Wilson Fermions and QCD at Nonzero Isospin Chemical Potential

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Contents

Motivation

- QCD at nonzero baryon chemical potential cannot be simulated on the lattice, but phase quenched QCD, also known as QCD at nonzero isospin chemical potential, can be simulated by probabilistic methods.
- Large pion densities may occur in neutron stars and heavy ion collisions.
- A better understanding of discretization errors helps us to extrapolate to the continuum limit.
- Nuclear matter as a theory of pions.

I. QCD at Nonzero Isospin Chemical Potential

Basic physics

Dirac spectra

Chiral Lagrangian

QCD at Nonzero Isospin Chemical Potential

$$Z_{\rm QCD}(m, \mu_I, T) = \sum_k e^{-\beta(E_k - n_k \mu_I)},$$

where n_k is the isospin charge of state k .

- The lightest states are the pions with $E_k/n_k = m_\pi/2$.
- When $\mu_I < m_\pi/2$ only states with $n_k = 0$ contribute to the low temperature partition function, and the partition function does not depend on μ_I .
- The chiral condensate does not depend on μ_I for $\mu_I < m_\pi/2$.
- The pion masses are μ_I -dependent. The pole mass of the charged pions follows from $-(p_0 - \mu_I)^2 + m_\pi^2 = 0$, so that the charged pions behave as $m_{\pi_k}(\mu) = m_{\pi_k} + q_k \mu_I$.

Bose Condensed Phase

- For $\mu > m_\pi/2$ it is advantageous to create as many pions as possible, and they Bose condense.
- With a repulsive two-body interaction we have

$$E_{\text{vac}} = \frac{n_I}{2} m_\pi + \frac{c_I}{2} n_I^2,$$

so that

$$\mu_I = \frac{dE_{\text{vac}}}{d\mu_I} = \frac{1}{2} m_\pi + c_I n_I.$$

QCD at Nonzero Isospin Chemical Potential

QCD at nonzero isospin chemical potential is the same as QCD at nonzero baryon chemical potential with the fermion determinant replaced by its absolute value (the phase quenched QCD partition function),

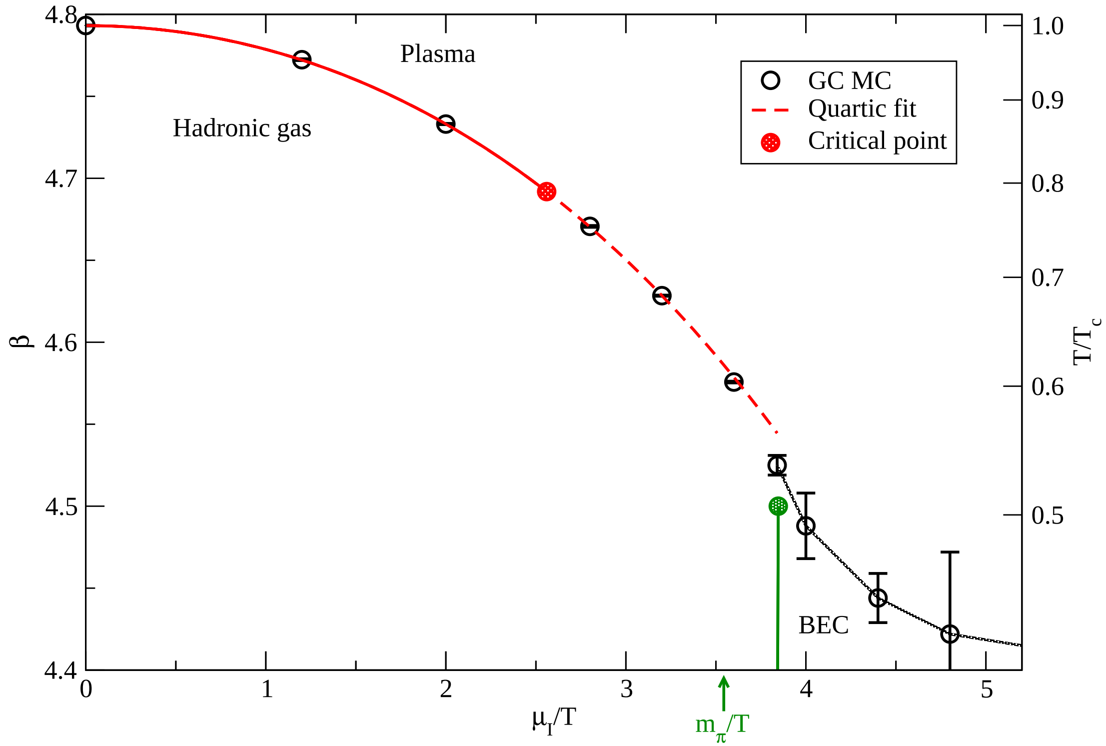
$$
\begin{aligned}
Z_{\rm QCD}(\mu_I) &= \langle \det(D+m+\mu_I\gamma_0)\det(D+m-\mu_I\gamma_0)\rangle \\
&= \langle |\det(D+m+\mu_I\gamma_0)|^2\rangle.
\end{aligned}
$$

