Goldstone bosons in crystalline chiral phases

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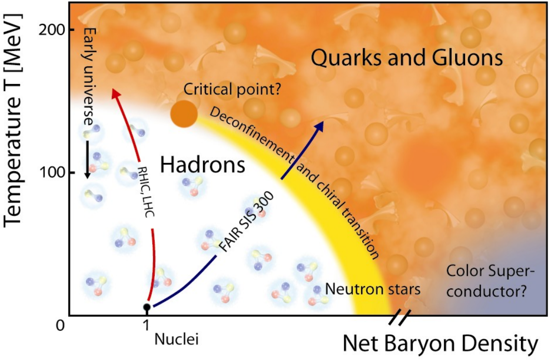
The QCD Phase Diagram

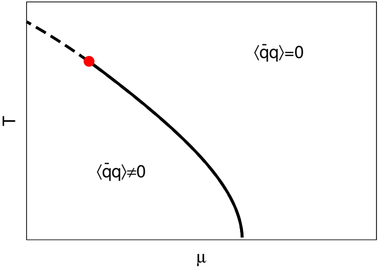


(GSI)

Simpler Phase Diagram

Focus on chiral symmetry

- Spontaneously broken in vacuum
- Order parameter:
  chiral condensate $\langle \overline{q} q \rangle$
- Believed to have first-order phase transition for low temperatures
- Critical endpoint



Most calculations: order parameter constant in space

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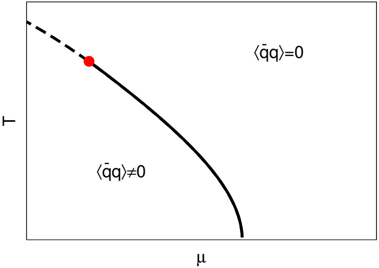

Most calculations: order parameter constant in space

What happens if we allow space dependence?

Inhomogeneous Phase

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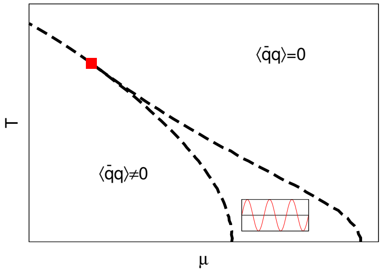
- Space dependent order parameter
- Popular for some time
  - Pion Condensation
  - (Color-) superconductivity
- Studied more recently in lower dimensional models (1+1 D Gross-Neveu model) [Schön, Thies, PRD (2000)]



Inhomogeneous Phase in Nambu–Jona-Lasinio Model

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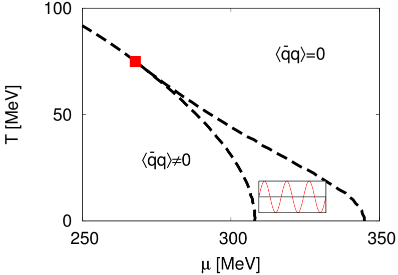
Nambu–Jona-Lasinio model

[Nickel, PRD (2009)]

- Critical endpoint replaced by Lifschitz point
- first order phase transition replaced by inhomogeneous region



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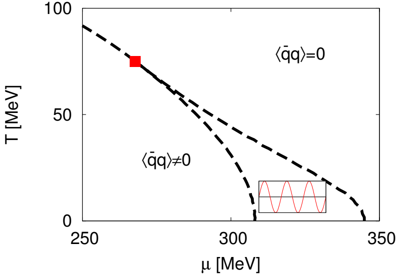

Here:

- Consider fluctuations around mean fields
- Goldstone bosons from spontaneous broken symmetries (chiral and spatial)
- Important for transport properties (e.g. cooling in neutron stars)
- Could lead to instabilities (Landau-Peierls instability)

Nambu–Jona-Lasinio Model

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- NJL Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi + G_S\left(\left(\overline{\psi}\psi\right)^2 + \left(\overline{\psi}i\gamma_5\tau^a\psi\right)^2\right)$$

- Derive thermodynamic properties from grand potential Ω
- Mean-field approximation

$$S(\vec{x}) = \langle\overline{\psi}\psi\rangle, \quad P(\vec{x}) = \langle\overline{\psi}i\gamma_5\tau^3\psi\rangle$$

