Inhomogeneous phases in effective models

based on arXiv:1312.3244 [nucl-th]

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Hirschegg, January 15th 2014

Outline

Motivation

With the extended linear sigma model a accurate description of low mass hadrons is possible.
Together with the parity doublet model it allows to calculate vacuum physics as well as nuclear matter conditions.

...BUT...

...is the nuclear matter ground state homogeneous?
How to address modulations beyond CDW?

The meson fields

scalar and pseudoscalar as well as vector and axial-vector fields

$$\Phi = (\sigma + \imath\eta_N)t_0 + (\vec{a}_0 + \imath\vec{\pi})\cdot\vec{t}\ ,$$
$$V^\mu = \omega^\mu t_0 + \vec{\rho}^\mu \cdot \vec{t} \quad \text{and} \quad A^\mu = f_1^\mu t_0 + \vec{a}_1^\mu \cdot \vec{t}\ ,$$

with $\vec{t} = \vec{\tau}/2$ and $t_0 = \mathbf{1}_2/2$.

transformation under chiral group $U(2)_R \times U(2)_L$

$$\Phi \to U_L \Phi U_R^\dagger,\ R^\mu \to U_R R^\mu U_R^\dagger,\ \text{and}\ L^\mu \to U_L L^\mu U_L^\dagger,$$

$R^\mu \equiv V^\mu - A^\mu$, $L^\mu \equiv V^\mu + A^\mu$.

D. Parganlija, F. Giacosa, D. H. Rischke, arXiv:1003.4934 [hep-ph],

D. Parganlija, P. Kovacs, G. Wolf, F. Giacosa and D. H. Rischke, arXiv:1208.2054 [hep-ph].

The Lagrangian for the mesons

$$
\begin{aligned}
\mathcal{L}_{\text{mes}} = & \operatorname{Tr}\left[(D_\mu \Phi)^\dagger (D^\mu \Phi) \quad - m^2 \Phi^\dagger \Phi \; - \lambda_2 \left(\Phi^\dagger \Phi\right)^2\right] - \lambda_1 \left[\operatorname{Tr}\left(\Phi^\dagger \Phi\right)\right]^2 \\
& + \operatorname{Tr}\left[H\left(\Phi^\dagger + \Phi\right)\right] \; + c\left(\det \Phi^\dagger - \det \Phi\right)^2 \\
& - \frac{1}{4}\operatorname{Tr}\left(L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}\right) + \operatorname{Tr}\left[\left(\frac{1}{2}m_1^2 + \Delta\right)\left(L_\mu L^\mu + R_\mu R^\mu\right)\right] \\
& + \frac{1}{2}h_1 \operatorname{Tr}\left(\Phi^\dagger \Phi\right)\operatorname{Tr}\left(L_\mu L^\mu + R_\mu R^\mu\right) + h_2 \operatorname{Tr}\left(\Phi^\dagger L^\mu L_\mu \Phi + \Phi R^\mu R_\mu \Phi^\dagger\right) + 2h_3 \operatorname{Tr}\left(\Phi R_\mu \Phi^\dagger L^\mu\right) \\
& + 2h_3 \operatorname{Tr}\left(\Phi R_\mu \Phi^\dagger L^\mu\right) - \imath \frac{g_2}{2}\left(\operatorname{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \operatorname{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}\right) + \ldots
\end{aligned}
$$

spontaneous symmetry breaking, explicit symmetry breaking, $U(1)_A$ anomaly.

covariant derivative and field strength tensors :

$D^\mu \Phi = \partial^\mu \Phi - \imath g_1 (\Phi R^\mu - L^\mu \Phi)$

$L^{\mu\nu} = \partial^\mu L^\nu - \partial^\nu L^\mu$

$R^{\mu\nu} = \partial^\mu R^\nu - \partial^\nu R^\mu$

The Lagrangian for the mesons

$$
\begin{aligned}
\mathcal{L}_{\text{mes}} =& \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{1}{4}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)^2 \\
&+ \frac{1}{2}m^2(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2 + \varepsilon\sigma + \frac{1}{2}m_\omega^2\omega_0^2 \\
&+ \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m_\chi^2\chi^2 + g\chi(\sigma^2+\pi^2)
\end{aligned}
$$

besides σ a second scalar resonance χ is present, possibly a tetraquark state or a $\pi\pi$ resonance identified with $f_0(500)$.

chiral symmetry is explicit and spontaneous broken, and only terms are considered that will eventually condensate.