Alford-Kapustin-Wilczek-1999

Therefore this partition function can be simulated on the lattice using probabilistic methods. A phase transition to a Bose condensed phase takes place at $\mu = m_\pi/2$.

KSTVZ-2000, Toublan-JV-2000, Son-Stephanov-2000

Phase Diagram at Nonzero Isospin Chemical Potential



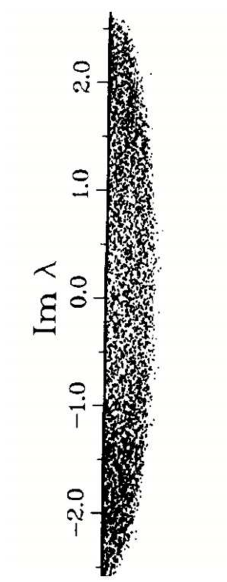
Phase diagram of QCD at nonzero isospin chemical potential (de Forcrand-Stephanov-Wenger-2007).

Agrees with earlier work by Kogut and Sinclair.

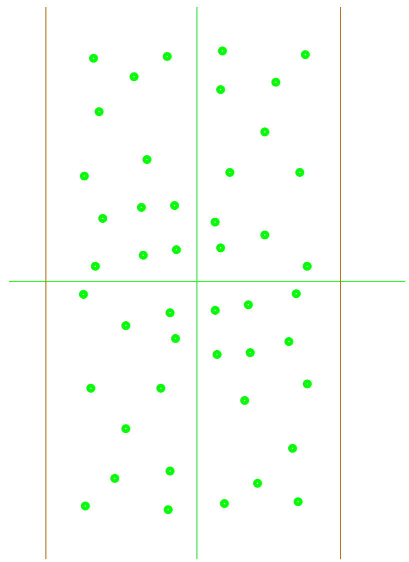
Lattice QCD Dirac Spectra at $\mu_I \neq 0$

The absolute value of the the fermion determinant does not affect the eigenvalue density of the Dirac operator very much, and behaves the same as the eigenvalue density of the quenched Dirac operator.

The Dirac operator at nonzero chemical potential $D = \gamma_\nu D_\nu + \mu\gamma_0$ is nonhermitian with eigenvalues scattered in the complex plane.

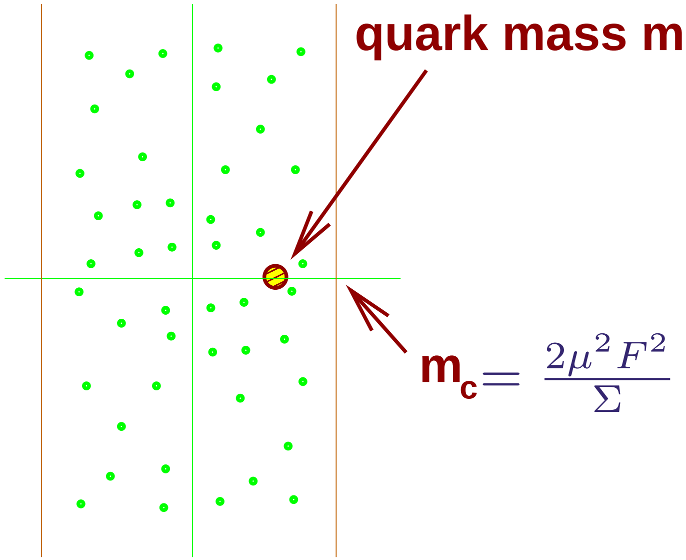


Barbour et al. 1986



Scatter plot of Dirac eigenvalues

Interpretation of the Condensation Phase Transition in Terms of Dirac Spectra



Scatter plot of Dirac eigenvalues

$$\mu^2 = \frac{1}{4} m_\pi^2 = \frac{m\Sigma}{2F^2},$$

can also be written as

$$m = \frac{2\mu^2 F^2}{\Sigma}.$$

The same result is obtaied from a mean field analysis of the phase quenched partition function.

