Born series

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- Lippmann-Schwinger equation: $\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{T}$

  $\Rightarrow \quad \hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \hat{V}\hat{G}_0\hat{V}\hat{G}_0\hat{V} + \dots$ ($\hat{=}$ Born series)

- Scattering amplitude: $f(\vec{k}\,', \vec{k}) = -\frac{\mu}{2\pi\hbar^2} \langle \vec{k}\,' | \hat{T}(E + i\varepsilon) | \vec{k} \rangle = \sum_{j=1}^{\infty} f^{(j)}(\vec{k}\,', \vec{k})$

  - Born approximation: $f^{(1)}(\vec{k}\,', \vec{k}) = -\frac{\mu}{2\pi\hbar^2} \langle \vec{k}\,' | \hat{V} | \vec{k} \rangle = -\frac{\mu}{2\pi\hbar^2} \tilde{V}(\vec{q}) \quad \checkmark$

  - nth-order Born approximation:

    $f^{(n)}(\vec{k}\,', \vec{k}) = -\frac{\mu}{2\pi\hbar^2} \langle \vec{k}\,' | \hat{V}\hat{G}_0\hat{V}\hat{G}_0 \dots \hat{V} | \vec{k} \rangle \qquad$ (n times $\hat{V}$)

    Insertion of $2(n-1)$ complete sets of momentum eigenstates

    $\longrightarrow$ same result as in section 2.4

- Green's operator: $\hat{G}_0(z) = \left(z - \hat{H}_0\right)^{-1}$
- Define: $\hat{G}(z) \equiv \left(z - \hat{H}\right)^{-1}$

$$= \left(z - \hat{H}_0\right)^{-1} + \left(z - \hat{H}_0\right)^{-1} \Big[\underbrace{(z - \hat{H}_0) - (z - \hat{H})}_{= \hat{H} - \hat{H}_0 = \hat{V}} \Big] \left(z - \hat{H}\right)^{-1}$$

$\Rightarrow$ $\boxed{\hat{G}(z) = \hat{G}_0(z) + \hat{G}_0(z) \hat{V} \hat{G}(z)}$ LSE for the Green's operator

$\rightarrow$ Born series: $\hat{G} = \hat{G}_0 + \hat{G}_0 \hat{V} \hat{G}_0 + \hat{G}_0 \hat{V} \hat{G}_0 \hat{V} \hat{G}_0 + ...$

$$= \hat{G}_0 + \hat{G}_0 \left(\hat{V} + \hat{V} \hat{G}_0 \hat{V} + ... \right) \hat{G}_0 \quad = \hat{G}_0 + \hat{G}_0 \hat{T} \hat{G}_0$$

$$\Rightarrow \boxed{\begin{aligned} \hat{G} &= \hat{G}_0 + \hat{G}_0 \hat{T} \hat{G}_0 \\ \hat{T} &= \hat{V} + \hat{V} \hat{G} \hat{V} \end{aligned}}$$

Diagrammatic interpretation

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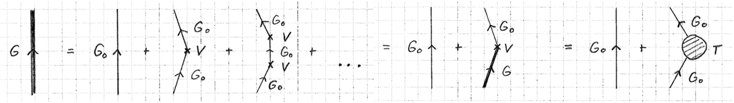

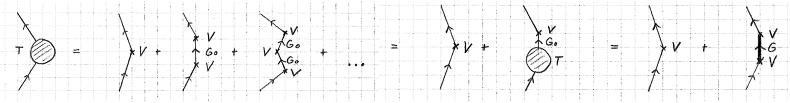

- G: full propagator $\leftrightarrow$ propagation of the wave from $\vec{r}$ to $\vec{r}\,'$
- G_0: bare propagator $\leftrightarrow$ free propagation without interaction
- T: full interaction
- V: bare interaction

2.6 The optical theorem

- continuity equation: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$
  - probability density: $\rho = |\psi|^2$
  - probability current density: $\vec{j} = \frac{\hbar}{2\mu i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$

- stationary state: $\frac{\partial\rho}{\partial t} = 0 \;\Rightarrow\; \vec{\nabla}\cdot\vec{j} = 0$

$$\Rightarrow \quad \vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = \psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^* = 0$$

- incoming + scattering wave: $\psi(\vec{r}) = \phi(\vec{r}) + \psi_s(\vec{r})$

$$\Rightarrow \quad \phi^*\vec{\nabla}^2\phi - \phi\vec{\nabla}^2\phi^* + \psi_s^*\vec{\nabla}^2\psi_s - \psi_s\vec{\nabla}^2\psi_s^*$$
$$+ \phi^*\vec{\nabla}^2\psi_s - \phi\vec{\nabla}^2\psi_s^* + \psi_s^*\vec{\nabla}^2\phi - \psi_s\vec{\nabla}^2\phi^* = 0$$

- incoming wave: $\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

$$\Rightarrow \quad \vec{\nabla}^2\phi = -\vec{k}^{\,2}\phi, \quad \vec{\nabla}^2\phi^* = -\vec{k}^{\,2}\phi^* \qquad \Rightarrow \quad \phi^*\vec{\nabla}^2\phi - \phi\vec{\nabla}^2\phi^* = 0 \qquad (*)$$

i.e., ϕ satisfies the continuity equation by itself.