  - keep space dependence, but neglect time dependence

$$\mathscr{L}_{MF} = \overline{\psi}S^{-1}\psi + V(\vec{x})$$

$$S^{-1} = i\partial\!\!\!/ - \underbrace{\left(m - 2G_S(S(\vec{x}) + i\gamma_5\tau^3 P(\vec{x}))\right)}_{=:M(\vec{x})}, \qquad V(\vec{x}) = -G_S\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

Space dependent Mass

- Space dependent mass

$$M(\vec{x}) = m - 2G_S\left(S(\vec{x}) + iP(\vec{x})\right)$$

- Crystal with unit cell vectors $\vec{n}_i$, $i = 1, 2, 3$
- Periodicity in mass

$$M(\vec{x}) = M(\vec{x} + \vec{n}_i)$$

- Fourier transformation

$$M(\vec{x}) = \sum_{\vec{q}_k} M_{\vec{q}_k} e^{i\vec{q}_k\vec{x}}$$

- Wave vector $\vec{q}_k$ spans reciprocal lattice: $\vec{q}_k\vec{n}_i = 2\pi N_{ki}, \quad N_{ki} \in \mathbb{Z}$

Grand Potential

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Arrive at grand potential

$$\Omega = -N_C N_F \frac{1}{V} \sum_{E_\lambda} T \ln \left[2\cosh\left(\frac{E_\lambda - \mu}{2T} \right) \right] + \frac{1}{V} \int d^3x \frac{|M(\vec{x}) - m|^2}{4G_S}$$

with eigenvalues E_λ of H in momentum space

$$H(\vec{p}_m, \vec{p}_{m'}) = \begin{pmatrix} -\vec{\sigma}\vec{p}_m \delta_{\vec{p}_m,\vec{p}_{m'}} & -\sum_{\vec{q}_k} M_{\vec{q}_k} \delta_{\vec{q}_k,(\vec{p}_m - \vec{p}_{m'})} \\ -\sum_{\vec{q}_k} M_{\vec{q}_k} \delta_{\vec{q}_k,(\vec{p}_{m'} - \vec{p}_m)} & \vec{\sigma}\vec{p}_m \delta_{\vec{p}_m,\vec{p}_{m'}} \end{pmatrix}$$

$\rightarrow$ Matrix in momentum space

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$\rightarrow$ Matrix in momentum space

Propagator has different incoming and outgoing momenta

$$S^{-1}(p_{out}, p_{in}) = \gamma^0 (p_0 \delta_{p_{out},p_{in}} - H(p_{out}, p_{in}))$$

Gap Equation

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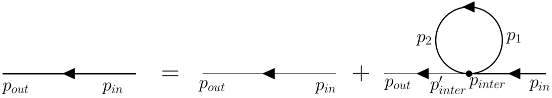


Gap equation

$$S^{-1}(p_{out}, p_{in}) = S_0^{-1}(p_{out}, p_{in}) - \Sigma(p_{out}, p_{in})$$

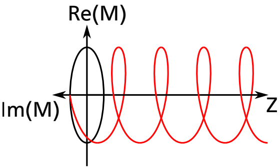
Self energy

$$\Sigma(p_{out}, p_{in}) = 2G_S \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\left[\mathbb{1}\,\mathrm{Tr}\left(\mathbb{1}\, iS(p_1, p_2)\right) + i\gamma^5\tau_3 \mathrm{Tr}\left(i\gamma_5\tau^3 iS(p_1, p_2)\right)\right]$$
$$\delta(p_{out} + p_2 - p_{in} - p_1)$$

Modulated Order Parameter

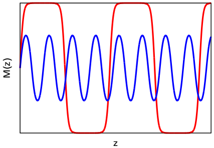
Different (periodical) modulations:

Chiral Density Wave



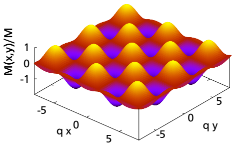
$$M(z) = M \ \exp(iqz)$$

Solitonic modulation



$$M(z) = \Delta\nu \ \mathrm{sn}(\Delta z|\nu)$$

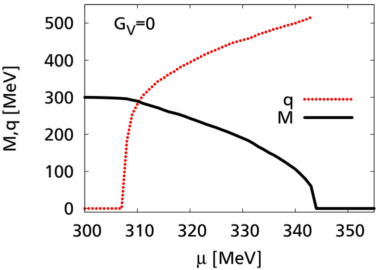
2D 'Egg' carton

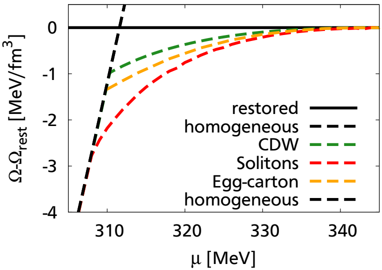


$$M(x, y) = M \ \cos(qx)\cos(qy)$$

[Carignano, Buballa, PRD (2012)]

Results





[Carignano, Buballa, PRD (2012)]

Phonons and Goldstone modes

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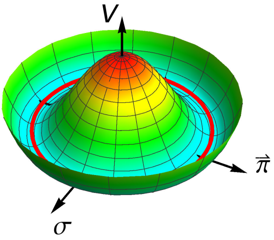
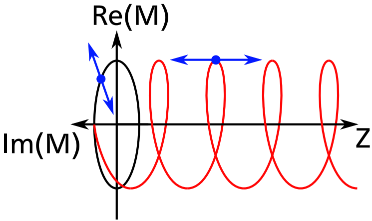
Homogeneous phase



Chiral symmetry broken
Pions as Goldstone modes

Inhomogeneous Phase



Rotational and translational symmetry broken

- fluctuations in amplitude
- fluctuations in space

Details in: [Lee, Nakano, Tsue, Tatsumi, Friman, PRD (2015)]

Long range correlations

[Lee, Nakano, Tsue, Tatsumi, Friman, PRD (2015)]

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In Ginzburg-Landau Model (valid for $M \approx 0$)
Long range correlations ($\phi = M/2G_S$)

$$f_{ij}(x) = \langle \phi_i(x)\phi_j^*(0) \rangle$$

in z-direction

$$\langle \phi(z\vec{e}_z) \cdot \phi^*(0) \rangle \propto \frac{1}{2}\Delta^2 \cos(qz) \left(\frac{z}{z_0}\right)^{-T/T_0}$$

and longitudinal

$$\langle \phi(x_\perp \vec{e}_\perp) \cdot \phi^*(0) \rangle \propto \frac{1}{2}\Delta^2 \left(\frac{x_\perp}{x_0}\right)^{-2T/T_0}$$

Algebraic decay

Full NJL model

Inverse propagator

$$S^{-1} = i\gamma^\mu \partial_\mu - \frac{1}{2}\left[(1 + \gamma_5\tau^3)M(\vec{x}) + (1 - \gamma_5\tau^3)M^*(\vec{x})\right]$$

Allow small perturbations in space with bosonic field $\vec{u}(x)$ up to quadratic order

$$\begin{aligned} M(\vec{x}) \to M_u(\vec{x}) &= M(\vec{x} + \vec{u}(x)) \\ &\approx \sum_{\{\vec{q}\}} M_{\vec{q}} e^{i\vec{q}\vec{x}} \left(1 + i\vec{q}\cdot\vec{u}(x) - \frac{1}{2}(\vec{q}\cdot\vec{u}(x))^2\right) \end{aligned}$$

Partition function

$$Z = \int i\mathscr{D}\psi^\dagger \int \mathscr{D}\psi \int \mathscr{D}\vec{u}\, \exp\left(\int_{x_E} \overline{\psi} S^{-1}[\vec{u}]\psi + V[\vec{u}]\right), \qquad V[\vec{u}] = \frac{|M_u(\vec{x})|^2}{4G_S}$$

Integrate out Fermions

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$$Z = \int i\mathscr{D}\psi^\dagger \int \mathscr{D}\psi \int \mathscr{D}\vec{u} \exp\left(\int_{x_E} \overline{\psi} S^{-1}[\vec{u}]\psi + V[\vec{u}]\right)$$

$$= \int \mathscr{D}\vec{u} \exp\left(\mathrm{Tr}\ln\left(S_{MF}^{-1} + \Sigma_1[\vec{u}] + \Sigma_2[\vec{u}^2]\right) + \int_{x_E} V_{MF}(\vec{x}) + V_1[\vec{u}] + V_2[\vec{u}^2]\right)$$

expand logarithm

$$\ln\left(S_{MF}^{-1} + \Sigma\right) = \ln\left(S_{MF}^{-1}(1 + S_{MF}\Sigma)\right)$$

$$= \ln S_{MF}^{-1} + S_{MF}\Sigma_1[\vec{u}] + S_{MF}\Sigma_2[\vec{u}^2] - \frac{1}{2} S_{MF}\Sigma_1[\vec{u}] S_{MF}\Sigma_1[\vec{u}] + \mathscr{O}(u^3)$$

