On the occation of this year's Nobel prize:

Hidden variables, EPR argument, and Bell inequalities

- <u>classical mechanics</u>: deterministic

<u>example</u>:
N particles with masses m_i and coordinates $\vec{r}^{(i)}(t)$,
and two-body forces $\vec{F}^{(ij)}(\vec{r}^{(i)}, \vec{r}^{(j)})$
= force of particle j on particle i

$\Rightarrow$ equations of motion

$$m_i \ddot{\vec{r}}^{(i)}(t) = \sum_{j \neq i} \vec{F}^{(ij)}(\vec{r}^{(i)}(t), \vec{r}^{(j)}(t)) \quad , i = 1..N$$

3N ordinary diff. equations of 2^{nd} order

$\Rightarrow$ <u>unique</u> solution for $\vec{r}^{(i)}(t)$, $i = 1, \dots, N$
if 6N initial conditions are fixed,

e.g. $\vec{r}^{(i)}(t=0)$, $\dot{\vec{r}}^{(i)}(t=0)$

(in practice:
initial values only known up to some precision $\rightsquigarrow$ "deterministic chaos")

- <u>quantum mechanics</u>

wave function evolves deterministically:

$$|\Psi(t)\rangle = \exp\left(-\frac{i}{\hbar} H\right)|\Psi(0)\rangle$$

↑

uniquely determined if $|\Psi(0)\rangle$ is known

<u>but</u>:

$|\Psi(t)\rangle$ yields only probabilities of observables

reminder:

$$|\Psi(t)\rangle = \sum_n c_n(t)\,|n\rangle \quad , \quad \hat{O}\,|n\rangle = \lambda_n\,|n\rangle$$

$\Rightarrow$ probability to measure λ_n at time t $= |c_n(t)|^2$

prediction with certainty only if $|\Psi\rangle$ is an eigenstate of $\hat{O}$

uncertainty relation:
Non-commuting operators do not have simultaneous eigenstates
$\Rightarrow$ cannot simultaneously be determined with infinite precision,

e.g. $[\hat{x}, \hat{p}] = i\hbar \quad \rightsquigarrow \quad \Delta x \, \Delta p \geq \frac{\hbar}{2}$

→ It is not possible to simultaneously measure all existing observables to arbitrary precission or to predict all possible observables at time $t > 0$ from a set of measurements at $t = 0$.

- Hidden variables

Could it be that there is an underlying theory which contains "hidden variables", i.e. parameters which cannot be measured, but nevertheless uniquely determine all observables, so that we could predict the observables if we knew these parameters?

The EPR argument

(Einstein, Podolsky, Rosen, Phys. Rev. 47 (1935), 777 - 780)

example: two particles in one dimension

position operators: $\hat{x}_1$, $\hat{x}_2$
momentum ": $\hat{p}_1$, $\hat{p}_2$

$$\Rightarrow \quad [\hat{x}_1, \hat{p}_1] = [\hat{x}_2, \hat{p}_2] = i\hbar$$

All other commutators vanish.

relative distance: $\hat{x}_{rel} = \hat{x}_1 - \hat{x}_2$
total momentum: $\hat{p}_{tot} = \hat{p}_1 + \hat{p}_2$

$$\Rightarrow \; [\hat{x}_{rel}, \hat{p}_{tot}] = [\hat{x}_1, \hat{p}_1] - [\hat{x}_2, \hat{p}_2] = 0$$

$\hookrightarrow$ We can (in principle) prepare a system with $p_{tot} = 0$ and $x_{rel} = x_0$.

This corresponds to a 2-particle wave function

$$\Psi(x_1, x_2) = \int dp \; e^{\frac{i}{\hbar}(x_1 - x_2 - x_0)p}$$

(modulo normalization)

$$\equiv \int dp \; \psi_p(x_2)\, u_p(x_1)$$

with

$$u_p(x_1) = e^{\frac{i}{\hbar} x_1 p}$$

$$\psi_p(x_2) = e^{-\frac{i}{\hbar}(x_2 + x_0)p}$$

We can interprete this as an expansion of Ψ in momentum eigenfunctions of particle 1,

$$\hat{p}_1 u_p(x_1) = \frac{\hbar}{i}\frac{\partial}{\partial x_1} e^{\frac{i}{\hbar}x_1 p} = p\, u_p(x_1)$$

with expansion coefficients $\psi_p(x_2)$

Now we <u>measure</u> the (a priori unknown) momentum p_1 and find $p_1 = p$

$\Rightarrow$ The wave function "collapses" to

$$\Psi(x_1, x_2) \longrightarrow \psi_p(x_2)\, u_p(x_1) = e^{\frac{i}{\hbar}(x_1 - x_2 - x_0)p}$$

$$\equiv \Psi'(x_1, x_2)$$

Since $p_{tot} = 0$ and we have not disturbed particle 2 (which we can assume to be far away from particle 1) by the measurement, we expect that particle 2 has momentum $p_2 = -p$.

