Perturbative Quantum Chromodynamics {QCD}

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A high-energy electron on collision course with...

...a quark, confined in the proton.

\[
\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = +2
\]

\[
\text{Delta (+2)}
\]

\[
\frac{-1}{3} - \frac{1}{3} - \frac{1}{3} = -1
\]

\[
\text{Delta (-1)}
\]
Perturbative Quantum Chromodynamics \{QCD\}

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Brief Historical Overview

- 1950s Invention of bubble chambers and spark chambers lead to observation of a large number of different hadrons
- 1961 Gell-Mann and Yuval Ne‘eman invented the „eightfold way“
- 1963 Gell-Mann proposed, hadrons are build from Quarks [Nobel Prize 1969]
- 1965 Moo-Young Han and other solved Violation of the Pauli Principle (d++Barion) by giving the Quarks an additional degree of freedom
- 1969 Parton model was verified for deep inelastic scattering at SLAC
- 1973 David Gross, David Politzer and Frank Wilczek discovered „asymptotic freedom“ [Nobel Prize 2004]
- 1979 Evidence of Gluons was discovered in three jet events at PETRA
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Experimental results and perturbative QCD

Coupling parameter of the strong force

\[ \alpha_s(Q^2) = \frac{12\pi}{(33 - 2N_f)\ln(Q^2 / \Lambda^2)} \]
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Experimental results and perturbative QCD

- R ratio of Hadron and Myon in e+e- annihilation

\[
R = \frac{\sigma(e^- e^+ \to \text{Hadrons})}{\sigma_{QED}(e^- e^+ \to \mu^- \mu^+)} = 3 \sum_j Q_j^2 (1 + (\alpha_s / \pi) + 1.4(\alpha_s / \pi)^2 - 12.8(\alpha_s / \pi)^3)
\]
Standard Model

Is a unified Quantum field theory consisting of:

- Electroweak theory (Glashow, Salam, Weinberg)
  - Electromagnetic theory
  - Weak theory
- Quantum chromodynamics (Strong Force)

- A field theory can be described by a Lagrangian density
  \[ L = L(\phi_i, \partial \phi_i) \]
- Solving the functional \( S \)
  \[ S = \int d^n x \cdot L(\phi_i, \partial \phi_i) \]
- Leads to the Euler-Lagrange equations
  \[ \frac{\partial L}{\partial \phi_i} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi_i)} = 0 \quad i = 0,1,\ldots,n \]
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Standard Model

Two important equations:

- **Klein-Gordon equation**
  \[
  \left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right] \phi(t, x) = 0
  \]
  - Describes bosons (Spin=0, gauge bosons)
  - Scalar field $\phi$

- **Dirac equation**
  \[
  \left[ i\hbar c \sum_{\mu=0}^{3} \gamma^\mu \frac{\partial}{\partial x^\mu} - \beta mc^2 \right] \psi(x) = 0
  \]
  - Describes fermions (Spin=1/2, leptons, quarks)
  - Spinor field $\psi$
Standard Model

Symmetries:

- Electroweak theory
  \[ SU(2)_I \times U(1)_Y \]
- Quantum Chromodynamics
  \[ SU(3)_C \]

Standard model is a gauge theory with symmetry:
\[ SU(2)_I \times U(1)_Y \times SU(3)_C \]
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Group Theory

U means unitary: \(UU^+ = U^+U = I = UU^{-1}\)

A unitary operator can be written as \(U = \exp(i \cdot \sum \alpha_i \cdot \tau_i) \quad \alpha_i \in \mathbb{R} \quad \tau_i \) is hermitean

- A unitary transformation leaves the normalization of the state vectors unchanged
- U(1): Group of all unitary linear operators in one dimension
- Example: \(U_{\alpha}z = e^{i\alpha}z\)
- U(1) is abelian: \(e^{i\alpha}e^{i\beta} = e^{i\beta}e^{i\alpha} \quad z \in C; \alpha, \beta \in R\)

SU means specific unitary: \(\text{Det}(U) = 1 \Rightarrow \text{Trace}(\tau) = 0\)

- SU(2): Group of all specific unitary linear operators in two dimensions
- Generators of the SU(2) group: \(\frac{1}{2} \ast \text{Pauli matrices}\)