The mirror assignment

mirror assignment

$$\psi_{1,R} \to U_R\ \psi_{1,R}\ , \qquad \psi_{1,L} \to U_L\ \psi_{1,L},$$
$$\psi_{2,R} \to U_L\ \psi_{2,R}\ , \qquad \psi_{2,L} \to U_R\ \psi_{2,L}$$

$$+m_0\left(\bar{\psi}_{2,L}\psi_{1,R} - \bar{\psi}_{2,R}\psi_{1,L} - \bar{\psi}_{1,L}\psi_{2,R} + \bar{\psi}_{1,R}\psi_{2,L}\right)$$

in our case $m_0 = a\chi \to$ additional nucleon mass contribution generated via the tetraquark field condensation

The mirror assignment

Baryon Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{bar}} = {} & \bar{\psi}_{1,L} \imath \not{D}_{1,L} \psi_{1,L} + \bar{\psi}_{1,R} \imath \not{D}_{1,R} \psi_{1,R} + \bar{\psi}_{2,L} \imath \not{D}_{2,L} \psi_{2,L} + \bar{\psi}_{2,R} \imath \not{D}_{2,R} \psi_{2,R} \\
& - \hat{g}_1 \left(\bar{\psi}_{1,L} \Phi \psi_{1,R} + \bar{\psi}_{1,R} \Phi^{\dagger} \psi_{1,L} \right) - \hat{g}_2 \left(\bar{\psi}_{2,L} \Phi^{\dagger} \psi_{2,R} + \bar{\psi}_{2,R} \Phi \psi_{2,L} \right) \\
& + a\chi \left(\bar{\psi}_{2,L} \psi_{1,R} - \bar{\psi}_{2,R} \psi_{1,L} - \bar{\psi}_{1,L} \psi_{2,R} + \bar{\psi}_{1,R} \psi_{2,L} \right)
\end{aligned}
$$

$D^{\mu}_{1,R} = \partial^{\mu} - \imath c_1 R^{\mu}$, $D^{\mu}_{1,L} = \partial^{\mu} - \imath c_1 L^{\mu}$, $D^{\mu}_{2,R} = \partial^{\mu} - \imath c_2 R^{\mu}$, and $D^{\mu}_{2,L} = \partial^{\mu} - \imath c_2 L^{\mu}$.

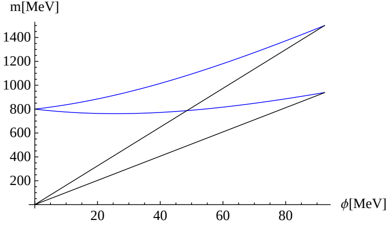
only the mesons π, σ, ω, and χ are retained

$$
\begin{aligned}
\mathcal{L}_{\text{bar}} = {} & \overline{\Psi}_1 \imath \gamma_{\mu} \partial^{\mu} \Psi_1 + \overline{\Psi}_2 \imath \gamma_{\mu} \partial^{\mu} \Psi_2 - \frac{\widehat{g}_1}{2} \overline{\Psi}_1 (\sigma + \imath \gamma_5 \tau^3 \pi) \Psi_1 - \frac{\widehat{g}_2}{2} \overline{\Psi}_2 (\sigma - \imath \gamma_5 \tau^3 \pi) \Psi_2 \\
& - g_{\omega} \overline{\Psi}_1 \imath \gamma_{\mu} \omega^{\mu} \Psi_1 - g_{\omega} \overline{\Psi}_2 \imath \gamma_{\mu} \omega^{\mu} \Psi_2 - a\chi (\overline{\Psi}_2 \gamma_5 \Psi_1 - \overline{\Psi}_1 \gamma_5 \Psi_2)
\end{aligned}
$$

S. Gallas, F. Giacosa, G. Pagliara, arXiv:1105.5003v1 [hep-ph]

Mirror assignment

not just nucleon N but also its chiral partner N^*
chiral eigenstates are not equal to mass eigenstates
$\Rightarrow$ mass eigenstates emerge after diagonalization