Separate different orders of u in the partition function

$$Z = \int \mathscr{D}u \exp(\omega_{MF} + \omega_1[\vec{u}] + \omega_2[\vec{u}^2])$$

Individual Contributions

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$\omega_1[u] = 0$ from gap equation

$$\omega_2[u^2] = \underbrace{\mathrm{Tr}\left[S_{MF}\Sigma_2[u^2]\right] + \int_{x_E} \frac{1}{2}\frac{M(\partial_z^2 M^*)u^{*2} + M^*(\partial_z^2 M)u^2}{4G_S}}_{=0 \text{ gap equation}}$$

$$- \frac{1}{2}\mathrm{Tr}\left(S_{MF}\Sigma_1[u]S_{MF}\Sigma_1[u]\right) + \int_{x_E} \frac{|(\partial_z M)u|^2}{4G_S}$$

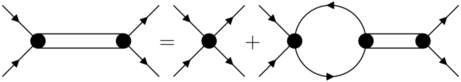
second part needs calculation of

$$\mathrm{Tr}\left(S_{MF}\Sigma_1[u]S_{MF}\Sigma_1[u]\right)$$

similar object to polarization loops for meson calculations

Mesons in NJL

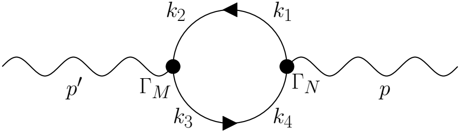
Start from Bethe-Salpeter equation



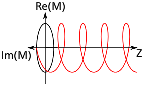
one can get the meson propagator

$$D_{MN}(p,p') = \frac{2G_S}{1 - 2G_S J_{MN}(p,p')}$$

with polarization loop $J_{MN}(p,p')$



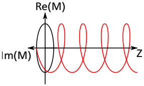
$\propto \mathrm{Tr}\,[\Gamma_M S \Gamma_N S]$



Chiral transformation for CDW

Hamiltonian Chiral Density Wave

$$H(p,p') = -\gamma_0\gamma^k p_k \delta(p-p') + \gamma_0 \frac{M}{2}\left[(\mathbb{1}+\gamma_5\tau^3)\delta(p-p'+q) + (\mathbb{1}-\gamma_5\tau^3)\delta(p-p'-q)\right]$$



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Apply a rotation in chiral space [Dautry, Nyman, Nucl. Phys. (1979)]

$$U(k,k') = \frac{1}{2}\left[(\mathbb{1}+\gamma_5\tau^3)\delta(k-k'-q/2) + (\mathbb{1}-\gamma_5\tau^3)\delta(k-k'+q/2)\right]$$

to the Hamiltonian

$$H'(p,p') = U^\dagger H U = \gamma_0\left[-\gamma^k p_k - \gamma^k\gamma_5\tau^3 q_k/2 + M\right]\delta(p-p')$$

From this get eigenvalues

$$E_\pm(\vec{p}) = \sqrt{\vec{p}^2 + M^2 + \vec{q}^2/4 \pm \sqrt{(\vec{p}\cdot\vec{q})^2 + \vec{q}^2 M^2}}$$

CDW propagator

CDW: analytic expression for propagator
Inverse propagator

$$S^{-1}(p, p') = \gamma^0 \left(p_0 \delta(p - p') - H(p, p') \right)$$

apply chiral transformation

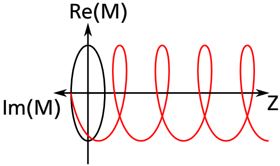
$$S = U S' U$$

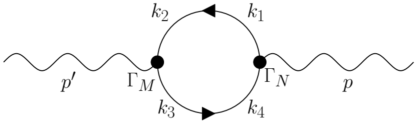
with

$$S'(k) = \frac{1}{N(k)} \left[A(k) + \gamma_5 \tau^3 B(k) + \gamma_\mu C^\mu(k) + \gamma_5 \tau^3 \gamma_\mu D^\mu(k) + \gamma_5 \tau^3 \gamma_\mu \gamma_\nu E^{\mu\nu}(k) \right]$$

