- total cross section:

$$\sigma_{\text{tot}} = \int d\Omega \, \frac{d\sigma}{d\Omega} = 2\pi \int_{-1}^{1} d\cos\theta \, \frac{d\sigma}{d\Omega}(\theta)$$

orthogonality of Legendre polynomials: $\int_{-1}^{1} dx \, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$

$$\Rightarrow \quad \boxed{\sigma_{\text{tot}} = \sum_\ell \sigma_\ell, \qquad \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\,\sin^2\delta_\ell \quad \text{“partial cross section”}}$$

no interference terms:

After integration over $d\Omega$ the direction of the momentum is no longer fixed.

$\Rightarrow$ The angular momentum can in principle be measured precisely.

- Optical theorem:

  Legendre polynomials: $P_\ell(1) = 1$

  $$\Rightarrow \quad f_k(\theta = 0) = \frac{1}{k} \sum_\ell (2\ell + 1)\, \sin\delta_\ell\, e^{i\delta_\ell}$$

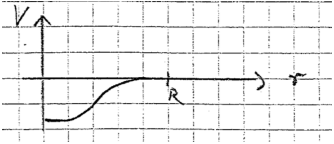
  $$\Rightarrow \quad \frac{4\pi}{k} \operatorname{Im} f_k(\theta = 0) = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1)\, \sin^2\delta_\ell = \sigma_{\text{tot}} \qquad \checkmark$$

- Each partial wave satisfies the optical theorem separately:

  $$\frac{4\pi}{k} \operatorname{Im} f_\ell\, P_\ell(\cos(0)) = \frac{4\pi}{k} \operatorname{Im} f_\ell = \frac{4\pi}{k^2} (2\ell + 1)\, \sin^2\delta_\ell = \sigma_\ell$$

Computation of δ_ℓ

- arbitrary potential (finite range: $V(r \geq R) = 0$)



- $\left(\frac{d^2}{dr^2} - U(r) - \frac{\ell(\ell+1)}{r^2} + k^2\right) u_\ell(r \leq R) = 0$

  in general to be solved numerically

- $u_\ell(r \geq R) = C_\ell\big(\cos\delta_\ell F_\ell(kr) - \sin\delta_\ell G_\ell(kr)\big)$

- boundary condition: continuity of the wave function and its derivative at $r = R$

  → "logarithmic derivative":

$$\left.\frac{u'_\ell(r\leq R)}{u_\ell(r\leq R)}\right|_{r=R} \stackrel{!}{=} \left.\frac{u'_\ell(r\geq R)}{u_\ell(r\geq R)}\right|_{r=R} = \left.\frac{\cos\delta_\ell \frac{d}{dr}F_\ell(kr) - \sin\delta_\ell \frac{d}{dr}G_\ell(kr)}{\cos\delta_\ell F_\ell(kr) - \sin\delta_\ell G_\ell(kr)}\right|_{r=R}$$

$$\Leftrightarrow \quad \tan\delta_\ell = \left.\frac{\frac{d}{dr}F_\ell(kr) - F_\ell(kr)\frac{u'_\ell(r)}{u_\ell(r)}}{\frac{d}{dr}G_\ell(kr) - G_\ell(kr)\frac{u'_\ell(r)}{u_\ell(r)}}\right|_{r=R}, \qquad F_\ell(kr) = kr\, j_\ell(kr), \quad G_\ell(kr) = kr\, n_\ell(kr)$$

- small energies: $kR \ll 1$

$$j_\ell(kR) \propto (kR)^\ell \quad \Rightarrow \quad F_\ell(kr)|_{r=R} \propto (kR)^{\ell+1} \quad \Rightarrow \quad \frac{d}{dr}F_\ell(kr)|_{r=R} \propto k\,(kR)^\ell$$

$$n_\ell(kR) \propto (kR)^{-(\ell+1)} \quad \Rightarrow \quad G_\ell(kr)|_{r=R} \propto (kR)^{-\ell} \quad \Rightarrow \quad \frac{d}{dr}G_\ell(kr)|_{r=R} \propto k\,(kR)^{-(\ell+1)}$$

$\longrightarrow$ "natural" behavior: $\delta_\ell \approx \sin\delta_\ell \approx \tan\delta_\ell \propto k^{2\ell+1} \propto E^{\ell+\frac{1}{2}}$

$\Rightarrow \quad \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell \propto E^{2\ell}$ $\longrightarrow$ fast convergence, $\ell = 0$ dominates

- "scattering length": $a \equiv -\lim\limits_{k\to 0}\frac{\delta_0}{k}$
  - hard sphere with radius R ($\to$ exercises): $\delta_0 = -kR \;\Rightarrow\; a = R$

    $\longrightarrow$ scattering length = radius of a hard sphere with the same low-energy behavior

Experimental determination of phase shifts (from a theorist's perspective)

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- Legendre polynomials: $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad \ldots$

  $$\Rightarrow \; f_k(\theta) = \sum_{\ell=0}^{\infty} f_\ell \, P_\ell(\cos\theta) = f_0 + f_1 \, \cos\theta + \ldots, \qquad f_\ell = \frac{2\ell+1}{k} \, \sin\delta_\ell \, e^{i\delta_\ell}$$

- differential cross section:

  $$\frac{d\sigma}{d\Omega}(\theta) = \left|f_k(\theta)\right|^2 = \left|f_0\right|^2 + (f_0^* f_1 + f_0 f_1^*) \cos\theta + \ldots = \left|f_0\right|^2 + 2\mathrm{Re}\, f_0^* f_1 \, \cos\theta + \ldots$$