S. Gallas, F. Giacosa, arXiv:1308.4817 [hep-ph]

naive assignment ($m_0 = 0$):

$$m_N = m_{\Psi_1} \propto \phi$$
$$m_{N^*} = m_{\Psi_2} \propto \phi$$

mirror assignment ($m_0 \neq 0$):

$$m_N = \frac{1}{2}\sqrt{\left(\hat{g}_1 + \hat{g}_2\right)^2 \phi^2 + 4m_0^2} + \frac{1}{4}\left(\hat{g}_1 - \hat{g}_2\right)\phi$$
$$m_{N^*} = \frac{1}{2}\sqrt{\left(\hat{g}_1 + \hat{g}_2\right)^2 \phi^2 + 4m_0^2} - \frac{1}{4}\left(\hat{g}_1 - \hat{g}_2\right)\phi$$

The chiral density wave

Ansatz for chiral density wave:

$$\langle\sigma\rangle \sim \phi\cos(2\ f\ x)\ , \qquad \langle\pi\rangle \sim \phi\sin(2\ f\ x)$$

effect on mesons and baryons

for the mesons additional contributions in the potential arise:

$$U_{\text{mes}}^{\text{mean-field}} = 2f^2\phi^2 + \frac{\lambda}{4}\phi^4 - \frac{1}{2}m^2\phi^2 - \varepsilon\phi\cos(2fx) - \frac{1}{2}m_\omega^2\bar{\omega}_0^2 + \frac{1}{2}m_\chi^2\bar{\chi}^2 - g\bar{\chi}\phi^2$$

the baryons develop a coordinate space dependence which can be transformed to a momentum dependence:

$$\psi_1 \to \exp[-\imath\gamma_5\tau_3 fx]\psi_1, \quad \psi_2 \to \exp[+\imath\gamma_5\tau_3 fx]\psi_2.$$

A. H., F. Giacosa, D. H. Rischke, arXiv:1312.3244 [nucl-th]

Effective potential

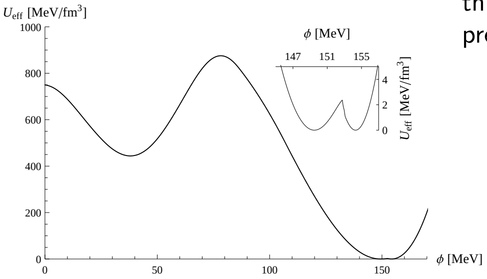
$$U_{\text{eff}}(\phi, \bar{\chi}, \bar{\omega}_0, f) = \sum_{k=1}^{4} \int \frac{2d^3p}{(2\pi)^3}[E_k(p) - \mu^*]\Theta[\mu^* - E_k(p)] + U_{\text{mes}}^{\text{mean-field}}$$

short notation $\mu^* = \mu - g_\omega \bar{\omega}_0$

meson mean fields are obtained by minimization U_{eff}

$$0 \stackrel{!}{=} \frac{\partial U_{\text{eff}}}{\partial \phi},\ 0 \stackrel{!}{=} \frac{\partial U_{\text{eff}}}{\partial \bar{\chi}},\ 0 \stackrel{!}{=} \frac{\partial U_{\text{eff}}}{\partial \bar{\omega}_0},\ 0 \stackrel{!}{=} \frac{\partial U_{\text{eff}}}{\partial f}$$

Potential at $\mu = 923$ MeV

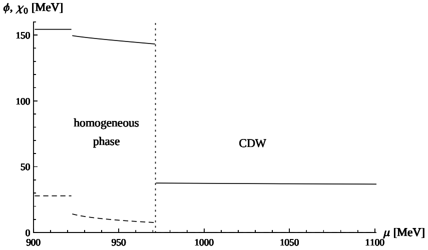


three distinguished minima are present:

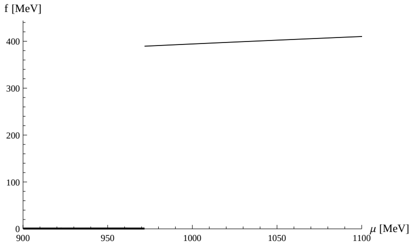
- vacuum ground state at $\phi = 154.4$ MeV
- homogeneous nuclear matter ground state at $\phi = 149.5$ MeV
- local minima at $\phi = 38.3$ MeV with $f \neq 0$

ϕ as a function of the extrema of $\bar{\chi}$, $\bar{\omega}_0$ and f

Dynamical quantities as a function of μ



solid line: ϕ, dashed line: $\bar{\chi}$



f is of the size of 1.5fm

At $\mu = 973$ MeV a transition to the inhomogeneous ground state is realized. The transition corresponds to a mixed phase between $2.4\rho_0$ to $10.4\rho_0$.