Toublan-JV-2000

Spontaneous Symmetry Breaking Pattern

- Normal phase: $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2) \rightarrow \mathrm{SU}_V(2)$
- The Isospin chemical potential occurs as $\mu_I \tau_3$ and breaks $\mathrm{SU}_V(2) \rightarrow \mathrm{U}_V(1)$.
- In the condensed phase, the chiral condensate becomes charged under isospin and spontaneously breaks $\mathrm{U}_V(1)$. This gives rise to one exactly massless Goldstone boson.

Chiral Lagrangian

In the limit of light quark masses, QCD at nonzero isospin chemical potential reduces to a theory of pions interacting by a chiral Lagrangian.

The chemical potential appears in the QCD Lagrangain as an imaginary vector potential. The QCD partition function is invariant under local gauge transformations of this vector potential, and the chiral Lagrangian also should have this gauge invariance resulting in

$$
\begin{aligned}
\mathcal{L} &= \frac{F_\pi^2}{4}\mathrm{Tr}\nabla_\mu U \nabla_\mu U^\dagger - \frac{1}{2} m\Sigma \mathrm{Tr}(U+U^\dagger), \\
\nabla_0 \Sigma &= \partial_0 \Sigma - \frac{1}{2}\mu_I[\tau_3, \Sigma].
\end{aligned}
$$

Kogut-Stephanov-Toublan-1999, KSTVZ-2000, Toublan-JV-2000, Son-Stephanov-2000

The important point is that the chemical potential does not introduce extra parameters.

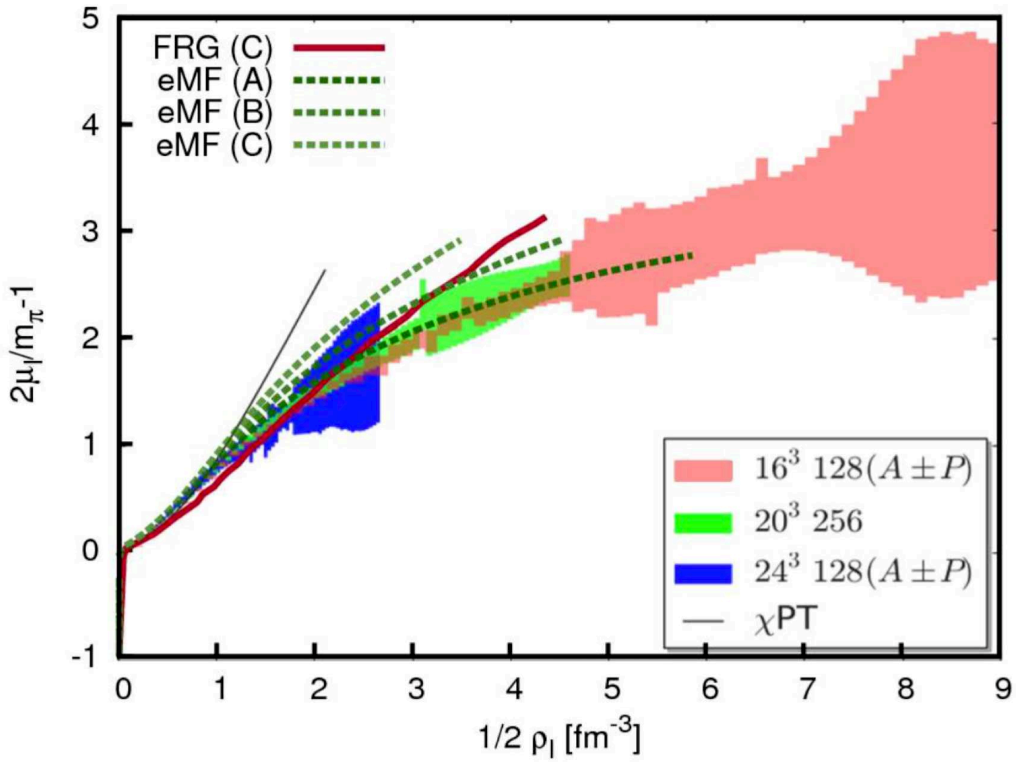


Figure 5: Isospin chemical potential over isospin density at zero temperature and baryon chemical potential in comparison to the lattice data from Ref. [17]. The isospin densities are rescaled here to adjust to the lattice parameters. This mainly compensates for the larger pion mass and decay constant on the lattice.