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

- SU(2) is non abelian:

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
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Group Theory

SU means specific unitary:  \( \text{Det}(U) = 1 \rightarrow \text{Trace}(\tau) = 0 \)

- **SU(3):** Group of all specific unitary linear operators in three dimensions
- **Generators of the SU(3) group:** \( \frac{1}{2} \) Gell-Mann matrices

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
\]

- **SU(3) is non abelian**
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Operators in QCD

- To change the color state of a quark we can construct operators from the Gell-Mann matrices

\[
I_+ = \frac{1}{2} (\lambda_1 + i \lambda_2) \quad I_- = \frac{1}{2} (\lambda_1 - i \lambda_2)
\]

\[
U_+ = \frac{1}{2} (\lambda_6 + i \lambda_7) \quad U_- = \frac{1}{2} (\lambda_6 - i \lambda_7)
\]

\[
V_+ = \frac{1}{2} (\lambda_4 + i \lambda_5) \quad V_- = \frac{1}{2} (\lambda_4 - i \lambda_5)
\]

- Two Gell-Mann matrices remain

\[
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\]

- They don’t change color!
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Operators in QCD

Example:
- Transition from green to red

\[
\begin{pmatrix}
R \\
G \\
B
\end{pmatrix}
\rightarrow
q_G = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\quad I_+ q_G = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + i \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = q_R
\]

- The quark colour changes from green to red
  - a specific gluon is absorbed

\[
q_R \xleftarrow{I_+} \quad \bar{G}R
\]

- So, you can relate a gluon to every operator!
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Gauge Theory

Gauge theory in Electrodynamics:
• Maxwell equations stay invariant under a transformation

\[ \phi \rightarrow \phi' = \phi - \frac{\partial X(r,t)}{\partial t} \hspace{1cm} A \rightarrow A' = A + \nabla X(r,t) \]

Example
• From the second Maxwell equation follows

\[ \nabla \cdot B = 0 \Rightarrow B = \nabla \times A \]

• Applying to the third equation yields

\[ \nabla \times (E + \frac{\partial A}{\partial t}) = 0 \]
\[ E + \frac{\partial A}{\partial t} = -\nabla \phi \]
\[ E = -\nabla \phi - \frac{\partial A}{\partial t} \]
\[ E = -\nabla (\phi + \frac{\partial X}{\partial t}) - \frac{\partial}{\partial t} (A + \nabla X(r,t)) \]

Maxwell _Equations_
\[ \nabla \cdot E = \rho \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \times B = j + \frac{\partial E}{\partial t} \]
Gauge Theory

Gauge theory in quantum mechanics:

- Schrödinger equation stays invariant under a transformation
  \[
  \phi \rightarrow \phi' = \phi - \frac{\partial X(r,t)}{\partial t} \quad A \rightarrow A' = A + \nabla X(r,t) \quad \psi \rightarrow \psi' = \exp(iqX)\psi
  \]

- Schrödinger equation for a particle with charge \(q\) in electromagnetic field \(\phi\)
  \[
  \left\{ \frac{1}{2m} (-i\nabla - qA)^2 + q\phi \right\} \psi(t,x) = i \frac{\partial \psi}{\partial t}
  \]

- We can combine \(\phi\) and \(A\) to the four vector
  \[
  A^\mu = (\phi, A)
  \]
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Gauge Theory

Gauge theory in QCD:

- The Lagrangian density is invariant
  \[ L = \bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi \]
- under a global transformation
  \[ \Psi' \rightarrow \exp(i\frac{g_s}{2} \lambda_i \beta_j)\Psi \]
  \[ L = \bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi = \bar{\Psi'}(i\gamma_\mu \partial^\mu - m)\Psi' \]
- If the transformation is local,
  \[ \Psi' \rightarrow \exp(i\frac{g_s}{2} \lambda_i \beta_j (x))\Psi \]
  \[ L = \bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi \neq \bar{\Psi'}(i\gamma_\mu \partial^\mu - m)\Psi' = \bar{\Psi}(i\gamma_\mu (i\frac{g_s}{2} \lambda_j (\partial^\mu \beta_j (x)) + \partial^\mu) - m)\Psi \]
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Gauge Theory

To make it invariant:

- $\partial^\mu$ is replaced by $D^\mu = \partial^\mu + i\frac{g_s}{2} \lambda_j G_j^\mu$, where $G_j^\mu$ are vector fields
- Gauge transformation of the vector fields

$$G'^\mu = G^\mu - \partial^\mu \beta_j - g_s f_{jkl} \beta_k G_j^\mu$$

- With the field-strength tensor

$$F_{j\nu}^{\mu} = \partial^\mu G_j^\nu - \partial^\nu G_j^\mu - g_s f_{jkl} G_k^\mu G_l^\nu$$

- We obtain the Lagrangian density of QCD

$$L = \overline{\Psi} (i\gamma^\mu D^\mu - m) \Psi - \frac{1}{4} F_{j,\mu\nu} F_j^{\mu\nu}$$

- The Dirac equation of a free particle obtained from the Lagrangian density is invariant under a local transformation

$$(i\gamma^\mu D^\mu - m) \Psi = 0 \Rightarrow (i\gamma^\mu D^\mu - m) \Psi' = 0$$
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Feynman Graphs

- With the $\gamma$-matrices
  \[ \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]

- The four gradient
  \[ \partial^\mu = (\frac{\partial}{\partial t}, -\nabla) \]

- We can write the Dirac equation of a particle with charge $e$ in the field $A^\mu$
  \[ (i\gamma^\mu \partial_\mu - m)\psi(x) = -e\gamma^\mu A_\mu(x)\psi(x) \]

- To solve this inhomogeneous differential equation we define a Green’s function $K$
  \[ (i\partial - m)K(x, x') = \delta^4(x - x') \]

- Solution of the Dirac equation
  \[ \psi(x) = -e \int K(x - x') A(x')\psi(x')d^4x' \]

Abbreviation
\[ \partial = \gamma^\mu \partial_\mu \]
\[ A = \gamma^\mu A_\mu \]
Feynman Graphs

• We can add any solution of the homogeneous Dirac equation to the solution

\[ \psi(x) = \phi(x) - e \int K(x - x') A(x') \psi(x') d^4 x' \]

• Perturbative calculation means, to handle the second term as small perturbation

\[ \psi^{(0)}(x) = \phi(x) \]

\[ \psi^{(1)}(x) = \phi(x) - e \int K(x - x') A(x') \psi^{(0)}(x') d^4 x' \]

\[ \psi^{(2)}(x) = \phi_i(x) - e \int d^4 x' K(x - x') A(x') \phi_i(x') + e^2 \int d^4 x' d^4 x'' K(x - x'') A(x'') K(x'' - x') A(x') \phi_i(x') \]

• Now the only thing we need is the Green’s function K

• We define the Fourier transform of K

\[ K(x - x') = (2\pi)^{-4} \int d^4 p \cdot \tilde{K}(p) \cdot \exp(-i \cdot p(x - x')) \]

• and find the electron propagator \( \tilde{K}(p) \)

\[ \tilde{K}(p) = \frac{p + m}{p^2 - m^2 + i\varepsilon} \]
Feynman Graphs

• With the electron propagator and integration we find the Green’s function for a particle moving forward in time to be

\[
K(x-x') = -i(2\pi)^{-3} \int d^3 p \cdot \exp[ip(x-x')-iE(t-t')] \cdot \frac{+\gamma^0 E - \gamma \cdot p + m}{2E}
\]

• And for a particle moving backwards in time

\[
K(x-x') = -i(2\pi)^{-3} \int d^3 p \cdot \exp[ip(x-x')+iE(t-t')] \cdot \frac{-\gamma^0 E - \gamma \cdot p + m}{2E}
\]
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Scattering in QED

- Defining the scattering matrix as
  \[ \psi_{\text{Scat}} = S \cdot \phi_i = \phi(x) - e \int K(x - x') A(x') \psi^{(0)} d^4 x' \]
- We can calculate the matrix elements in first order
  \[ S_{fi}^{(1)} = -e \int d^4 x' \int d^3 x_2 \phi_f^+ (x_2) K(x_2 - x') A(x') \phi_i (x') \]
- To be
  \[ S_{fi}^{(1)} = ie \int d^4 x' \phi_f^* (x') A(x') \phi_i (x') \]
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Scattering in QED