Gross-Neveu Lagrangian in $1+1$ dimensions

$$\mathscr{L}_{\mathrm{GN}} = \sum_{i=1}^{N} \bar{\psi}_i \left(\imath \gamma_\mu \partial^\mu + \gamma_0 \mu - m_0 \right) \psi_i + \frac{1}{2} g^2 \left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^2$$

- in $1+1$ dimensions renormalizable
- discrete chiral symmetry
- asymptotic freedom
- in the large N limit chiral symmetry is broken

M. Thies, K. Urlichs, arXiv:hep-th/0302092

M. Wagner, arXiv:hep-ph/0704.3023v1 [hep-lat]

GN in a box

the partition function:

$$Z = \int \left(\prod_{i=1}^{N} D\bar{\psi}_i D\bar{\psi}_i \right) e^{-S} \,, S = \int d^2x \left(\frac{1}{2g^2}\sigma^2 + \sum_{i=1}^{N} \bar{\psi}_i Q \psi_i \right)$$

$$Q = \gamma_\mu \partial^\mu + \gamma_0 \mu + \sigma$$

for arbitrary operator Q the action reads:

$$\frac{S}{N} = \frac{1}{2\lambda} \int d^2x \ \sigma^2 - \frac{1}{2} \ln \left[\det \left(Q^\dagger Q \right) \right]$$

homogeneous case: $\det \left(Q^\dagger Q \right) = \left(k_0^2 + k^2 + \sigma^2 - \mu^2 \right)^2 - (2\mu k_0)^2$

Setting the stage

all variables in terms of a physical scale σ_0:

- chiral condensate: $\hat{\sigma} = \frac{\sigma}{\sigma_0}$
- chemical potential: $\hat{\mu} = \frac{\mu}{\sigma_0}$
- the spacial lattice size leads to momentum cutoff: $L \to \hat{f}$ with N modes
- the temporal lattice size leads to temperature cutoff: $L_0 \to \hat{s} = \frac{1}{N_0 \hat{t}}$ with N_0 modes

discretized version

$$\frac{S}{N} = \frac{\pi}{\lambda} N N_0 \frac{\hat{s}}{\hat{f}} \hat{\sigma}^2 - \frac{1}{2} \sum_{n_0=0}^{N_0-1} \sum_{n=-N}^{N} \ln \left\{ \left[\left(\frac{2\pi}{\hat{s} N_0} \left(\frac{1}{2} + n_0 \right) \right)^2 + \left(\hat{f} \frac{n}{N} \right)^2 + \hat{\sigma}^2 - \hat{\mu}^2 \right]^2 + \left[2\hat{\mu} \frac{2\pi}{\hat{s} N_0} \left(\frac{1}{2} + n_0 \right) \right]^2 \right\}$$

Adding physics to the numerics

consider the gap equation $0 \overset{!}{=} \frac{\partial(S/N)}{\partial\hat{\sigma}}$:

in the vacuum the chiral symmetry is broken:

$$\hat{t} = \hat{t}_0 \to 0 : \qquad \hat{\sigma} \to \hat{\sigma}_0, \quad N_0 \to N_{00}$$

the number N_{00} corresponds to lowest temperature available

at a critical temperature chiral symmetry is restored:

$$\hat{t} = \hat{t}_c : \qquad \hat{\sigma} = 0, \quad N_0 \to N_{0c}$$

the number N_{0c} corresponds to the critical temperature

Combining both allows to connect the vacuum value $\hat{\sigma}_0$ to the critical temperature $\hat{t}_c$. Since no physical scale is present it is possible to set $\hat{\sigma}_0 = 1$.

Fixing N, N_{00}, N_{0c}, $\hat{s}$ and $\hat{f}$

parameters that are limited by computational power

N_{00} and N: the larger the values the closer to the continuum limit

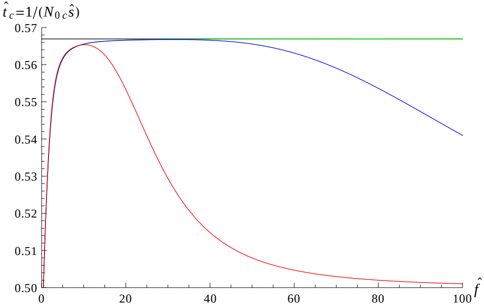
parameters that connect numerics to physics