Detmold-Orginos-Shi-2012, Kamikado-Strodthhof-von Smekal-Wambach-2012

II. Phases of Wilson Fermions

Spectral Properties of Wilson Fermions

Aoki Phase

First Order Scenario

Wilson Dirac operator

Wilson introduced the Wilson term to eliminate doublers

$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{1}{2}a\nabla^*_\mu\nabla_\mu \equiv D + W.$$

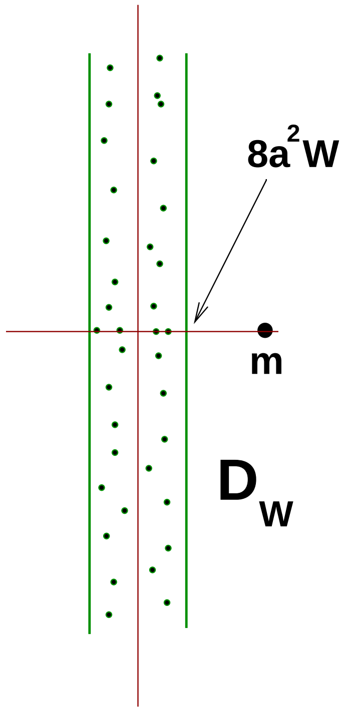
$$\{D_W, \gamma_5\} \neq 0.$$

$$D_W = \gamma_5 D_W^\dagger \gamma_5.$$

Block structure

$$D_W = \begin{pmatrix} aA & id \\ id^\dagger & aB \end{pmatrix}$$

with $A^\dagger = A,\ B^\dagger = B.$



Gap of the Wilson Dirac Spectrum

Structure of the Wilson Dirac operator

$$D_W = \begin{pmatrix} aA & C \\ -C^\dagger & aB \end{pmatrix} \qquad \text{with} \qquad A^\dagger = A, \quad B^\dagger = B.$$

For $a = 0$ we have that ‘

$$\gamma_5(D_W + m) = \begin{pmatrix} m & C \\ C^\dagger & m \end{pmatrix}.$$

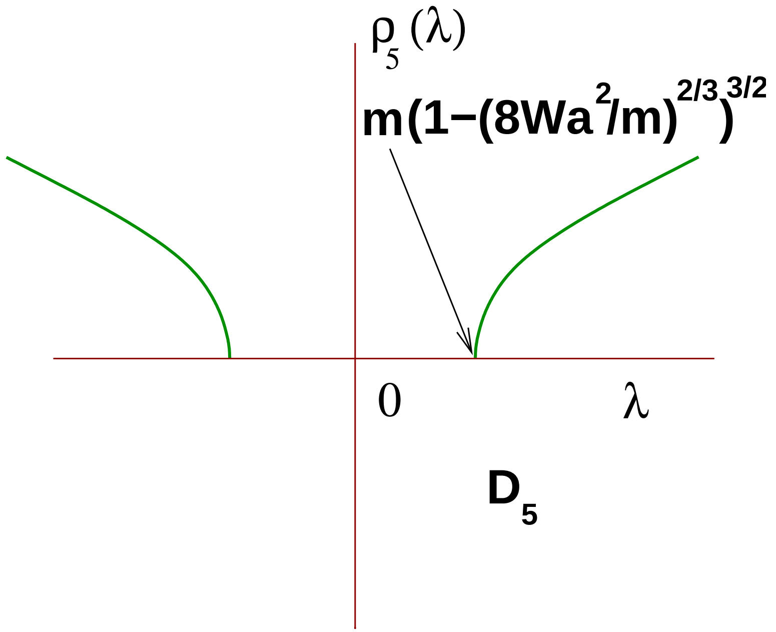
By diagonalizing C we find that the eigenvalues are given by

$$(m - \lambda)(-m - \lambda) - c_k^* c_k = 0 \quad \Longrightarrow \quad \lambda = \pm\sqrt{m^2 + c_k^* c_k}$$

That is why $\gamma_5(D_W + m)$ has a gap $[-m, m]$ for $a = 0$. For $a \neq 0$ states intrude inside the gap.