Indeed:

$$\hat{p}_2 \Psi'(x_1, x_2) = \frac{\hbar}{i} \frac{\partial}{\partial x_2} \Psi'(x_1, x_2) = -p \, \Psi'(x_1, x_2)$$

(because the "coefficient" $\psi_p(x_2)$ is an eigenfunction of $\hat{p}_2$ with eigenvalue $-p$)

Alternatively we may measure the (again a priory unknown) position x_1 of particle 1, e.g. finding $x_1 = x$. Then, since we know the relative distance $x_1 - x_2 = x_0$, we expect for particle 2

$$x_2 = x_1 - x_0 = x - x_0$$

Formally we get this by expanding

$$\Psi(x_1, x_2) = \int dx \, \varphi_x(x_2) \, v_x(x_1)$$

in eigenfunctions of $\hat{x}_1$ with eigenvalue x,

$$v_x(x_1) = \delta(x - x_1)$$

$$\Rightarrow \hat{x}_1 v_x(x_1) = x_1 \delta(x - x_1) = x \, \delta(x - x_1) = x \, v_x(x_1)$$

$$\Rightarrow \Psi(x_1, x_2) = \varphi_{x_1}(x_2) \overset{!}{=} \int dp \, e^{\frac{i}{\hbar}(x_1 - x_2 - x_0)p}$$

$$\Rightarrow \varphi_x(x_2) = \int dp \, e^{\frac{i}{\hbar}(x - x_2 - x_0)p}$$

recall: $\int dk \, e^{ikx} = 2\pi \, \delta(x)$

$$\Rightarrow \varphi_x(x_2) = 2\pi\hbar \, \delta(x - x_2 - x_0)$$

= eigenfunction of $\hat{x}_2$ with eigenvalue $x - x_0$

c) After the measurement of $x_1 = x$ the wave function collapses to

$$\Psi(x_1, x_2) \rightarrow \Psi''(x_1, x_2) = \varphi_x(x_2) \, v_x(x_1)$$
$$= 2\pi\hbar \, \delta(x - x_2 - x_0) \, \delta(x - x_1)$$

$$\Rightarrow \hat{x}_2 \Psi''(x_1, x_2) = (x - x_0) \, \Psi''(x_1, x_2) \quad \checkmark$$

So, in summary:

By measuring $p_1 = p$
we can predict with certainty $p_2 = -p$
and by measuring $x_1 = x$
we can predict with certainty $x_2 = x - x_0$,
even if particle 2 is far away,
so that it cannot have been affected
by the measurement of particle 1.

A possible solution is that quantum mechanics is not complete, but there is an underlying theory where both particles have both, well defined positions and well defined momenta, but as hidden variables, so we do not know them until they are measured. So, for instance, it is not the measurement of $p_1 = p$ which causes $p_2 = -p$, but both properties already existed before the measurement, after the system with $p_{tot} = 0$ has been prepared.

<u>similar argument by D. Bohm:</u>

decay of a spin-0 particle in two spin-$\frac{1}{2}$ particles

2 spin operators $\quad \vec{S}^{(i)} = \begin{pmatrix} S_x^{(i)} \\ S_y^{(i)} \\ S_z^{(i)} \end{pmatrix}, \quad i = 1,2$

$$[S_e^{(j)}, S_m^{(k)}] = \delta_{jk}\, i\hbar \sum_n \varepsilon_{emn} S_n^{(j)}$$

$\Rightarrow$ Two different spin components of the same particle cannot simultaneously measured precisely.

Assuming that there are no spin-changing interactions, the two spin-$\frac{1}{2}$ particles must have total spin 0, i.e. they form a state

$$|S, M_S\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

$\Rightarrow$ The spin projections s_z of the particles must be opposite.

$\Rightarrow$ If we measure $s_z^{(1)} = +\frac{\hbar}{2}$
we must find $s_z^{(2)} = -\frac{\hbar}{2}$

(The state collapses to $|\uparrow\rangle|\downarrow\rangle$)

But since $|0,0\rangle$ is invariant under rotations, this must be true for spin projections along any directions, e.g.

$$s_x^{(1)} = \pm\frac{\hbar}{2} \quad \Rightarrow \quad s_x^{(2)} = \mp\frac{\hbar}{2}$$

etc.

So we may again ask whether the results of these measurements are encoded in hidden variables already before the measurement is taken.