The Bell inequalities

In 1964, John Bell found a way to find out experimentally whether or not there are hidden parameters.

Consider again two spin-$\frac{1}{2}$ particles in a total spin-0 state $|0,0\rangle$.
We measure the spin projection of the two particles along two arbitrary directions, characterized by the unit vectors $\vec{\alpha}$ and $\vec{\beta}$, respectively. More precisely, we define

$N(\vec{\alpha};\vec{\beta})$ = relative probability to find particle 1 in a state with spin projection $+\frac{\hbar}{2}$ along $\vec{\alpha}$ and particle 2 in a state with spin projection $+\frac{\hbar}{2}$ along $\vec{\beta}$.

quantum-mechanical evaluation:

As usual, we represent the spin operator by Pauli matrices

$$\vec{s} = \frac{\hbar}{2}\vec{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2) eigenstates of s_z = eigenstates of σ_z :

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad , \quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow \quad \sigma_x|\uparrow\rangle = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle \quad , \quad \sigma_x|\downarrow\rangle = |\uparrow\rangle$$

$$\sigma_y|\uparrow\rangle = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i|\downarrow\rangle \quad , \quad \sigma_y|\downarrow\rangle = -i|\uparrow\rangle$$

and of course

$$\sigma_z|\uparrow\rangle = |\uparrow\rangle \quad , \quad \sigma_z|\downarrow\rangle = -|\downarrow\rangle$$

projection operator on a state with spin projection $+\frac{\hbar}{2}$ along $\vec{\alpha}$:

$$P_{\vec{\alpha}} = \frac{1}{2}\left(1 + \vec{\alpha}\cdot\vec{\sigma}\right)$$

$$\Rightarrow \quad N(\vec{\alpha},\vec{\beta}) = \langle 0,0|\, P_{\vec{\alpha}}^{(1)}\, P_{\vec{\beta}}^{(2)}\, |0,0\rangle$$

$$= \frac{1}{4}\langle 0,0|\,(1+\vec{\alpha}\cdot\vec{\sigma}^{(1)})(1+\vec{\alpha}\cdot\vec{\sigma}^{(2)})\,|0,0\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

and $\vec{\sigma}^{(i)}$ acts only on the i-th particle.

Result of the explicit evaluation (→ exercise)

$$N(\vec{\alpha};\vec{\beta}) = \frac{1}{4}(1 - \vec{\alpha}\cdot\vec{\beta})$$
$$= \frac{1}{4}(1-\cos\varphi) = \frac{1}{2}\sin^2\frac{\varphi}{2}$$

with $\varphi = \measuredangle(\vec{\alpha},\vec{\beta})$

(special cases: $\varphi = 0 \Rightarrow N(\vec{\alpha},\vec{\beta}) = 0$
$\varphi = \pi \Rightarrow N(\vec{\alpha},\vec{\beta}) = \frac{1}{2}$

This makes sense: The probability to find particle 1 in a state with spin projection $+\frac{\hbar}{2}$ along $\vec{\alpha}$ is $\frac{1}{2}$. But then the spin projection of particle 2 along the same axis must be $-\frac{\hbar}{2}$.)

<u>assuming hidden variables</u>

If there are hidden variables which determine the outcome of a measurement before it is taken, the probability of finding particle 2 with spin projection $+\frac{\hbar}{2}$ along $\vec{\beta}$ must be equal to the probability of finding particle 1 with spin projection $-\frac{\hbar}{2}$ along $\vec{\beta}$ (because if we measure particle 2 and particle 1 along the same axis, this is what is always found, see above).

But then we can say that

$N(\vec{\alpha};\vec{\beta})$ = probability of finding particle 1 in a state with spin projection
$+\frac{\hbar}{2}$ along $\vec{\alpha}$
and
$-\frac{\hbar}{2}$ along $\vec{\beta}$

Now let's introduce a third axis $\vec{\gamma}$ and define

$N(\vec{\alpha},\vec{\gamma};\vec{\beta})$ = probability of finding particle 1 in a state with spin projection
$+\frac{\hbar}{2}$ along $\vec{\alpha}$ and $\vec{\gamma}$
and
$-\frac{\hbar}{2}$ along $\vec{\beta}$

$N(\vec{\alpha};\vec{\gamma},\vec{\beta})$ = ...
$+\frac{\hbar}{2}$ along $\vec{\alpha}$
$-\frac{\hbar}{2}$ " $\vec{\gamma}$ and $\vec{\beta}$

$$\Rightarrow \quad N(\vec{\alpha};\vec{\beta}) = N(\vec{\alpha},\vec{\gamma};\vec{\beta}) + N(\vec{\alpha};\vec{\gamma},\vec{\beta})$$

Of course

$$N(\vec{\alpha},\vec{\gamma};\vec{\beta}) \leq N(\vec{\gamma};\vec{\beta})$$

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$$\Rightarrow \quad \boxed{N(\vec{\alpha};\vec{\beta}) \leq N(\vec{\alpha};\vec{\gamma}) + N(\vec{\gamma};\vec{\beta})}$$

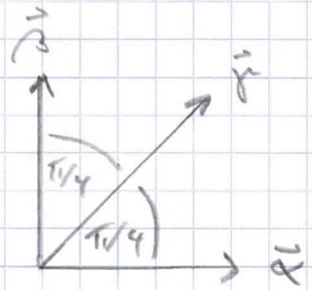
(Bell inequality, 1964)

This can be tested experimentally by performing a series of measurements of the spins of two particles along different directions.

<u>example:</u>

Take $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ in one plane with

$$\sphericalangle(\vec{\alpha},\vec{\gamma}) = \sphericalangle(\vec{\gamma},\vec{\beta}) = \frac{\pi}{4} \quad , \quad \sphericalangle(\vec{\alpha},\vec{\beta}) = \frac{\pi}{2}$$



Then quantum mechanics predicts

$$N(\vec{\alpha};\vec{\beta}) = \frac{1}{2}\sin^2\frac{\pi}{4} = \frac{1}{4}$$

$$N(\vec{\alpha};\vec{\gamma}) + N(\vec{\gamma};\vec{\beta}) = 2\cdot\frac{1}{2}\sin^2\frac{\pi}{8} = 0.146\ldots$$

$\Rightarrow$ In quantum mechanics one finds

$$N(\vec{\alpha};\vec{\beta}) > N(\vec{\alpha};\vec{\gamma}) + N(\vec{\gamma};\vec{\beta})$$

in contradiction to the Bell inequality!

Analogous experiments with photons confirm quantum mechanics and exclude hidden parameters (under certain assumptions, like locality, causality...).