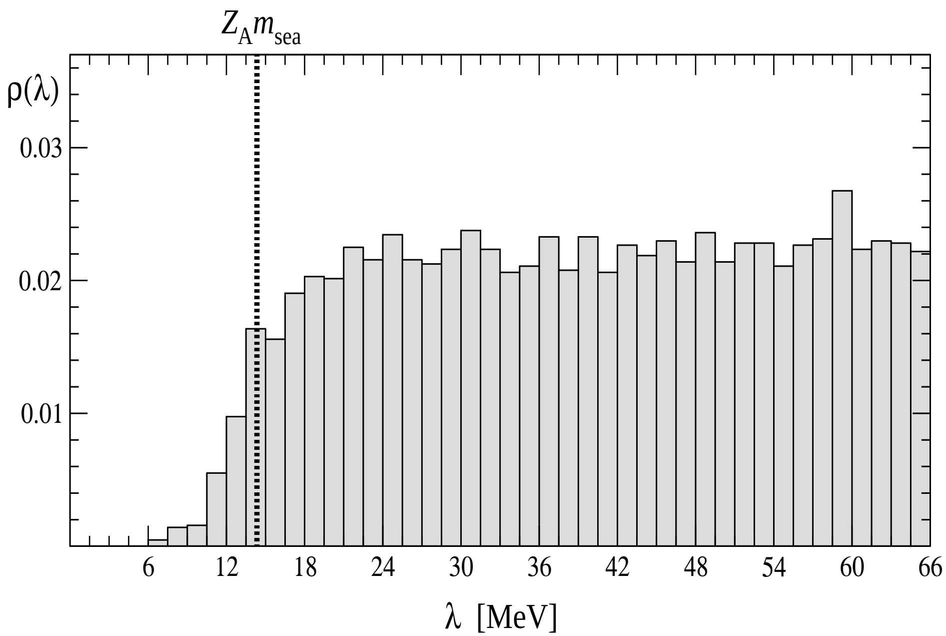
Spectrum of the Hermitian Wilson Dirac Operator

$$D_5 \equiv \gamma_5(D_W + m) = D_5^\dagger.$$



The eigenvalues of D_5 are NOT paired as $\pm\lambda_k$.

Lattice Results for the Wilson Dirac Spectrum



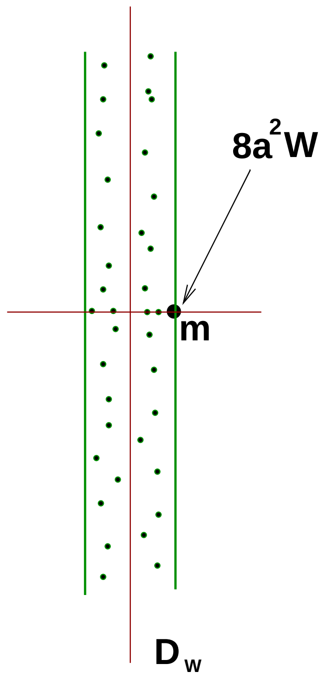
Spectral density of $\gamma_5(D_W + m)$ on a 48×24^3 lattice. Lüscher-2007

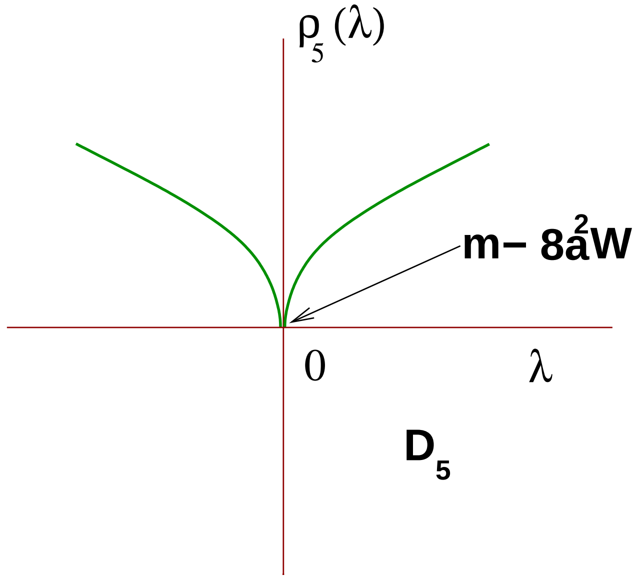
- Dirac spectrum has a gap.
- A Gaussian tail intrudes inside the gap.