diagonal in momentum space

Pions in the inhomogeneous phase





For the CDW the π_3 (with $\Gamma_{\pi_3} = i\gamma_5\tau^3$) is no NG boson
Instead

$$\Gamma_{\pi_1} = i\gamma_5\tau^1, \qquad \Gamma_{\pi_2} = i\gamma_5\tau^2, \qquad \Gamma_{\bar{\pi}}(z) = -\mathbb{1}\sin qz + i\gamma_5\tau^3\cos qz$$

$$J_{\bar{\pi}\bar{\pi}}(p,p') = i\left(\prod_{j=1}^{4}\int\frac{d^4k_j}{(2\pi)^4}\right)\mathrm{Tr}\left[\Gamma_{\bar{\pi}}(p+k_4-k_1)S(k_1,k_2)\Gamma_{\bar{\pi}}(k_2-k_3-p')S(k_3,k_4)\right]$$

Integrating δ-functions

Fourier transformation

$$\Gamma_{\bar{\pi}}(p) = \frac{i}{2}\left[(-\mathbb{1} + \gamma_5\tau^3)\delta(p-q) + (\mathbb{1} + \gamma_5\tau^3)\delta(p+q)\right]$$

Inserting $S = US'U$

Integrating over internal momenta

$$J_{\bar{\pi}\bar{\pi}}(p, p') = i\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[i\gamma_5\tau^3 S'(k) i\gamma_5\tau^3 S'(k-p)\right]\delta(p-p')$$

yields NG Boson via gap equation

$$\lim_{p=p'\to 0}(1 - 2G_S J_{\bar{\pi}\bar{\pi}}(p, p')) = 0$$

$$\Rightarrow \quad p_0 = m_{\bar{\pi}} = 0$$

Summary and Outlook

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Mean-Field Calculations:

- Crystalline phase replaces first order phase transition and critical endpoint in phase diagram
- One dimensional modulations favored over two dimensional

Bosonic exitations

- Explicit construction of Goldstone modes in CDW
- Conceptional difficult due to non-diagonal structure of propagator
- Calculations simplified by chiral transformations
- Goldstone mode identified

Outlook

- Calculate dispersion relations of Goldstone Bosons
- Derive transport propeerties
- Applications for beyond mean-field calculations

Thank you

Full CDW propagator

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Of course more complicated for full CDW propagators

$$S' = \frac{1}{N}\left[A + \gamma^5\tau^3 B + \gamma_\mu C^\mu + \gamma^5\tau^3\gamma_\mu D^\mu + \gamma^5\tau^3\gamma_\mu\gamma_\nu E^{\mu\nu}\right]$$

$$A = M\left(k^2 - M^2 - \frac{1}{4}q^2\right), \quad B = -Mq\cdot k$$

$$C_\mu = k_\mu\left(k^2 - M^2 + \frac{1}{4}q^2\right) - \frac{1}{2}q_\mu\ q\cdot k$$

$$D_\mu = -k_\mu\ q\cdot k + \frac{1}{2}q_\mu\left(k^2 + M^2 + \frac{1}{4}q^2\right)$$

$$E_{\mu\nu} = q_\mu k_\nu M$$

$$N = \left(k^2 - M^2 - \frac{1}{4}q^2\right)^2 + q^2k^2 - (q\cdot k)^2$$

Spinodials

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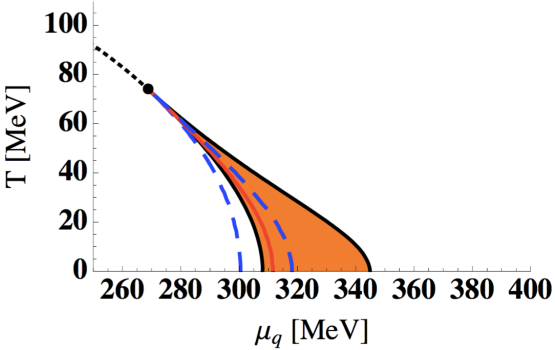

[Nickel, PRD (2009)]