$$\Rightarrow \psi_s^*\vec{\nabla}^2\psi_s - \psi_s\vec{\nabla}^2\psi_s^* = -\left(\phi^*\vec{\nabla}^2\psi_s - \phi\vec{\nabla}^2\psi_s^*\right) - \left(\psi_s^*\vec{\nabla}^2\phi - \psi_s\vec{\nabla}^2\phi^*\right)$$

$$= -2\mathrm{i}\,\mathrm{Im}\left(\phi^*\vec{\nabla}^2\psi_s - \psi_s\vec{\nabla}^2\phi^*\right) \qquad (\text{since } a - a^* = 2\mathrm{i}\,\mathrm{Im}\,a)$$

$$\stackrel{(*)}{=} -2\mathrm{i}\,\mathrm{Im}\left(\phi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\phi^*\right)$$

$$= -2\mathrm{i}\,\mathrm{Im}\,\phi^*\left(\vec{\nabla}^2 + \vec{k}^{\,2}\right)\psi$$

$$= -2\mathrm{i}\,\frac{2\mu}{\hbar^2}\,\mathrm{Im}\,\phi^*\,V\,\psi \qquad (\text{since } \ (-\tfrac{\hbar^2}{2\mu}\vec{\nabla}^2 + V)\psi = \tfrac{\hbar^2\vec{k}^{\,2}}{2\mu}\psi)$$

- integrate over an (infinite) volume $\mathcal{V}$:

$$\int_{\mathcal{V}} d^3r \left(\psi_s^* \vec{\nabla}^2 \psi_s - \psi_s \vec{\nabla}^2 \psi_s^*\right) = -2\mathrm{i}\, \frac{2\mu}{\hbar^2}\, \mathrm{Im} \int_{\mathcal{V}} d^3r\, \phi^* V \psi$$

- left-hand side

$$= \int_{\mathcal{V}} d^3r\, \vec{\nabla} \cdot \left(\psi_s^* \vec{\nabla} \psi_s - \psi_s \vec{\nabla} \psi_s^*\right) = \frac{2\mu i}{\hbar} \int_{\mathcal{V}} d^3r\, \vec{\nabla} \cdot \vec{j}_s = \frac{2\mu i}{\hbar} \int_{\partial\mathcal{V}} dS\, \vec{n} \cdot \vec{j}_s$$

$$= \frac{2\mu i}{\hbar} \times \text{ total number of scattered particles per time}$$

$$= \frac{2\mu i}{\hbar}\, \sigma_{\text{tot}}\, |\vec{j}_{\text{in}}|, \qquad \sigma_{\text{tot}} \equiv \sigma = \text{total cross section}, \qquad |\vec{j}_{\text{in}}| = \frac{\hbar k}{\mu}$$

$$= 2ik\, \sigma_{\text{tot}}$$

- right-hand side $= -2\mathrm{i}\, \frac{2\mu}{\hbar^2}\, \mathrm{Im} \int d^3r\, e^{-i\vec{k}\cdot\vec{r}} V(\vec{r})\, \psi(\vec{r}) = 8\pi \mathrm{i}\, \mathrm{Im}\, f(\vec{k}, \vec{k})$

scattering amplitude: $f(\vec{k}\,', \vec{k}) = -\frac{\mu}{2\pi\hbar^2} \int d^3r\, e^{-i\vec{k}\,'\cdot\vec{r}}\, V(\vec{r})\, \psi_{\vec{k}}(\vec{r})$

$$\Rightarrow \quad \sigma_{\text{tot}} = \frac{4\pi}{k} \, \mathrm{Im} \, f(\vec{k}, \vec{k})$$

- $f(\vec{k}, \vec{k}) \equiv f_k(\theta = 0) \equiv f_k(0)$: scattering amplitude in forward direction

$\Rightarrow$ "optical theorem": $$\boxed{\sigma_{\text{tot}} = \frac{4\pi}{k} \, \mathrm{Im} \, f_k(0)}$$

- qualitative interpretation:

  The scattered particles are missing in forward direction.
  In the wave description this corresponds to a destructive interference of incoming and scattering wave, which is related to the imaginary part of the scattering amplitude.

- Born series: $f_k(\theta,\varphi) = \sum_{n=1}^{\infty} f_k^{(n)}(\theta,\varphi)$
- Write $V(\vec{r}) = g v(\vec{r})$, $g = const.$ („coupling constant"), e.g. $g = e$

$$\Rightarrow \quad f_k^{(n)} \propto g^n$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \left| f_k^{(1)} + f_k^{(2)} + \ldots \right|^2 = \underbrace{\left| f_k^{(1)} \right|^2}_{\propto g^2} + \underbrace{\left(f_k^{(1)\,*} f_k^{(2)} + f_k^{(1)} f_k^{(2)\,*} \right)}_{\propto g^3} + \ldots$$

$$\Rightarrow \quad \sigma_{\text{tot}} = \underbrace{\sigma_{\text{tot}}^{(1)}}_{\propto g^2} + \cdots = \frac{4\pi}{k} \operatorname{Im} \left(\underbrace{f_k^{(1)}(0)}_{\propto g} + \underbrace{f_k^{(2)}(0)}_{\propto g^2} + \ldots \right)$$

Compare powers of g:

- $\operatorname{Im} f_k^{(1)}(0) = 0$
- $\sigma_{\text{tot}}^{(1)} = \int d\Omega \, |f_k^{(1)}(\theta,\varphi)|^2 = \frac{4\pi}{k} \operatorname{Im} f_k^{(2)}(0) \quad \Rightarrow$ $\operatorname{Im} f_k^{(2)}(0) \neq 0$, unless $f_k^{(1)}(\theta,\varphi)$ vanishes for all angles.