Phase Transition for Wilson Fermions

- A phase transition occurs when the quark mass hits the cloud of eigenvalues.
- Since $\det D_W = \det \gamma_5 D_W$, the gap of the Hermitian Dirac operator has to close as well at this point.
- The new phase is known as the Aoki phase.

Onset of the Aoki Phase





(Used units where $m_\pi^2 = m$)

A second order phase transition to the Aoki phase occurs at $m_\pi^2 = 8Wa^2$.

Difference Between Spectra of the Wilson Dirac Operator and the Dirac operator at $\mu_I \neq 0$

- At nonzero chemical potential we still have that $\{\gamma_5, D(\mu)\} = 0$ so that the complex eigenvalues of $D(\mu)$ occurs in pairs $\pm\lambda_k$.
- Because $D_W^\dagger = \gamma_5 D_W \gamma_5$, its complex eigenvalues occur in complex conjugate pairs.
- The spectrum of D_W is not constrained by the imaginary axis and can fluctuate perpendicular to this axis.

Spontaneous Breaking of Isospin and Parity

Since $\gamma_5 D_W$ is Hermitian, the two flavor partition function is positive definite and can be simulated by probabilistic methods.

When the gap is closed, the condensate corresponding to $\gamma_5 D_W$ is given by the discontinuity

$$
\begin{aligned}
& \mathrm{Tr} \frac{1}{\gamma_5(D_W+m)+is} - \mathrm{Tr} \frac{1}{\gamma_5(D_W+m)-is} \\
= \quad & \sum_k \frac{-2is}{\lambda_k^2+s^2} \\
= \quad & -2\pi i \rho(0).
\end{aligned}
$$

This corresponds to the chiral condensate

$$\langle \bar{\psi} \gamma_5 \tau_3 \psi \rangle \neq 0.$$

Symmetry Breaking Pattern

Therefore the vector symmetry is broken spontaneously according to

$$\mathrm{SU}_V(2) \to \mathrm{U}_V(1).$$

Therefore, the Aoki phase has two massless Goldstone bosons with a nonzero isospin charge.

Caveats

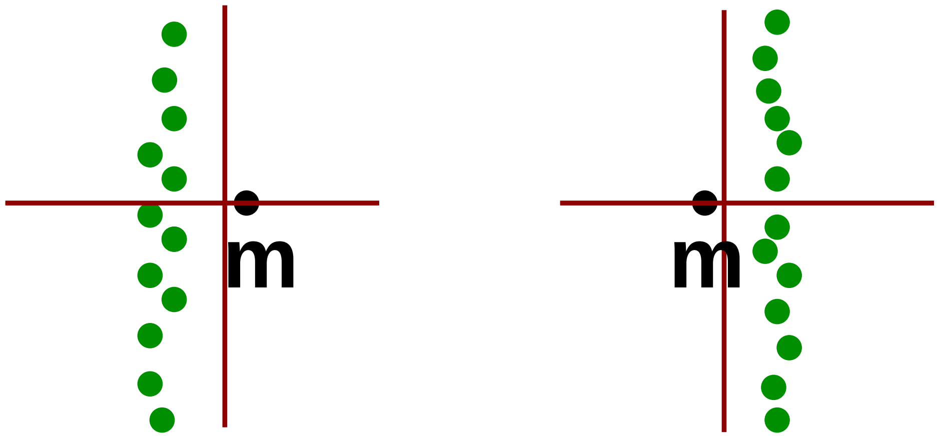
- This argument assumes that the fermion determinant does not affect the cloud of eigenvalues. Therefore, in the quenched case, there always is a transition to the Aoki phase in the approach to the chiral limit.
- For dynamical fermions there will be a feed back from the fermion determinant. Configurations with eigenvalues close to the quark mass are suppressed.
- However, the Wilson Dirac operator is nonhermitian and eigenvalues are not constrained to the imaginary axis in any way.