- Matrix element for this example is

\[
S_{fi}^{(1)} = ie^2 \int d^4q \delta^4(p_4 - p_2 + q) \delta^4(p_3 - p_1 - q) \cdot (2\pi)^4 \cdot \bar{u}(p_3) \gamma_\mu u(p_1) \cdot \frac{-g^{\mu\nu}}{q^2 + i\epsilon} \bar{u}(p_4) \gamma_\nu u(p_2)
\]

\[\begin{align*}
\bar{u}(p_3) & \quad -ie\gamma_\mu \delta^4(p_3 - p_1 - q) \\
\bar{u}(p_4) & \quad -ie\gamma_\nu \delta^4(p_4 - p_2 + q) \\
u(p_1) & \quad \frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \\
u(p_2) & \quad \end{align*}\]

**Solution of Dirac Equation**

\[
\begin{align*}
u_1(p) = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E + m} \\ \frac{p_x + ip_y}{E + m} \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
u_2(p) = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E + m} \\ -\frac{p_x - ip_y}{E + m} \end{pmatrix}
\end{align*}
\]
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Feynman Graphs in QED

- Incoming (outgoing) fermion: $u(p), \bar{u}(p)$
- Incoming (outgoing) photon: $\epsilon_{\mu}(k), \epsilon_{\mu}^*(k)$
- Virtual photon: $-g_{\mu\nu} \frac{1}{q^2 + i\epsilon}$
- Virtual electron: $\frac{p + m}{p^2 - m^2 + i\epsilon}$
- Lepton-Photon-Vertex $-ie \gamma_\nu \delta^4(\p_f - \p_i + q)$

**Diagram:**

- **Photon**
- **Vertex**
- **Fermion**
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Feynman Graphs in QED

- Elementary processes

*Time*

\[ Emission \quad Absorption \quad Production \quad Annihilation \]
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Feynman Graphs in QCD

- Incoming (outgoing) fermion: \( u(p), \bar{u}(p) \)
- Incoming (outgoing) photon: \( \varepsilon_\mu(k), \varepsilon_\mu^*(k) \)
- Virtual gluon: \( \frac{-g^{\mu\nu}\delta_{ab}}{q^2 + i\varepsilon} \)
- Virtual electron: \( \frac{p + m}{p^2 - m^2 + i\varepsilon} \)
- Quark-Gluon-Vertex

\[ -ig\gamma^k\lambda^a\delta^4(p_f - p_i + q) \]

\( \sim g^2 \)

\( \sim g \)
Asymptotic Freedom

- As seen, we can describe processes in QED in low order approximations.
- The approximation converges fast, because the coupling constant is small:
  \[
  \alpha = \frac{1}{4\pi \varepsilon_0 \frac{e^2}{\hbar c}} \approx \frac{1}{137} << 1
  \]

- In QCD it is not sure, that the perturbative calculation converges.
- Therefore, a very important discovery was:
  \[
  \alpha_s(Q^2) = \frac{12\pi}{(33 - 2N_f) \ln(Q^2 / \Lambda^2)}
  \]
  \[
  Q \to \infty \Rightarrow \alpha \to 0
  \]
Asymptotic Freedom

• There is also a $Q$ dependence in the QED
  – Not as strong
  – inverse behaviour

$$Q^2 \to \infty \Rightarrow \alpha(Q^2) < \alpha(\infty)$$

Example:

• Positive charge in dielectric oil

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \quad F = \frac{qq'}{4\pi\varepsilon_0 r^2}$$

• Following QED there are vacuum fluctuation creating virtual electron positron pairs, that are causing this effect of shielding
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Asymptotic Freedom

- Experiments at LEP support this theory

\[
\alpha(M_Z^2) = \frac{1}{128.9} \quad \alpha(Q^2 \approx \text{small}) = \frac{1}{137}
\]

Feynman diagram
Asymptotic Freedom

- The $Q$ dependence in QCD is stronger
  - $\alpha$ is called running coupling constant
  - Two effects contribute
    - Quark-Antiquark production
    - Emission of a Gluon
- Quark-Antiquark production would dominate only if we had more than 16 quark flavours!
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Asymptotic Freedom

- Feynman diagram quark-antiquark production

- Feynman diagram emission of a gluon

GluonLoop
Thank you for your attention!