The Quark Propagator in Superconducting Phases

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Outline

1. Motivation and introduction
2. The quark propagator in the chirally unbroken phase
3. The quark propagator in the 2SC/CFL phase for massless quarks
4. The quark propagator in the 2SC/CFL phase for massive quarks
5. Summary and outlook
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cold and dense quark matter expected to be a color superconductor
⇒ investigations in

- NJL-type models
- weak coupling at asymptotically large densities

quark propagator provides information about
- dynamical symmetry breaking
- pressure, i.e. thermodynamical potential
- excitation spectrum
- ...

⇒ Extend successful truncation scheme of DSE’s in vacuum to finite densities!
Dyson-Schwinger equation for quark propagator

\[ -1 = -1 + \Box \]

our approach:

- approximate gluon propagator and quark-gluon vertex separately

other approaches:

- weak coupling expansion
  (T. Schäfer, F. Wilczek, 1999; R. Pisarski, D. Rischke, 1999; D. Hong et al., 1999)

- phenomenological coupling
  (C. Roberts, S. Schmidt, 2000)
Approximation of quark-gluon vertex

**abelian vertex construction**

\[ \Gamma^a(p, q; k = p - q) \rightarrow i g \Gamma(k^2) \gamma_\mu \frac{\lambda^a}{2} \]
Approximation of quark-gluon vertex

**abelian vertex construction**

\[
\Gamma_a(p, q; k = p - q) \quad \longrightarrow \quad ig \Gamma(k^2) \gamma_\mu \frac{\lambda^a}{2}
\]

**Γ(k^2) determined by**

DSE studies including Yang-Mills sector in Landau gauge:
chosen such that
- propagator multiplicative renormalizable
- correct anomalous dimension of mass function

analysis of lattice QCD data for propagators in Landau gauge:
invert DSE for Γ(k^2) for given quark and gluon propagator
Approximation of gluon propagator

Vacuum gluon propagator in Landau gauge

\[ D_{\mu\nu}^{ab\text{ vac}}(k) = \frac{Z(k^2)}{k^2} T_{\mu\nu}(k) \delta^{ab} \]
Approximation of gluon propagator

vacuum gluon propagator in Landau gauge

\[ D^{ab \, \text{vac}}_{\mu\nu}(k) = \frac{Z(k^2)}{k^2} T_{\mu\nu}(k) \delta^{ab} \]

incorporate medium effects of particle-hole excitations

add medium polarisation of bare quarks to inverse vacuum propagator:

\[ D^{ab \, \text{int}}_{\mu\nu} = D^{ab \, \text{vac}}_{\mu\nu}^{-1} + \Gamma \]
Approximation of gluon propagator

**Vacuum gluon propagator in Landau gauge**

\[
D_{\mu\nu}^{ab\text{ vac}}(k) = \frac{Z(k^2)}{k^2} T_{\mu\nu}(k) \delta^{ab}
\]

**Incorporate medium effects of particle-hole excitations**

Add medium polarisation of bare quarks to inverse vacuum propagator:

\[
D_{\mu\nu}^{ab -1} = D_{\mu\nu}^{ab\text{ vac} -1} + D_{\mu\nu}^{G(k)} P_T^{\mu\nu}(k) + D_{\mu\nu}^{F(k)} P_L^{\mu\nu}(k) \delta^{ab}
\]

**Medium gluon propagator in Landau gauge**

\[
D_{\mu\nu}^{ab}(k) = \left( \frac{Z(k^2)}{k^2 + G(k)} P_T^{\mu\nu}(k) + \frac{Z(k^2)}{k^2 + F(k)} P_L^{\mu\nu}(k) \right) \delta^{ab}
\]
Truncated Dyson-Schwinger equation

\[ S^{-1}(p) = Z_2 S_0^{-1}(p) + \frac{Z_2}{3\pi^3} \int d^4q \gamma_\mu S(q) \gamma_\nu \left( \frac{\alpha_s(k^2)}{k^2 + G(k)} P^T_{\mu\nu} + \frac{\alpha_s(k^2)}{k^2 + F(k)} P^L_{\mu\nu} \right) \]