First Order Scenario and Dirac Spectra

- Because of the fermion determinant the eigenvalues will be repelled from the quark mass.
- The eigenvalues of the Wilson Dirac operator can move perpendicular to the imaginary axis.
- When the quark mass crosses zero the Dirac spectrum jumps to the other side of the imaginary axis. This results in a first order phase transition.
- This is known as the first order scenario.

  Sharpe-Singleton-1998

First Order Scenario and Collective Fluctuations



The fuzzy string of eigenvalues cannot move through the mass because the fermion determinant would vanish at this point. This gives a first order phase transition.

Because the Dirac eigenvalues are at a finite distance from the quark mass, the pion mass is always finite.

Chiral Lagrangian for Wilson Fermions

Up to low-energy constants, the chiral Lagrangian for Wilson fermions is uniquely determined by the transformation properties of QCD. Since the Wilson term has the same invariance properties as the mass matrix one easily finds

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \frac{F^2}{4}\mathrm{Tr}\partial_\mu U \partial_\mu U^\dagger - \frac{1}{2} m V \Sigma \mathrm{Tr}(U + U^\dagger) \\
& + a^2 V W_8 \mathrm{Tr}(U^2 + U^{-2}) + a^2 W_6 [\mathrm{Tr}(U + U^\dagger)]^2 + a^2 W_7 [\mathrm{Tr}(U - U^\dagger)]^2 .
\end{aligned}
$$

Sharpe-Singleton-1998, Rupak-Shoresh-2002,
Bär-Rupak-Shoresh-2004,Damgaard-Splittorff-JV-2011

- This Lagrangian is uniquely determined by the transformation properties of the QCD partition function (spurion formalism).

III. Wilson Fermions at Nonzero Isospin Chemical Potential

Chiral Lagrangian

Symmetries

Phase Diagram

Wilson Fermions at Nonzero Isospin Chemical Potential

- The Aoki phase has two Goldstone bosons with nonzero isospin charge.
- They will condense for an infinitesimal isospin chemical potential.
- In the Aoki phase the Wilson QCD partition function is nonanlytic at $\mu_I = 0$.

Chiral Lagrangian

It is clear how to write down the chiral Lagrangian. Just replace the derivatives in the Wilson chiral Lagrangian by covariant derivatives.

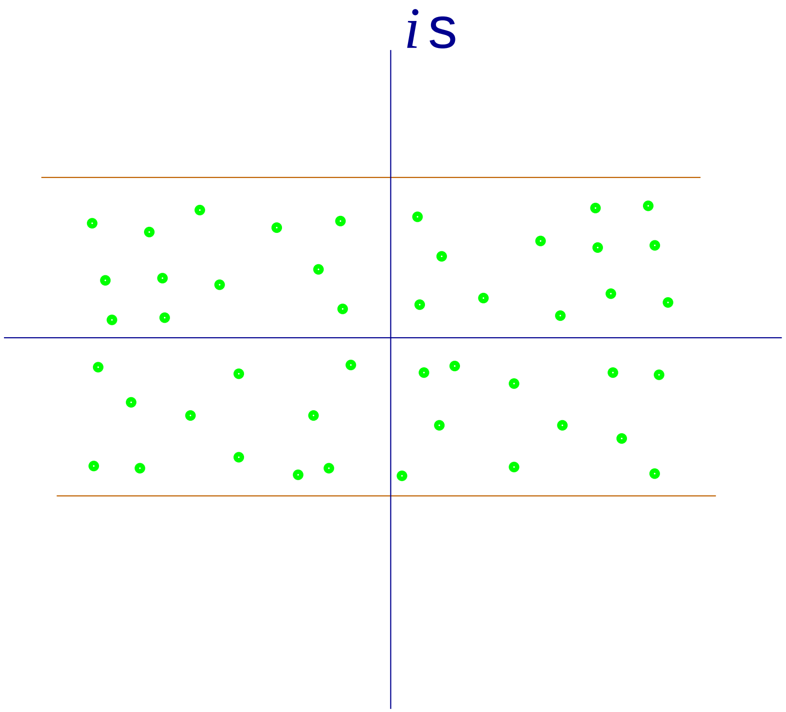
The phases of this Lagrangian can be determined by mean field theory.

What Happens to the Aoki Phase?