→ determine $\delta_\ell(E)$ by fitting angular distributions

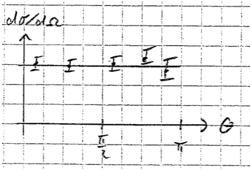
- practical difficulties:
  - uncertainties in the measurements
  - sign ambiguities (see below)

- $f_\ell = \frac{2\ell+1}{k} \sin \delta_\ell e^{i\delta_\ell}$

- small energies (only $\ell = 0$ relevant):

  $f_k(\theta) \approx f_0 = \frac{1}{k} \sin \delta_0 e^{i\delta_0}$

  $\Rightarrow \quad \frac{d\sigma}{d\Omega} \approx \frac{1}{k^2} \sin^2 \delta_0$ (isotropic scattering)



- somewhat larger energies:

  $f_k(\theta) \approx f_0 + f_1 \cos \theta = \frac{1}{k} \sin \delta_0 e^{i\delta_0} + \frac{3}{k} \sin \delta_1 e^{i\delta_1} \cos\theta$

  $\Rightarrow \quad \frac{d\sigma}{d\Omega} \approx \frac{1}{k^2} \Big(A + B \cos \theta \Big), \qquad A = \sin^2 \delta_0,$

  $B = 6 \sin \delta_0 \sin \delta_1 \cos(\delta_0 - \delta_1)$

2.8 Inelastic channels

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- We now allow for internal excitations of the target.

→ inelastic processes: $E' = E - \Delta E < E$

  - E: energy of the incoming particle
  - E': energy of the scattered particle
  - ΔE: excitation energy of the target (in the CM frame)

- further possibilities:
  - The projectile gets absorbed by the target.
  - New particles are produced.
  - The projectile changes its particle nature.
- common feature:

  reduction of the outgoing current in the elastic channel ($E' = E$)

- elastic scattering: $u_\ell(r \to \infty) \propto e^{2i\delta_\ell} e^{i(kr-\ell\frac{\pi}{2})} - e^{-i(kr-\ell\frac{\pi}{2})}$

  outgoing incoming spherical waves

  real scattering phase δ_ℓ consistent with the optical theorem

  $\Rightarrow$ particle number conserved

- presence of inelastic channels

  → fewer outgoing particles in the elastic channel:

  $$u_\ell^{\mathrm{el}}(r \to \infty) \propto \eta_\ell\, e^{2i\delta_\ell} e^{i(kr-\ell\frac{\pi}{2})} - e^{-i(kr-\ell\frac{\pi}{2})}, \qquad 0 \leq \eta_\ell \leq 1$$

  "inelasticity"

  - strictly speaking the wrong word:

    $\eta_\ell = 1$ corresponds to elastic scattering

- Wave function for the elastic channel: $\psi^{\text{el}}_{\vec{k}}(\vec{r}) = \sum_\ell \frac{u^{\text{el}}_\ell(r)}{r} P_\ell(\cos\theta)$

- Asymptotic behavior:

$$\psi^{\text{el}}_{\vec{k}}(\vec{r}) \stackrel{r\to\infty}{\longrightarrow} e^{ikz} + f^{\text{el}}_k(\theta)\,\frac{e^{ikr}}{r}, \qquad f^{\text{el}}_k(\theta) = \sum_\ell f^{\text{el}}_\ell\, P_\ell(\cos\theta)$$

$$u^{\text{el}}_\ell(r\to\infty) \propto \eta_\ell\, e^{2i\delta_\ell} e^{i(kr-\ell\frac{\pi}{2})} - e^{-i(kr-\ell\frac{\pi}{2})}$$

- Expand the plane wave in Legendre polynomials (as in the purely elastic case)

$$\to \quad \boxed{f^{\text{el}}_\ell = \frac{2\ell+1}{2ik}\left(\eta_\ell\, e^{2i\delta_\ell} - 1\right)}$$

- Elastic cross section: $\sigma_{\text{el}} = \int d\Omega\, |f^{\text{el}}_k(\theta)|^2$

$$\to \quad \boxed{\sigma_{\text{el}} = \frac{\pi}{k^2}\sum_\ell (2\ell+1)\left|\eta_\ell\, e^{2i\delta_\ell} - 1\right|^2}$$

- total crossection: $\sigma_{\text{tot}} = \sigma_{\text{el}} + \sigma_{\text{inel}}$

  Still: Particles which are scattered elastically, inelastically, absorbed or transformed, are missing in forward direction at elastic energies.

  $\longrightarrow$ optical theorem: $\sigma_{\text{tot}} = \frac{4\pi}{k} \,\text{Im}\, f_k^{\text{el}}(0)$

$$\Rightarrow \quad \boxed{\sigma_{\text{tot}} = \frac{2\pi}{k^2} \sum_\ell (2\ell+1)\,(1 - \eta_\ell \cos 2\delta_\ell)}$$

- inelastic cross section: $\sigma_{\text{inel}} = \sigma_{\text{tot}} - \sigma_{\text{el}}$

$$\Rightarrow \quad \sigma_{\text{inel}} = \frac{\pi}{k^2} \sum_\ell (2\ell+1) \Big\{ 2\,(1 - \eta_\ell \cos 2\delta_\ell) - (\eta_\ell\, e^{-2i\delta_\ell} - 1)(\eta_\ell\, e^{2i\delta_\ell} - 1) \Big\}$$

$$\Rightarrow \quad \boxed{\sigma_{\text{inel}} = \frac{\pi}{k^2} \sum_\ell (2\ell+1)\big(1 - \eta_\ell^2\big)}$$

(independent of δ_ℓ)