- similar to HDL
- strong running coupling

\[ \alpha_s(k^2) = \frac{Z_1 F}{Z_2^2} g^2 Z(k^2) \Gamma(k^2) \]

screening and damping included through

\[ m_g(k^2)^2 = \frac{N_f \mu^2 \alpha_s(k^2)}{\pi} \]
\[ F(k) = 2 m_g(k^2)^2 + \ldots \]
\[ G(k) = \frac{\pi}{2} m_g(k^2)^2 \frac{k_4}{|k|} + \ldots \]
Strong running coupling $\alpha_s$

- with abelian vertex construction DSE studies underestimate chiral symmetry breaking in vacuum ($M_q \approx 170\text{MeV}$) 
  (C.S. Fischer, R. Alkofer, 2003)
- lattice studies cannot constrain deep infrared 
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The unbroken phase - I

quasiparticle propagator in chiral limit:

\[ \begin{align*}
S^{-1} &= S^{+\ -1} \gamma_4 \Lambda^+ + S^{-\ -1} \gamma_4 \Lambda^- \\
S^{+\ -1} &= -ip_4 + \mu - |\vec{p}| + \Sigma^+ \\
&= -ip_4 \left( 1 - \frac{\text{Im} \Sigma^+}{p_4} \right) + \mu - |\vec{p}| + \text{Re} \Sigma^+
\end{align*} \]

Static magnetic gluon exchange renders unbroken phase a non-Fermi liquid. leading non-analytic contribution:

\[ \text{Im} \Sigma^+ \approx -\frac{4}{3\pi} \int_{k > |p-p_F|} dk \frac{\alpha_s(k^2) k}{k^2 + \frac{\pi}{2} m(k^2)^2 |p_4|} \]

\[ \rightarrow \begin{align*}
p = p_F & \quad \frac{4}{9} \frac{\alpha_s(\Lambda_1^2)}{\pi} \ln \left( \frac{|p_4|}{\Lambda_2} \right)
\end{align*} \]

\[ \Lambda_1 \propto |p_4|^{\frac{1}{3}} \] dominating scale of contributions
for coupling from DSE studies and $\mu = 1\text{GeV}$:

- logarithmic divergence constrained to small energies (temperatures)
- sensitive to infrared behavior of strong coupling
- dependence on $\vec{p}$ included
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Nambu-Gor’kov formalism

In Nambu-Gor’kov space with bispinors $\Psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}$, the quark DSE

$$S^{-1} = Z_2 S_0^{-1} + Z_1 F \Sigma$$

includes the gap-equation

$$\Sigma = \begin{pmatrix} \Sigma^+ \\ \Phi^- \\ \Phi^+ \\ \Sigma^- \end{pmatrix} = - \int \frac{d^4 q}{(2\pi)^4} \Gamma^\mu_0 a S(q) \Gamma^\nu_b (p, q) D_{\mu \nu}^{ab} (p - q).$$

→ room for diquark condensation (through finite $\Phi$’s)

→ apply same truncations as in the unbroken case
Dirac structure

$T$- and $\chi$- symmetric, even-parity and color-flavor symmetric
(R. Pisarski, D. Rischke, 1999)

\[
\Sigma_i = -i \vec{p} \cdot \vec{\gamma} \Sigma_{A,i} - i \omega_p \gamma_4 \Sigma_{C,i} = \gamma_4 \left( \Sigma_i^+ \Lambda^+ + \Sigma_i^- \Lambda^+ \right)
\]

\[
\phi_i = \left( \phi_{C,i} + \gamma_4 \hat{p} \cdot \vec{\gamma} \phi_{A,i} \right) \gamma_5 = \gamma_5 \left( \phi_i^+ \Lambda^+ + \phi_i^- \Lambda^- \right)
\]
Dirac and Color-Flavor structure in chiral limit

**Dirac structure**

\( T- \) and \( \chi- \) symmetric, even-parity and color-flavor symmetric
(R. Pisarski, D. Rischke, 1999)

\[
\begin{align*}
\Sigma_i &= -i \hat{p} \cdot \vec{\gamma} \Sigma_{A,i} - i \omega_p \gamma_4 \Sigma_{C,i} = \gamma_4 \left( \Sigma_i^+ \Lambda^+ + \Sigma_i^- \Lambda^- \right) \\
\phi_i &= \left( \phi_{C,i} + \gamma_4 \hat{p} \cdot \vec{\gamma} \phi_{A,i} \right) \gamma_5 = \gamma_5 \left( \phi_i^+ \Lambda^+ + \phi_i^- \Lambda^- \right)
\end{align*}
\]