At nonzero isospin chemical potential, the Dirac operator

$$\gamma_5(D_W c + m + \mu\gamma_0)$$

becomes nonhermitian.



$$\langle\bar{\psi}\gamma_5\tau_3\psi\rangle = \lim_{s\to 0}\lim_{V\to\infty}\sum_k \frac{-2is}{\lambda_k^2 + s^2} = 0.$$

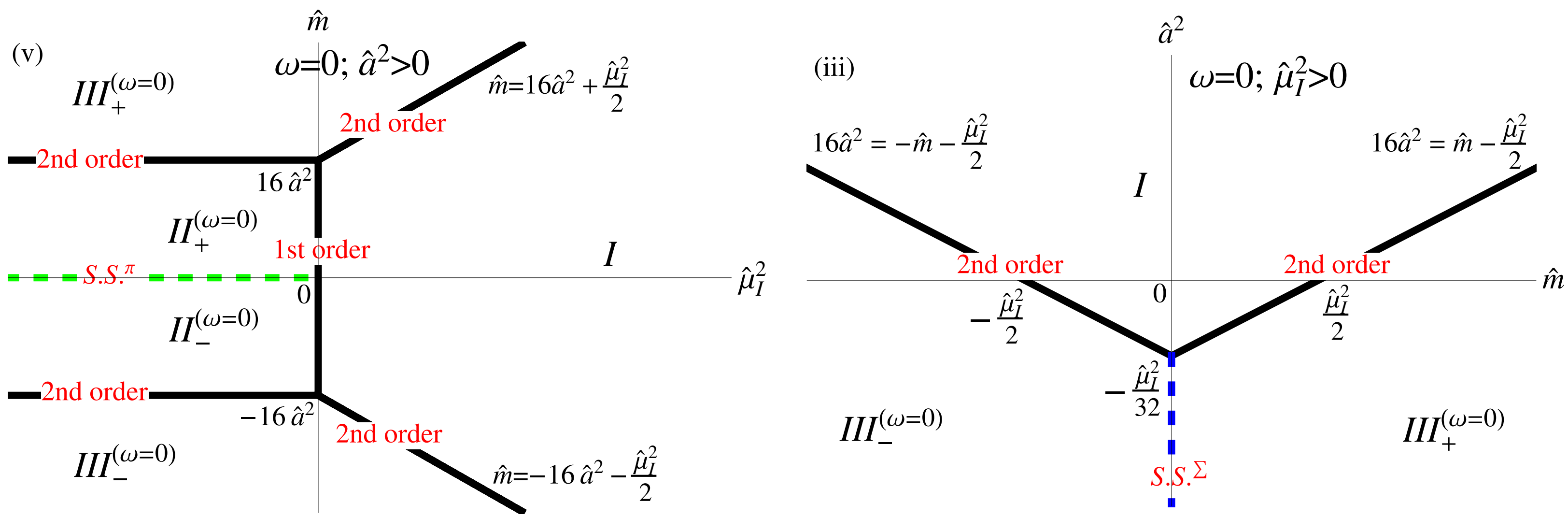
Critical Value of the Pion Mass

- A nonzero μ_I broadens the cloud of eigenvalues.
- The quark mass enters the cloud of eigenvalues at

$$m_\pi^2 = 4\mu_I^2 + 16Wa^2.$$

- Inside this phase the $\mathrm{U}_V(1)$ symmetry is spontaneously broken and we have one exactly massless Goldstone boson as is the case for $a = 0$.

Phase Diagram



Phase I : Condensed phase with one exactly massless Goldstone boson.

Phase III : Normal phase

Phase II : Phase with spontaneously broken parity and isospin, but all pions are massive because $\mu_I \tau_3$ breaks isospin. An imaginary chemical potential does not change the width of the strip of eigenvalues.

Janssen-Kieburg-Splittorff-JV-Zafeiropoulos-2015

IV. Conclusions

- The phases of QCD at nonzero isospin chemical potential as well as of QCD with Wilson fermions can be understood in terms of Dirac spectra. This complements the use of chiral Lagrangians and mean field theory.

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IV. Conclusions

- The phases of QCD at nonzero isospin chemical potential as well as of QCD with Wilson fermions can be understood in terms of Dirac spectra. This complements the use of chiral Lagrangians and mean field theory.
- The Aoki phase is destroyed by an infinitesimally small isospin chemical potential.
- In the chiral limit, the Wilson QCD partition function is nonanalytic at $\mu_I = 0$.