Simplifying assumptions

TECHNISCHE UNIVERSITÄT DARMSTADT

1. only prozesses $A + B \to A + B$,

   i.e., no production, absorption or decay of particles

2. only elastistic scattering,

   i.e., no internal excitations of projectile or target (will be relaxed later)

3. no coherence effects by scattering from multiple target particles

   (like Bragg scattering at crystals)

   - theoretical treatment: only one target particle
   - comparison with the experiment:

     divide measured counting rate by the number of target particles
   - prerequisite: $\Delta x \ll d$

     Δx: spatial extension of the wave packet of the projectile
     d: distance of the target particles

Example:

scattering electrons ($mc^2 = 511$ keV)
from nuclei in a crystal ($d = 1$ Å $= 10^5$ fm)

Momentum uncertainty:

$$\Delta p > \frac{\hbar}{2\Delta x} \gg \frac{\hbar}{2d} = \frac{\hbar c}{2dc} = \frac{200 \text{ MeV fm}}{2 \cdot 10^5 \text{ fm } c} = 1 \text{ keV}/c$$

Energy:

$$E = \frac{\vec{p}^{\,2}}{2m} \quad \Rightarrow \quad \Delta E = \frac{p\,\Delta p}{m} + \mathcal{O}((\Delta p)^2) \quad \Rightarrow \quad \frac{\Delta E}{E} \approx 2\frac{\Delta p}{p}$$

$$\Rightarrow \quad \frac{\Delta E}{E} \approx 2\frac{\Delta pc}{pc} = \frac{2 \text{ keV}}{\sqrt{2mc^2\,E}}$$

e.g., $E = 10$ keV $\quad \Rightarrow \quad \frac{\Delta E}{E} \approx \frac{2 \text{ keV}}{\sqrt{1000 \text{ keV} \cdot 10 \text{ keV}}} = 2\%$

4. no spin

5. The potential $V(\vec{r})$ describing the interaction depends only on the difference $\vec{r}$ between projectile's and target's positions.

   Separation of relative and center-of-mass motion

   $\rightarrow$ equivalent one-body problem in the CM frame:

   scattering of a single particle with reduced mass $\mu = \left(\frac{1}{m_p} + \frac{1}{m_t}\right)^{-1}$

   from the potential $V(\vec{r})$

6. The scattering process can be treated as stationary problem.
   $\rightarrow$ time independent Schrödinger equation
   - visualization:
     diffraction of a continuously incoming wave of water or light at a barrier
   - precondition:
     The potential has a finite range or its value drops fast enough,
     and the wave packets are much larger than the range of the potential

2.2 The scattering amplitude

TECHNISCHE UNIVERSITÄT DARMSTADT

- time dependent Schrödinger equation:

  $$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(\vec{r})\right)\Psi(t,\vec{r}) = i\hbar\frac{\partial}{\partial t}\,\Psi(t,\vec{r})$$

- stationary solution: $\Psi(t,\vec{r}) = \psi(\vec{r})e^{-\frac{i}{\hbar}Et}$

  → time independent Schrödinger equation: $\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\,\psi(\vec{r})$

- goal: find solutions in the continuum (= not bound states) with waves coming in from and scattered to infinite distances

- spatially localized potential: $V(\vec{r}) \xrightarrow{|\vec{r}|\to\infty} 0$

  → $E > 0$ for unbound solutions

- E can be fixed to arbitrary values by corresponding preparation of the beam.

  → Boundary value problem for given $E > 0$

- Ansatz: $\psi(\vec{r}) = \psi_{\text{in}}(\vec{r}) + \psi_{\text{sc}}(\vec{r})$

- ψ_{in}: Solution in absence of the potential;
  determined by the particle source

  → choose plane wave with energy E going in z direction:

$$\psi_{\text{in}}(\vec{r}) \propto e^{i\vec{k}\cdot\vec{r}} = e^{ikz}, \qquad \vec{k} = k\vec{e}_z, \quad \vec{p} = \hbar\vec{k}, \quad E = \frac{\hbar^2\vec{k}^{\,2}}{2\mu}$$

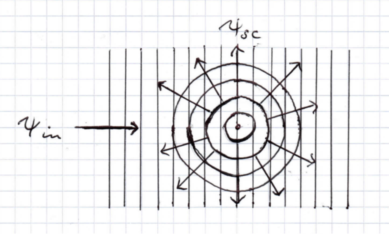
- ψ_{sc}: correction due to the potential

$$V(\vec{r}) \overset{|\vec{r}|\to\infty}{\longrightarrow} 0 \quad \Rightarrow \quad -\frac{\hbar^2}{2\mu}\nabla^2\psi_{\text{sc}}(\vec{r}) = E\,\psi_{\text{sc}}(\vec{r}) \quad \text{for} \quad |\vec{r}| \to \infty$$

(i.e., $\psi_{\text{sc}}(\vec{r})$ is asymptotically a solution of the free Schrödinger equation)

- intuitive expectation:

  ψ_{sc} is not a plane wave but asymptotically outgoing in radial direction



$$\psi_{sc}(\vec{r}) \overset{r\to\infty}{\longrightarrow} f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}, \qquad r \equiv |\vec{r}|\,, \quad f_k\text{: „scattering amplitude”}$$

$$\Rightarrow\ \nabla^2 f_k(\theta,\varphi)\,\frac{e^{ikr}}{r} = \Delta\, f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r} f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}\right] + \frac{1}{r^2\sin^2\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta} f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}\right) + \frac{\partial^2}{\partial\varphi^2} f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}\right]$$

$$= -k^2\, f_k(\theta,\varphi)\,\frac{e^{ikr}}{r} \qquad + \mathcal{O}\!\left(\tfrac{1}{r^2}\right)$$

$\longrightarrow$ asymptotically a solution of the free Schrödinger equation $\checkmark$

- total asymptotics: $\psi(\vec{r}) \overset{r\to\infty}{\longrightarrow} \mathcal{N}\left(e^{ikz} + f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}\right)$
  - $\mathcal{N}$: normalization constant (drops out in calculations of the cross section $\rightarrow$ can be omitted)
- alternative notation for the scattering amplitude: $f_k(\theta,\varphi) \equiv f(\vec{k}\,', \vec{k})$
  - $\vec{k}$ = wave vector of the incoming wave $(= k\vec{e}_z)$
  - $\vec{k}\,'$ = wave vector of the outgoing wave
    ($|\vec{k}\,'| = |\vec{k}|$ for elastic scattering, $(\theta,\varphi) \hat{=}$ direction of $\vec{k}\,'$)
- isotropic potentials (almost always assumed in these lectures):

$$V(\vec{r}) = V(r) \quad \Rightarrow \quad f_k(\theta,\varphi) = f_k(\theta)$$

- Relation to the cross section:

$$d\sigma = \frac{\vec{j}_{\text{sc}} \cdot d\vec{S}}{|\vec{j}_{\text{in}}|} = \frac{(\vec{j}_{\text{sc}})_r \, r^2 \, d\Omega}{|\vec{j}_{\text{in}}|} \qquad (r \to \infty)$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \lim_{r\to\infty} \frac{(\vec{j}_{\text{sc}})_r \, r^2}{|\vec{j}_{\text{in}}|}$$

- Probability current: $\vec{j} = \frac{\hbar}{2\mu i}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$
  - incoming wave: $\vec{j}_{\text{in}} = \frac{\hbar k}{\mu} \, \vec{e}_z$
  - scattered wave:

$$(\vec{j}_{\text{sc}})_r = \frac{\hbar}{2\mu i}\left(\psi_{\text{sc}}^* \frac{\partial}{\partial r}\psi_{\text{sc}} - \psi_{\text{sc}} \frac{\partial}{\partial r}\psi_{\text{sc}}^*\right) \xrightarrow{r\to\infty} \frac{\hbar k}{\mu} \, |f_k|^2 \frac{1}{r^2}$$

$$\Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega}(\theta, \varphi) = |f_k(\theta, \varphi)|^2}$$

2.3 Green's functions

TECHNISCHE UNIVERSITÄT DARMSTADT

- Aim: Solve the Schrödinger equation

$$\Big(-\frac{\hbar^2}{2\mu}\nabla^2 + V(\vec{r})\Big)\psi_{\vec{k}}(\vec{r}) = E_k\,\psi_{\vec{k}}(\vec{r}) \equiv \frac{\hbar^2 k^2}{2\mu}\,\psi_{\vec{k}}(\vec{r})$$

$$\Leftrightarrow \quad \big(\nabla^2 + k^2\big)\psi_{\vec{k}}(\vec{r}) = \frac{2\mu}{\hbar^2} V(\vec{r})\,\psi_{\vec{k}}(\vec{r})$$

with the boundary condition

$$\psi_{\vec{k}}(\vec{r}) \overset{r\to\infty}{\longrightarrow} e^{ikz} + f_k(\theta,\varphi)\,\frac{e^{ikr}}{r} \equiv e^{i\vec{k}\cdot\vec{r}} + f(\vec{k}\,',\vec{k})\,\frac{e^{ikr}}{r}$$

- Green's function: $\big(\nabla^2 + k^2\big)G_k(\vec{r},\vec{r}\,') = \delta^3(\vec{r}-\vec{r}\,')$ (= definition of G_k)

→ general solution of the Schrödinger equation (formally):

$$\boxed{\psi_{\vec{k}}(\vec{r}) = \varphi_{\vec{k}}(\vec{r}) + \int d^3r'\, G_k(\vec{r},\vec{r}\,')\,\frac{2\mu}{\hbar^2}\,V(\vec{r}\,')\,\psi_{\vec{k}}(\vec{r}\,')}$$

$\varphi_{\vec{k}}(\vec{r}) = e^{ikz} \equiv e^{i\vec{k}\cdot\vec{r}}$: solution of the homogeneous equation
$\big(\nabla^2 + k^2\big)\varphi_{\vec{k}}(\vec{r}) = 0$ with the correct boundary condition