1. Basics of nonrelativistic quantum mechanics

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- Goals:
  - general introduction to this course
  - among other things: preparation for the relativistic quantum mechanics

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- energy $E = \hbar\omega$ (Einstein 1905; photoelectric effect),
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$\Rightarrow$ Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V\right)\psi \equiv H\psi$

  - Hamiltonian: $H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V$

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Obviously, this also works with a real potential V.

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probability density: $\rho = \psi^*\psi = |\psi|^2$

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- Quantum mechanics: Q = probability to find a particle in $\mathcal{V}$
  (if correctly normalized as $\int_{\mathbb{R}^3} d^3r\,|\psi|^2 = 1$)

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(ii) Observables are represented by hermitian operators $\hat{O}$.

(iii) Possible measurements correspond to the eigenvalues of $\hat{O}$.

(iv) The corresponding eigenstates $|n\rangle$, i.e., $\hat{O}|n\rangle = \lambda_n|n\rangle$, form a complete orthonormal basis of the Hilbert space.

$$\langle m|n\rangle = \delta_{mn}, \qquad \sum_n |n\rangle\langle n| = \mathbb{1} \; .$$

(For continuous spectra one can generalize this to δ-functions and integrals.)

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  = sum over the possible measurements, weighted by their probability ✓

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  $|\vec{r}\rangle$: eigenstate of the coordinate operator
- momentum-space rep.: $\tilde{\psi}(\vec{p}, t) = \langle\vec{p}|\psi(t)\rangle$,
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1.4 Schrödinger vs. Heisenberg picture

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