmatrix} ct \\ -\vec{x} \end{pmatrix}$$

$$\Rightarrow \quad x^2 \equiv x^\mu x_\mu \equiv \sum_{\mu=0}^{3} x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (ct)^2 - \vec{x}^{\,2} = (c\tau)^2$$

- Einstein summation convention:
  - greek indices: $a^\mu b_\mu \equiv \sum_{\mu=0}^{3} a^\mu b_\mu$
  - latin indices: $a^k b_k \equiv \sum_{k=1}^{3} a^k b_k$

Metric tensor

TECHNISCHE UNIVERSITÄT DARMSTADT

- Relation between covariant and contravariant four-vectors:

$$x_\mu = g_{\mu\nu} x^\nu$$

$$x^\mu = g^{\mu\nu} x_\nu$$

$$(g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(metric tensor)

- Act with the metric tensor on itself:

$$g^\mu{}_\nu = g^{\mu\lambda} g_{\lambda\nu}, \quad g_\mu{}^\nu = g_{\mu\lambda} g^{\lambda\nu} \quad \Rightarrow \quad (g^\mu{}_\nu) = (g_\mu{}^\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad g^\mu{}_\nu = g_\mu{}^\nu \equiv \delta^\mu{}_\nu \equiv \delta_\mu{}^\nu \equiv \delta^\nu_\mu$$ (motivated by the Kronecker symbol)