**color-flavor structure of gap functions (similar for \( \Sigma^+ \))**

2SC with 3 flavors: \( \Phi^+ = \frac{Z_2}{Z_{1F}} \phi_{2SC} \lambda_2 \otimes \tau_2 \)

CFL:
\[
\Phi^+ = \frac{Z_2}{Z_{1F}} \left( \phi_3 \sum_A \lambda_A \otimes \tau_A + \phi_6 \sum_S \lambda_S \otimes \tau_S \right)
\]
analytical result in weak coupling (Q. Wang, D. Rischke, 2001):

$$\phi_{weak}^+ = 512 \pi^4 \left( \frac{2}{N_f g^2} \right)^{\frac{5}{2}} e^{-\pi^2+\frac{4}{8}} \mu e^{\frac{-3\pi^2}{\sqrt{2}g}} \times \begin{cases} 1 \end{cases} \frac{2SC}{2^{-1/3}CFL}$$

\[ \]
... with $\alpha_s$ from lattice QCD studies:

- huge deviations from extrapolated weak coupling result!
  $\phi_{2SC}^+(\mu \approx 400\text{MeV}) > 60\text{MeV}$!
- ratio of $\phi_3^+$ to $\phi_{2SC}^+$ similar to weak coupling result
- large values for anti-quasiparticle pairing
Momentum dependence of gap-functions

relative dependence of $\phi_{2SC}^+$ on $|\vec{p}|$ on Fermi surface ($p_4 = 0$):

- weak coupling regime: gap function concentrated around Fermi momentum (colinear scattering dominates)
- strong coupling regime: no scale separation ($\Lambda_{QCD} \approx p_F$)
dispersion relation for massless fermions:

\[ S^+(|\vec{p}|, p_4) = - \int_{-\infty}^{\infty} d\omega \frac{\rho(|\vec{p}|, \omega)}{2\pi} \frac{\rho(|\vec{p}|, \omega)}{-ip_4 + \mu - \omega} \]

Fredholm equation (1st kind):
→ need positivity of \( \rho \) for inversion
→ MEM
Spectral functions via Maximum Entropy Method

dispersion relation for massless fermions:

\[ S^+ (|\vec{p}|, p_4) = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho(|\vec{p}|, \omega)}{-ip_4 + \mu - \omega} \]

Fredholm equation (1st kind):
→ need positivity of \( \rho \) for inversion
→ MEM

for gapped 2SC at \( \mu = 1 \text{GeV} \):
Spectral functions via Maximum Entropy Method

dispersion relation for massless fermions:

\[ S^+ (|\vec{p}|, p_4) = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho(|\vec{p}|, \omega)}{-ip_4 + \mu - \omega} \]

Fredholm equation (1st kind):
\[ \rightarrow \text{need positivity of } \rho \text{ for inversion} \]
\[ \rightarrow \text{MEM} \]

- access to dispersion relation and width of quasiparticles
- particle/holes vanishing below/above Fermi momenta
- small group velocity (\( \approx .5c \))
density given by

\[ \rho = \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left( Z_2 \gamma_4 S^+(p) \right) = \sum_i \frac{g_i}{(2\pi)^3} \int d^3 p \, n_i(p) =: \sum_i \frac{g_i}{6\pi^2} p_i^3 \]