$\hat{s}$ determines the critical temperature $\hat{t}_c$ via the coupled gap equation.

parameters to optimize the convergence

N_{0c} and $\hat{f}$ are chosen in such a way that $\frac{\partial \hat{t}_c}{\partial N_{0c}} \overset{!}{=} 0$ and $\frac{\partial \hat{t}_c}{\partial \hat{f}} \overset{!}{=} 0$

Fixing $\hat{f}$ as a function of N



- we set $N_{00} = 512$ and $N_{0c} = 66$
- vary $\hat{f}$ and calculate $\hat{s}$
- find maximum for $\hat{t}_c$

red line: $N = 12$, blue line $N = 48$,green line: $N = 144$, black line: analytic result

Inhomogeneous condensates

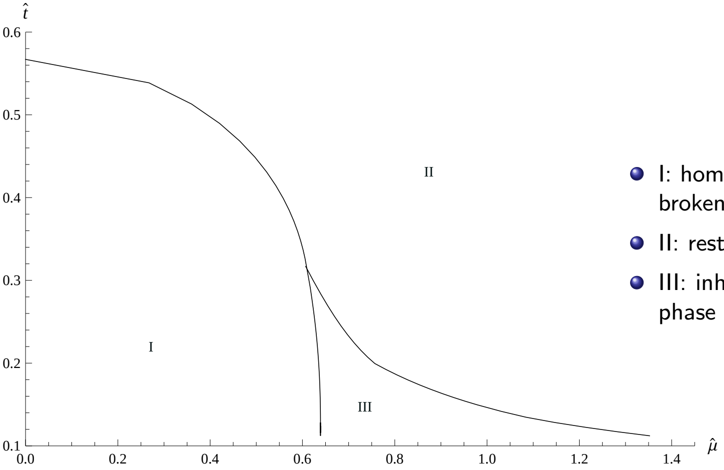
$$\hat{\sigma}(x) = \sum_{n=-N}^{N} c_n e^{\imath n x}$$

Q operator with arbitrary condensate

$$\begin{aligned} Q =& \delta_{k_0,k_0'}\delta_{k,k'}\left(\imath k_0 + \mu + \imath\gamma_0\gamma_1 k - \gamma_0 c_0\right) \\ &- \delta_{k_0,k_0'}\gamma_0 \sum_{n=1}^{N}\left(c_n\delta_{k,k'+n} + c_n^*\delta_{k+n,k'}\right) \end{aligned}$$

$$\frac{S}{N} = \frac{1}{2\lambda}\int d^2x\ \sigma^2 - \frac{1}{2}\ln\left[\det\left(Q^\dagger Q\right)\right]$$

GN phase diagram $N_0 = 94$ and $N = 144$



- I: homogeneous broken phase
- II: restored phase
- III: inhomogeneous phase

$$M3(\hat{t}, \hat{\mu}) = (0.6082, 0.3183)_{\text{analytic}} \simeq (0.6097, 0.3140)_{N_0=96,\ N=144}$$

The NJL Lagrangian in a box

The NJL Lagrangian

$$Z = N \int D\psi_i^\dagger D\psi_i \exp\left[\int_0^\beta d\tau \int d^3x \, \mathscr{L}_{\text{mean-field}}\right]$$

$$= N' \det Q \exp\left[-\int_0^\beta d\tau \int d^3x \, \frac{\lambda}{2} m_0^2\right]$$

with

$$\mathscr{L}_{\text{mean-field}} = \bar{\psi}_i \left(\partial\!\!\!/ + m_0\right) \psi_i - \frac{\lambda}{2} m_0^2$$

looks similar but it is not renormalizable and computational effort increases by a factor N^2. The regularization schema of choice is Pauli-Villars.