- unbroken phase non-Fermi liquid
- Luttinger’s theorem not valid, i.e. \( p_i \neq p_F \)
Coherence lengths

extract coherence length from diquark correlator

\[
\langle \langle \psi(x)^T CM_i \psi(y) \rangle \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} M_i \sum_{e=\pm} T^+_i(e)(p) \Lambda^e_p
\]

by

\[
\xi^2_{i,e} = \frac{\int d^4 p |\nabla_{\vec{p}} T^+_i(e)(p)|^2}{\int d^4 p |T^+_i(e)(p)|^2}
\]

\[\xi_+/\lambda \sim O(10) \quad \rightarrow \text{BEC-BCS transition?}\]

\[\text{even smaller with coupling from lattice studies}\]
For $\mu, \phi, \Lambda_{QCD} \ll p$ the gap-equation takes the form

$$\sum_i \phi_{C,i}^+(p) M_i \sim -\frac{3}{16\pi} \sum_i \lambda_i^T M_i \lambda \left( \frac{\alpha_s(p^2)}{p^2} \int_{p^2} dq^2 \phi_{C,i}^+(q) + \int_{p^2} dq^2 \frac{\alpha_s(q^2) \phi_{C,i}^+(q)}{q^2} \right)$$

with regular asymptotic form

$$\phi_{C,i}^+(p) \propto \frac{1}{p^2} \left( \ln \left( \frac{p^2}{\Lambda^2} \right) \right)^{\gamma_{\phi}^{-1}}$$

and

$$\gamma_{\phi} = \begin{cases} \frac{\gamma_m}{2} & \text{attractive channel} \\ -\frac{\gamma_m}{4} & \text{repulsive channel} \end{cases},$$

where $\gamma_m = 12/(33 - 2N_f)$. 
Which is the preferred phase?

effective potential, i.e. the negative pressure, via "Cornwall-Jackiw-Tomboulis" formalism:

$$\mathcal{A}[S] = -\frac{1}{2} \text{Tr}_{p,D,c,f,N} \ln S^{-1} + \frac{1}{4} \text{Tr}_{p,D,c,f,N} \left( 1 - S^{-1}_0 S \right)$$

pressure difference between 2SC and CFL to unbroken phase:
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Dirac and Color-Flavor structure with massive strange quarks

**Dirac structure**

Self-energy and gap-function in $T$-symmetric, even-parity and color-flavor symmetric phase parametrized by (R. Pisarski, D. Rischke, 1999):

\[
\Sigma_i = -i \vec{p} \cdot \vec{\gamma} \Sigma_{A,i} - i \gamma_4 (p_4 + i \mu) \Sigma_{C,i} + \Sigma_{B,i} + \gamma_4 \vec{p} \cdot \vec{\gamma} \Sigma_{D,i}
\]

\[
\phi_i = (\gamma_4 \hat{p} \cdot \vec{\gamma} \phi_A + \gamma_4 \phi_B + \phi_C + \hat{p} \cdot \vec{\gamma} \phi_D) \gamma_5
\]

→ selfconsistent, dynamical treatment of mass function!

→ is there a simple physical interpretation for this functions?

**Color-flavor structure**

Extend ansatz from (M. Alford, J. Berges, K. Rajagopal, 1999) to become selfconsistent.
Dependence of pressure difference on $m_s$

-2e-05
0
2e-05
4e-05

\[ \Delta p \] [GeV^4]

-2e-05
0
2e-05
4e-05

$m_s(\nu = 2\text{GeV})$ [MeV]

-2e-05
0
50
100
150
200

- first order phase transition at critical strange quark mass

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Critical strange quark mass

- light quark screen interaction also in strange quark sector
  → only small dynamical chiral symmetry breaking (different to NJL)!!!
  → 2SC never favored?

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Summary

- Selfconsistent solution of DSE by approximating gluon propagator, incorporating medium effects
- Access to excitation spectrum via MEM
- Huge deviations from extrapolated weak coupling results
- 2SC phase for physical strange quark mass?

Outlook

- Spin-1 phases
- Temperature
- Neutrality
- Incorporate Meissner effect
  (quasiparticle RPA, fully selfconsistent incl. of Yang-Mills sector)
Extend ansatz from (M. Alford, J. Berges, K. Rajagopal, 1999):

\[
\Sigma = \begin{pmatrix}
  b + e & b & c_1 \\
  b & b + e & c_1 \\
  c_2 & c_2 & d
\end{pmatrix}, \quad \Phi = \begin{pmatrix}
  b + e & b & c_1 \\
  b & b + e & c_1 \\
  c_2 & c_2 & d
\end{pmatrix}
\]

where rows and columns correspond to

\[(color, flavor) = (r, u), (g, d), (b, s), (r, d), (g, u), (r, s), (b, u), (g, s), (b, d).\]