The NJL Lagrangian in a box

brute force calculation of the determinant is no longer effective
$\rightarrow$ solving for the energy eigenstates is much faster

$$\gamma_0(\imath\vec{\gamma}\vec{\partial} - \sigma(z))\psi(\vec{x}) = E\psi(\vec{x})$$

$$\begin{aligned} \gamma_0(\imath\vec{\gamma}\vec{\partial} - \sigma(z))\gamma_0(\imath\vec{\gamma}\vec{\partial} - \sigma(z))\psi(\vec{x}) &= E^2\psi(\vec{x}) \\ \Leftrightarrow \left[\partial_1\partial^1 + \partial_2\partial^2 + (-\imath\gamma_3\partial^3 - \sigma(z))(\imath\gamma_3\partial^3 - \sigma(z))\right]\psi(\vec{x}) &= E^2\psi(\vec{x}) \end{aligned}$$

a separation Ansatz allows to calculate the three parts individually

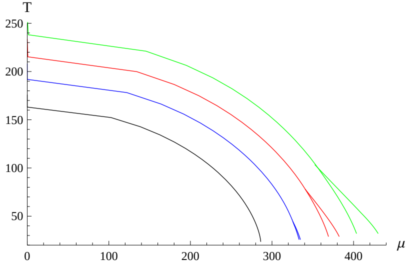
Performance for $m_0 = 241$ MeV and $\Lambda_{PV} = 859$ MeV

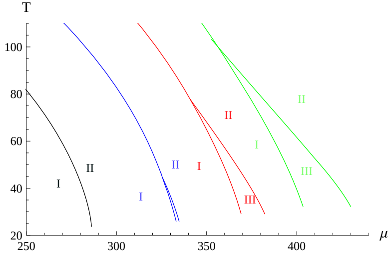
N_i	N_{0c}	$\hat{s}$	$\hat{f}$	$\hat{t}_c$	$\hat{t}_0$
12	11	$1.346 \cdot 10^{-1}$	$1.150 \cdot 10^{1}$	$6.757 \cdot 10^{-1}$	$8.458 \cdot 10^{-2}$
24	12	$1.232 \cdot 10^{-1}$	$1.868 \cdot 10^{1}$	$6.767 \cdot 10^{-1}$	$8.458 \cdot 10^{-2}$
48	13	$1.137 \cdot 10^{-1}$	$3.083 \cdot 10^{1}$	$6.768 \cdot 10^{-1}$	$9.166 \cdot 10^{-2}$
72	13	$1.137 \cdot 10^{-1}$	$4.180 \cdot 10^{1}$	$6.769 \cdot 10^{-1}$	$9.166 \cdot 10^{-2}$
108	14	$1.055 \cdot 10^{-1}$	$5.709 \cdot 10^{1}$	$6.769 \cdot 10^{-1}$	$9.871 \cdot 10^{-2}$
144	14	$1.055 \cdot 10^{-1}$	$7.151 \cdot 10^{1}$	$6.769 \cdot 10^{-1}$	$9.871 \cdot 10^{-2}$

$N_{00} = 96$

all values are in units of $m_0 = 241$ MeV

Phase diagram





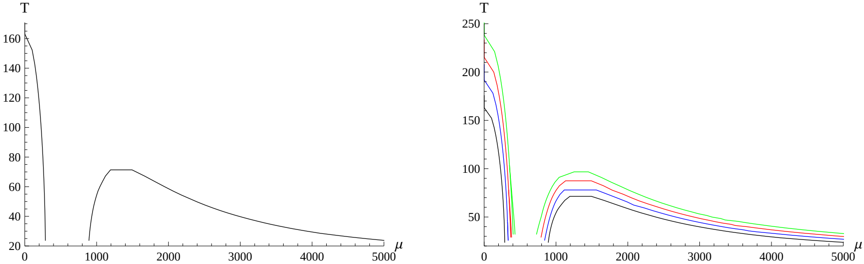
- I: homogeneous broken phase
- II: restored phase
- III: inhomogeneous region

$m_0 = 241,\ 300,\ 350,\ 400$ MeV, $\Lambda_{PV} = 859$ MeV, $N_i = 144$ and $N_{00} = 96$

compare to:

D. Nickel, arXiv:0906.5295 [hep-ph]

New measure of the continent



$m_0 = 241,\ 300,\ 350,\ 400$ MeV, $\Lambda_{PV} = 859$ MeV, $N_i = 144$ and $N_{00} = 96$

compare to:
S. Carignano, M. Buballa, arXiv:1111.4400 [hep-ph]

S. Carignano, M. Buballa, arXiv:1203.5343 [hep-ph]

Summary and Outlook

- relevance of inhomogeneous phases in the parity doublet model
- solid method to calculate inhomogeneous phases
- implementation for NJL-model in $3 + 1$ dimensions

- Quark-Meson model
- $2 + 1$ dimensions
- higher dimensional modulations
- magnetic fields
- diquark condensation
- ...